Journal of Optimization, Differential Equations and Their Applications

ISSN (print) 2617-0108

ISSN (online) 2663-6824

Volume 31

Issue 1

2023

Online Edition at http://model-dnu.dp.ua

Journal of Optimization, Differential Equations and Their Applications

Editor-in-Chief
Peter I. Kogut
Department of Mathematical Analysis and Optimization
Oles Honchar Dnipro National University
72, Gagarin av., Dnipro 49010, Ukraine
(+380) 67631-6755

p.kogut@i.ua

EDITORIAL BOARD

Journal of Optimization, Differential Equations and Their Applications welcomes application-oriented articles with strong mathematical content in scientific areas such as deterministic and stochastic ordinary and partial differential equations, mathematical control theory, modern optimization theory, variational, topological and viscosity methods, qualitative analysis of solutions, approximation and numerical aspects. JODEA also provides a forum for research contributions on nonlinear differential equations and optimal control theory motivated by applications to applied sciences.

JODEA is published twice a year (in June and December).
ISSN (print) 2617–0108, ISSN (on-line) 2663–6824, DOI 10.15421/14233101.

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 1–21, DOI 10.15421/142301

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

OPTIMAL CONTROL PROBLEM FOR NON-LINEAR DEGENERATE PARABOLIC VARIATION INEQUALITY: SOLVABILITY AND ATTAINABILITY ISSUES

Nina V. Kasimova,* Olha P. Kupenko,† Iryna M. Tsyganivska‡

Abstract. We investigate the optimal control problem with respect to coefficients of the degenerate parabolic variational inequality. Since problems of this type can have the Lavrentieff effect, we consider the optimal control problem in a class of so-called H-admissible solutions. We substantiate the attainability of H-optimal pairs via optimal solutions of some nondegenerate perturbed optimal control problems.

Key words: parabolic variation inequality, optimal control problem, control in coefficients, approximation, existence result, attainability result.

2010 Mathematics Subject Classification: 49J20, 49J21, 49J40, 47J20, 58J37.

Communicated by Prof. V.V. Semenov

1. Introduction

The purpose of this paper is to investigate optimal control problem associated with a degenerate parabolic inequality. The control is a matrix of coefficients in the main part of elliptic operator. It is well known that degenerate control problems of this type may admit nonuniqueness of admissible solution classes, which implies non-uniqueness of optimal solutions of particular kind and the optimal control problem in the coefficients can be stated in different forms depending on the choice of the class of admissible solutions (for example W- or H-solutions if we consider the weighted Sobolev space W or its subspace H as the phase space, correspondingly) (see [1], [2] and references there). These spaces allow to enlarge the class of boundary value problems and variational inequalities which are solvable by functional-analytical methods. In fact, we consider variational inequality with some degenerate weight function which is not bounded away from zero and infinity but only satisfying some local integrability conditions. Under these assumptions the nonlinear differential operator in our inequality is not coercive in the classical sense.

*Faculty of Mechanics and Mathematics, Department of Integral and Differential Equations, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, Kyiv, 01601, Ukraine, kasimova@knu.ua

†System Analysis and Control Department, Dnipro University of Technology, 19, Yavornitskii av., 49005 Dnipro, Ukraine, kupenko.olga@gmail.com

‡Faculty of Mechanics and Mathematics, Department of Algebra and Computer Mathematics, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, Kyiv, 01601, Ukraine, itsy8009@knu.ua

Since the range of OCPs in coefficients is very wide, including as well optimal shape design problems, optimization of certain evolution systems, some problems originating in mechanics and others, this topic has been widely studied by many authors (see [1]- [3], [6] and others).

As F. Murat showed (see [7]), in general, such problems have no solution even if the original elliptic equation is non-degenerate. It turns out that this feature is typical for the majority of problems for optimal control in coefficients. So,we have to restrict our optimization problem by introducing some additional control constrains (see, for instance, [8]). An optimal control problem for a variational inequality with the so-called anisotropic p-Laplacian in the principle part of this inequality is studied in [9] where the authors showed that the original problem is well-posed and derived existence of optimal pairs. In [10] an optimal control problem associated to Dirichlet boundary value problem for non-linear elliptic equation on a bounded domain is considered. In [6] the authors study the existence of optimal solutions in coefficients associated to a linear degenerate elliptic equa-tion with mixed boundary condition where by control variable they mean a weight coefficient in the main part of the elliptic operator. The sufficient conditions of the existence of weak solutions to one class of Neumann boundary value problems (BVP) are obtained in [11], and moreover, the authors propose a way for their approximation. In [12] the existence of H-optimal solutions for optimal control problem in coefficients for degenerate variational elliptic inequalities of monotone type in the class of so-called generalized solenoidal controls was proved. The solvability results for optimal control problems for degenerate elliptic and parabolic variation inequalities one can find in [13–16].

Taking into account a wide spectrum of application of the optimal control theory, in particular, we deal with possibilities of some types of approximation of original problems by those that are better researched and converge to the original problems in a suitable way. As for problems similar to the one studied in the given paper, in application a degenerate weight ρ occurs as the limit of a sequence of non-degenerate weights ρ for which the corresponding "approximate" optimal control problem is solvable. Thus, naturally, it arises the question: if limit points of the family of admissible solutions to the perturbed problems appear to be admissible solutions to the original problem, whether all optimal solutions are attainable in this sense? Note that for some optimal control problems the attainability and approximability questions remain in the focus of attention. In particular, similar questions were raised in [17] where the author studies the attainability issue for optimal control problem in coefficients for degenerate variational inequality of monotone type in the class of H-admissible solutions. In [2] the authors prove the existence of W-solutions to the optimal control problem and provide way for their approximation. In [18, 19] the author investigates the attainability issue for optimal control problem for degenerate linear elliptic and parabolic inequalities respectively.

Here we concentrate on the solvability of optimal control problem in coefficients for degenerate parabolic inequality in the so-called class of H-admissible solutions. Moreover, we are interested about attainability of H-optimal solutions to degene-rate problems via optimal solutions of non-degenerate problems.

2. Preliminaries and Notations

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary. For any subset $E \subset \Omega$ we denote by $|E|$ its N-dimentional Lebesgue measure $\mathcal{L}(E)$. The space $W_0^{1,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the classical Sobolev space $W^{1,1}(\Omega)$. Let p be a real number such that $2 \leq p < \infty$ and let q be its conjugate, namely $p^{-1} + q^{-1} = 1$. We say that a weight function $\rho = \rho(x)$ is degenerate in $\mathbb{R}^N$ if

$$\rho(x) > 0 \text{ a.e. in } \mathbb{R}^N \text{ and } \rho + \rho^{-1/(p-1)} \in L^1_{loc}(\mathbb{R}^N), \tag{2.1}$$

and the sum $\rho + \rho^{-1/(p-1)}$ does not belong to $L^\infty(\Omega)$, in general. For a given $\Omega \in \mathbb{R}^N$ we associate to this function the weighted Sobolev space $W = W(\Omega, \rho dx)$ which is a set of functions $y \in W_0^{1,1}(\Omega)$ for which the norm

$$\|y\|_\rho = \left(\int_\Omega \left(|y|^p + \rho \sum_{i=1}^N \left| \frac{\partial y}{\partial x_i} \right|^p \right) dx \right)^{1/p} \tag{2.2}$$

is finite.

Together with W let us consider the space $H = H(\Omega, \rho dx)$ which is the closure of $C_0^\infty(\Omega)$ in W.

Note that the spaces W and H are reflexive Banach spaces with respect to the norm $\|\cdot\|_\rho$ due to the estimate

$$\int_\Omega |\nabla y|\, dx \leq \left(\int_\Omega \rho |\nabla y|_p^p \, dx \right)^{1/p} \left(\int_\Omega \rho^{-1/(p-1)} \, dx \right)^{p/(p-1)} \leq C \|y\|_\rho,$$

where $|\eta|_p = \left(\sum_{k=1}^N |\eta_k|^p \right)^{1/p}$ is a Hölder norm of order p in $\mathbb{R}^N$. It is clear that $H \subseteq W$.

Since the smooth functions are in general not dense in the weight Sobolev space W, it follows that $H \neq W$; that is, for a "typical" degenerate weight ρ the identity $W = H$ is not always valid (for the corresponding examples we refer to [3,5]). However, if ρ is a non-degenerate weight function, that is, ρ is bounded between two positive constants, then it is easy to verify that $W = H = W_0^{1,p}(\Omega)$. We recall that the dual space of H is $H^* = W^{-1,-p/(p-1)}(\Omega, \rho^{-1/(p-1)} dx)$ (for more details see [6]).

Remark 2.1. Assume that there exists a value $\nu \in \left(\frac{N}{p}, +\infty \right) \cap \left[\frac{1}{p-1}, +\infty \right)$ such that $\rho^{-\nu} \in L^1(\Omega)$. Then the following result takes place (see [6]): relation (2.1) implies that

$$|||y|||_{\rho,\Omega} = \left[\int_\Omega \sum_{i=1}^N \left| \frac{\partial y}{\partial x_i} \right|^p \rho \, dx \right]^{1/p}$$

is a norm of the space H equivalent to (2.2) and the embedding

$$H \hookrightarrow L^p(\Omega)$$

is compact and dense.

Parabolic Variational Inequalities. Following Lions [20], let us cite some well-known results concerning solvability and solution uniqueness for non-degenerate non-linear parabolic variational inequalities which will be useful in the sequel.

Let $\mathcal{V}$ be reflexive Banach space and $\mathcal{H}$ be Hilbert space and

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*. \tag{2.3}$$

Let us consider such operator Λ that:

$$\begin{gathered} -\Lambda \text{ is an infinitesimal generating operator of a semigroup} \\ s \to G(s) \text{ in } \mathcal{V}, \mathcal{H}, \mathcal{V}^*, \text{ which is a compressive semigroup in } \mathcal{H}. \end{gathered} \tag{2.4}$$

Let us consider a non-linear operator $\mathcal{A}$ such that

$$\begin{gathered} \mathcal{A} : \mathcal{V} \to \mathcal{V}^* \text{ is a pseudomonotone operator, i.e.} \\ \text{it is bounded and if } y_k \to y \text{ weakly in } \mathcal{V}, \\ y_k, y \in \mathcal{K} \text{ and } \overline{\lim_{k\to\infty}} \langle \mathcal{A}(y_k), y_k - y\rangle_{\mathcal{V}} \le 0 \text{ then} \\ \underline{\lim_{k\to\infty}} \langle \mathcal{A}(y_k), y_k - v\rangle_{\mathcal{V}} \ge \langle \mathcal{A}(y), y - v\rangle_{\mathcal{V}} \; \forall v \in \mathcal{V}, \end{gathered} \tag{2.5}$$

and

$$\begin{gathered} \mathcal{A} \text{ is a coercive operator :} \\ \text{there exists such element } v_0 \in \mathcal{K} \text{ that} \\ \frac{\langle \mathcal{A}(v), v - v_0\rangle_{\mathcal{V}}}{\|v\|_{\mathcal{V}}} \to \infty \text{ as } \|v\| \to \infty, \end{gathered} \tag{2.6}$$

where

$$\mathcal{K} \text{ is a convex closed set in } \mathcal{V}. \tag{2.7}$$

Using operators, spaces and sets discussed above, and taking $\Lambda = \frac{d}{dt}$, we can consider the following problem for variational parabolic inequalities in its "weak" statement (see for details [20]): find $u \in \mathcal{K}$ such that

$$\begin{gathered} \langle \Lambda v, v - u\rangle_{\mathcal{V}} + \langle \mathcal{A}(u), v - u\rangle_{\mathcal{V}} \ge \langle f, v - u\rangle_{\mathcal{V}} \\ \forall v \in \mathcal{K},\ v' \in \mathcal{V}^*,\ v(0) = 0, \end{gathered} \tag{2.8}$$

where $f \in \mathcal{V}^*$.

Let us consider some "consistency conditions" for Λ and $\mathcal{K}$: $\forall v \in \mathcal{K}$ there exists some "regularizing" sequence v_j which satisfies the following conditions:

$$\begin{gathered} v_j \in \mathcal{K},\ v_j' \in \mathcal{V}^*,\ v_j(0) = 0, \\ v_j \to v \text{ in } \mathcal{V},\ j \to \infty, \\ \overline{\lim_{j\to\infty}} \langle \Lambda v_j, v_j - v\rangle_{\mathcal{V}} \le 0. \end{gathered} \tag{2.9}$$

Theorem 2.1. *[20, Theorem 9.1] If for convex set $\mathcal{K}$ and semigroup $G(s)$ we have*

$$G(s)\mathcal{K} \subset \mathcal{K} \ \forall s \geq 0,$$

then (2.9) *takes place.*

Theorem 2.2. *[20, Theorem 9.2] Let conditions* (2.3), (2.4), (2.5), (2.6) *with $v_0 \in \mathcal{K}$ such that $v_0' \in \mathcal{V}$, $v_0(0) = 0$, and* (2.9) *are fulfilled. Then $\forall f \in \mathcal{V}^*$ there exists the solution $u \in \mathcal{K}$ for the variational evolution inequality* (2.8).

Theorem 2.3. *[20, Theorem 9.4] Let conditions of Theorem 2.2 are fulfilled. Let us assume that $\forall u, v \in \mathcal{K}$:*

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{\mathcal{V}} \leq 0 \Rightarrow u = v. \tag{2.10}$$

Then the inequality (2.8) *admits a unique solution.*

Smoothing. Throughout the paper ε denotes a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence, while writing $\varepsilon \geq 0$, we also consider its limit $\varepsilon = 0$.

Definition 2.1. We say that a weight function ρ with properties (2.1) is approximated by non-degenerate weight functions $\{\rho^\varepsilon\}_{\varepsilon \geq 0}$ on Ω if:

$$\rho^\varepsilon(x) > 0 \text{ a.e. in } \Omega, \ \rho^\varepsilon + (\rho^\varepsilon)^{-1} \in L^\infty(\Omega), \ \forall \varepsilon > 0, \tag{2.11}$$

$$\rho^\varepsilon \to \rho, \ (\rho^\varepsilon)^{-1/(p-1)} \to \rho^{-1/(p-1)} \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{2.12}$$

Remark 2.2. The family $\{\rho^\varepsilon\}_{\varepsilon>0}$ satisfying properties (2.11)-(2.12) is called the non-degenerate perturbation of the weight function ρ.

Examples of such perturbations can be constructed using the classical smoothing. For instance, let Q be some positive compactly supported function such that $Q \in L^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} Q(x)dx = 1$, and $Q(x) = Q(-x)$. Then, for a given weight function $\rho \in L^1_{loc}(\mathbb{R}^N)$, we can take $\rho^\varepsilon = (\rho)_\varepsilon$, where

$$(\rho)_\varepsilon(x) = \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} Q\left(\frac{x-z}{\varepsilon}\right) \rho(z)\, dz = \int_{\mathbb{R}^N} Q(z)\rho(x + \varepsilon z)\, dz. \tag{2.13}$$

In this case, we say that the perturbation $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ of the original degenerate weight function ρ is constructed by the "direct" smoothing scheme.

Lemma 2.1. *[12] If $\rho, \rho^{-1/(p-1)} \in L^1_{loc}(\mathbb{R}^N)$ then the "direct" smoothing $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ possesses properties* (2.11)-(2.12).

Radon measures and convergence in variable spaces. By a nonnegative Radon measure on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω. The space of all nonnegative Radon measures on Ω will be denoted by $\mathcal{M}_+(\Omega)$. If μ is a nonnegative Radon measure on Ω, we will use $L^r(\Omega, d\mu)$, $1 \le r \le \infty$, to denote the usual Lebesque space with respect to the measure μ with the corresponding norm $\|f\|_{L^r(\Omega,\, d\mu)} = \left(\int_\Omega |f(x)|^r \, d\mu\right)^{1/r}$.

Let $\{\mu_\varepsilon\}_{\varepsilon>0}$, μ be Radon measure such that $\mu_\varepsilon \rightharpoonup^* \mu$ in $\mathcal{M}_+(\Omega)$: that is,

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi \, d\mu_\varepsilon = \int_\Omega \varphi \, d\mu \ \forall \varphi \in C_0^\infty(\mathbb{R}^N), \tag{2.14}$$

where $C_0^\infty(\mathbb{R}^N)$ is the space of all compactly supported continuous functions. A typical example of such measures is $d\mu_\varepsilon = \rho^\varepsilon(x)\, dx$, $d\mu = \rho(x)\, dx$, where $0 \le \rho^\varepsilon \rightharpoonup \rho$ in $L^1(\Omega)$. Let us recall the definition and main properties of convergence in the variable L^p-space.

1. A sequence $\{v_\varepsilon \in L^p(\Omega,\, d\mu_\varepsilon)\}$ is called bounded if

$$\varlimsup_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon < +\infty.$$

2. A bounded sequence $\{v_\varepsilon \in L^p(\Omega, d\mu_\varepsilon)\}$ converges weakly to $v \in L^p(\Omega, d\mu)$ if $\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon \varphi d\mu_\varepsilon = \int_\Omega v\varphi d\mu$ for any $\varphi \in C_0^\infty(\Omega)$ and we write $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)$.

3. The strong convergence $v_\varepsilon \to v$ in $L^p(\Omega, d\mu_\varepsilon)$ means that $v \in L^p(\Omega, d\mu)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon z_\varepsilon \, d\mu_\varepsilon = \int_\Omega vz \, d\mu \text{ as } z_\varepsilon \rightharpoonup z \text{ in } L^q(\Omega, d\mu_\varepsilon). \tag{2.15}$$

The following convergence properties in variable spaces hold:

(a) *Compactness criterium*: if a sequence is bounded in $L^p(\Omega, d\mu_\varepsilon)$, then this sequence is compact with respect to the weak convergence.

(b) *Property of lower semicontinuity*: if $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)$, then

$$\varliminf_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon \ge \int_\Omega v^p \, d\mu. \tag{2.16}$$

(c) *Criterium of strong convergence*: $v_\varepsilon \to v$ if and only if $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon = \int_\Omega v^p \, d\mu. \tag{2.17}$$

Concluding this section, we recall some well-known results concerning the convergence in the variable space $L^p(\Omega, d\mu_\varepsilon)$.

Lemma 2.2. *If $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a non-degenerate perturbation of the weight function $\rho(x) \ge 0$, then: (A_1) $(\rho^\varepsilon)^{-1} \to \rho^{-1}$ in $L^q(\Omega, \rho^\varepsilon dx)$. ($A_2$) $[v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)]$ $\Rightarrow [v_\varepsilon \rightharpoonup v$ in $L^1(\Omega)]$. (A_3) If a sequence $\{v_\varepsilon \in L^p(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ is bounded, then the weak convergence $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, \rho^\varepsilon dx)$ is equivalent to the weak convergence $\rho^\varepsilon v_\varepsilon \rightharpoonup \rho v$ in $L^1(\Omega)$. (A_4) If $a \in L^\infty(\Omega)$ and $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, \rho^\varepsilon dx)$, then $av_\varepsilon \rightharpoonup av$ in $L^p(\Omega, \rho^\varepsilon dx)$.*

Variable Sobolev Spaces. Let $\rho(x)$ be a degenerate weight function and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of the function ρ in the sense of Definition 2.1. We denote by $H(\Omega, \rho^\varepsilon dx)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\rho^\varepsilon}$. Since for every ε the function ρ^ε is non-degenerate, the space $H(\Omega, \rho^\varepsilon dx)$ coincides with the classical Sobolev space $W_0^{1,p}(\Omega)$.

Definition 2.2. We say that a sequence $\{y_\varepsilon \in H(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ converges weakly to an element $y \in W$ as $\varepsilon \to 0$, if the following hold: (i) This sequence is bounded. (ii) $y_\varepsilon \rightharpoonup y$ in $L^p(\Omega)$. (iii) $\nabla y_\varepsilon \rightharpoonup \nabla y$ in $L^p(\Omega, \rho^\varepsilon dx)^N$.

Theorem 2.4. *[12] Let $\rho^\varepsilon = (\rho)_\varepsilon$ be a direct smoothing of a degenerate weight $\rho \in L^1_{loc}(\mathbb{R}^N)$ and let $y^\varepsilon \in H(\Omega, \rho^\varepsilon dx)$, $y^\varepsilon \rightharpoonup y$ in $L^p(\Omega)$, $\nabla y^\varepsilon \rightharpoonup v$ in $L^p(\Omega, \rho^\varepsilon dx)^N$. Then $y \in H$ and $v = \nabla y$.*

Functional spaces. For some interval S and some Banach space $\{X, \|\cdot\|_X\}$ we can consider the set of all measurable by Bochner functions $u \in (S \to X)$ $L^p(S;X)$, $1 \le p < \infty$ for which $\int_s \|u(s)\|^p ds < \infty$.

Theorem 2.5. *[21, Theorem 1.11] The set $L^p(S;X)$, $1 \le p < \infty$ which forms a linear space with natural linear operations becames a Banach space with norm*

$$\|u\|_{L^p(S;X)} = \left(\int_S \|u(s)\|^p \, ds \right)^{1/p}. \tag{2.18}$$

Remark 2.3. Taking into account the definition of $L^p(S;X)$, Theorem 2.5 and properties of Bochner's integral (see [21]), the properties of the given section are valid for $L^p(S;X)$ as well as for X.

Compensated Compactness Lemma in Variable Lebesque and Sobolev spaces. Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function ρ.

In order to discuss the attainability of H-optimal solutions we use the following result, which we can obtain applying similar suggestions to [12, 22].

Lemma 2.3. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function $\rho(x) > 0$. Suppose that sequences $\{\vec{f}_\varepsilon\}_{\varepsilon>0}$ and $\{g_\varepsilon\}_{\varepsilon>0}$ are such that:*

(i) $\frac{\partial g_\varepsilon}{\partial t} - \operatorname{div}(\rho^\varepsilon \vec{f}_\varepsilon) = 0$ in the sense of distributions in $\Omega \times [0,T]$;

(ii) $\vec{f}_\varepsilon \rightharpoonup \vec{f}$ in $L^q(0,T; L^q(\Omega, \rho^\varepsilon dx)^N)$ as $\varepsilon \to 0$;

(iii) g_ε is bounded in $L^\infty(0,T; L^2(\Omega))$ and $g_\varepsilon \rightharpoonup g$ in $L^p(0,T; H(\Omega, \rho^\varepsilon dx))$ as $\varepsilon \to 0$.

If $p > \frac{2N}{N+2}$, then

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \vec{f}_\varepsilon \cdot \nabla g_\varepsilon \varphi \rho^\varepsilon \, dx dt = \int_0^T \int_\Omega \vec{f} \cdot \nabla g \varphi \rho \, dx dt, \qquad \forall \varphi \in C_0^\infty(\Omega \times [0,T]). \tag{2.19}$$

3. Setting of the Optimal Control Problem

The OCP, we consider in this paper, is to minimize the descrepancy between a given distribution $z_\partial \in L^p(0,T;L^p(\Omega))$ and the solution y of the degenerate variational inequality by choosing an appropriate matrix $U \in L^\infty(\Omega;\mathbb{R}^{N\times N})$, namely we deal with the following minimization problem:

$$I(U,y) = \int_0^T \int_\Omega |y(t,x) - z_\partial(t,x)|^p \, dxdt \to \inf, \tag{3.1}$$

$$U \in M_p^{\alpha,\beta}(\Omega), y \in \hat{\mathcal{K}}, \tag{3.2}$$

$$\begin{aligned} \langle v', v-y\rangle_{L^p(0,T;W)} + \langle - \ \mathrm{div}\left(U(x)\rho(x)[(\nabla y)^{p-2}]\nabla y\right), v-y\rangle_{L^p(0,T;W)} \\ + \langle |y|^{p-2}y, v-y\rangle_{L^p(0,T;W)} \geq \langle f, v-y\rangle_{L^p(0,T;W)} \end{aligned} \tag{3.3}$$

$$v \in \hat{\mathcal{K}}, \quad v' \in L^q(0,T;L^q(\Omega)), \quad v(0,x) = 0,$$

where $f \in L^q(0,T;L^q(\Omega))$ is a fixed element, $M_p^{\alpha,\beta}(\Omega) \subset L^\infty(\Omega;\mathbb{R}^{N\times N})$ is a class of admissible controls, $\hat{\mathcal{K}} \subset L^p(0,T;W)$ is a closed convex subset and

$$[\eta^{p-2}] = \mathrm{diag}\{|\eta_1|^{p-2}, |\eta_2|^{p-2}, \dots, |\eta_N|^{p-2}\} \ \forall \eta \in \mathbb{R}^N.$$

Let α and β be constants such that $0 < \alpha \leq \beta < +\infty$. We define $M_p^{\alpha,\beta}(\Omega)$ as a set of all symmetric matrices $U(x) = \{a_{ij}(x)\}_{1\leq i,j\leq N}$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$ such that the following conditions of growth, monotonicity, and strong coercivity are fulfilled:

$$|a_{i,j}(x)| \leq \beta \text{ a.e. in } \Omega \ \forall i,j \in \{1,\dots,N\}, \tag{3.4}$$

$$\left(U(x)\left([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta\right), \zeta - \eta\right)_{\mathbb{R}^N} \geq 0 \text{ a.e. in } \Omega \ \forall \zeta, \eta \in \mathbb{R}^N, \tag{3.5}$$

$$\left(U(x)[\zeta^{p-2}]\zeta, \zeta\right)_{\mathbb{R}^N} = \sum_{i,j=1}^N a_{i,j}(x)|\zeta_j|^{p-2}\zeta_j\zeta_i \geq \alpha|\zeta|_p^p \text{ a.e. in } \Omega. \tag{3.6}$$

Remark 3.1. It is easy to see that $M_p^{\alpha,\beta}(\Omega)$ is a nonempty subset of the space $L^\infty(\Omega;\mathbb{R}^{N\times N})$ and its typical representatives are diagonal matrices of the form $U(x) = \mathrm{diag}\{\delta_1(x), \delta_2(x), \dots, \delta_N(x)\}$, where $\alpha \leq \delta_i(x) \leq \beta$ a.e. in Ω $\forall i \in \{1,\dots,N\}$.

For every fixed control $U \in M_p^{\alpha,\beta}(\Omega)$ let us define a non-linear operator $\mathcal{A} : L^p(0,T;H) \to L^q(0,T;H^*)$ in the following way:

$$\begin{aligned} \langle \mathcal{A}(y), v\rangle_{L^p(0,T;H)} = \int_0^T \int_\Omega \sum_{i,j=1}^N \left(a_{i,j}(x) \left|\frac{\partial y}{\partial x_j}\right|^{p-2} \frac{\partial y}{\partial x_j}\right) \frac{\partial v}{\partial x_i} \rho \, dxdt \\ + \int_0^T \int_\Omega |y|^{p-2} yv \ dxdt. \end{aligned} \tag{3.7}$$

,

Definition 3.1. We say that a matrix $U = [a_{i,j}]$ is an admissible control to degenerate problem (3.2)-(3.3) if $U \in U_{ad}$, where the set U_{ad} is defined as follows

$$U_{ad} = \{U = [\vec{a}_1, \dots, \vec{a}_N] \in M_p^{\alpha,\beta}(\Omega) | \ | \operatorname{div}(\rho \vec{a}_i)| \le \gamma_i, \text{ a.e. in } \Omega, \forall i = \overline{1, N}\}. \tag{3.8}$$

Here $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ is a strictly positive vector.

Definition 3.2. Let K be the convex closed subset of H, $0 \in K$, $\mathcal{K} = \{v | \, v \in L^p(0,T;H), v(t) \in K \text{ a.e.}\}$ be the convex closed subset of $L^p(0,T;H)$. We say that a function $y = y(U,f) \in \mathcal{K}$ is an H-solution to degenerate variational inequality (3.2)-(3.3), if

$$\begin{gathered} \langle v', v - y \rangle_{L^p(0,T;H)} + \langle \mathcal{A}(y), v - y \rangle_{L^p(0,T;H)} \ge \langle f, v - y \rangle_{L^p(0,T;H)} \\ v \in \mathcal{K}, \quad v' \in L^q(0,T;L^q(\Omega)), \quad v(0,x) = 0. \end{gathered} \tag{3.9}$$

Definition 3.3. We say that the set Ξ_H is the set of admissible pairs to the optimal control problem (3.1)-(3.3), (3.8) if

$$\Xi_H = \{(U,y) \in U_{ad} \times L^p(0,T;H) | \, y \in \mathcal{K}, (U,y) \text{ are related by (3.9)}\}.$$

Remark 3.2. We can inroduce a W-solution and the set Ξ_W by the similar way.

Hence for given control object described by relations (3.2)-(3.3) with both fixed control constraints ($U \in U_{ad}$) and fixed cost functional (3.1), we have two different statements of the original optimal control problem, namely

$$\left\langle \inf_{(U,y) \in \Xi_W} I(U,y) \right\rangle \text{ and } \left\langle \inf_{(U,y) \in \Xi_H} I(U,y) \right\rangle.$$

As a matter of fact, there is no comparison between these problems, in general. Indeed, having assumed that $W \ne H$ for a given degenerate weight function $\rho \ge 0$, we can come to the effect which is usually called the Lavrentieff phenomenon. It means that for some $U \in U_{ad}$ and $f \in L^q(0,T;L^q(\Omega))$ an H-solution $y_H(U,f)$ to problem (3.2)-(3.3) does not coincide with its W-solution $y_W(U,f)$. In this paper we deal with H-solutions to problem (3.2)-(3.3).

Definition 3.4. We say that a pair $(U_0, y_0) \in L^\infty(\Omega; \mathbb{R}^{N \times N}) \times L^p(0,T;H)$ is an H-optimal solution to problem (3.1)-(3.3), (3.8) if $(U_0, y_0) \in \Xi_H$ and $I(U_0, y_0) = \inf_{(U,y) \in \Xi_H} I(U,y)$.

Definition 3.5. We say that a sequence $\{(U_k, y_k) \in \Xi_H\}_{k \in \mathbb{N}}$ is bounded if

$$\sup_{k \in \mathbb{N}} [\|U_k\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \|y_k\|_{L^p(0,T;L^p(\Omega))} + \|\nabla y_k\|_{L^p(0,T;L^p(\Omega,\rho dx)^N)}]$$

is finite.

4. Existence of H-Optimal Solutions

In this section we show that considered optimal control problem (3.1)-(3.3) for degenerate parabolic variational inequality with monotone operator is regular in the class of H-admissible solutions. Imposing additional control constrains (3.8) and using the special version of compensated compactness lemma (Lemma 2.3) we prove that the set of H-admissible solutions for problem (3.2)-(3.3) is sequentially closed. And using the direct method of Calculus of Variations we prove the existence of H-optimal solutions for considered problem.

Theorem 4.1. *For every control $U \in M_p^{\alpha,\beta}(\Omega)$ and every $f \in L^q(0,T;L^q(\Omega))$ there exists a unique H-solution to degenerate parabolic variational inequality* (3.2)-(3.3).

Proof. Let $U \in M_p^{\alpha,\beta}(\Omega)$ be a fixed matrix. Let us consider the following elliptic operator $A_1 : H \to H^*$:

$$\langle A_1(y), v\rangle_H = \sum_{i,j=1}^{N} \int_\Omega \left(a_{i,j}(x) \left| \frac{\partial y}{\partial x_j} \right|^{p-2} \frac{\partial y}{\partial x_j} \right) \frac{\partial v}{\partial x_i} \rho \, dx + \int_\Omega |y|^{p-2} y v \, dx.$$

Then taking into account (3.6) from [12, Lemma 1] we have the next coercivity property for operator A_1:

$$\langle A_1(y), y\rangle_H \geq \min\{\alpha, 1\} \|y\|_\rho^p. \tag{4.1}$$

Hence from (3.7) and (2.18) we have that

$$\langle \mathcal{A}(y), y\rangle_{L^p(0,T;H)} \geq \min\{\alpha, 1\} \|y\|^p_{L^p(0,T;H)}, \tag{4.2}$$

where $\|y\|_{L^p(0,T;H)} = \int_0^T \|y\|_\rho^p dt$.

Let us fix an element $v_0 \in \mathcal{K}$ such that $v_0' \in L^q(0,T;L^q(\Omega))$, $v_0(0,x) = 0$ and show the coercivity property (2.6). For all $y \in \mathcal{K}$ we consider the following pairing, by estimate (4.2), we have:

$$\langle \mathcal{A}(y), y - v_0\rangle_{L^p(0,T;H)} \geq \min\{\alpha, 1\} \|y\|^p_{L^p(0,T;H)} - |\langle \mathcal{A}(y), v_0\rangle_{L^p(0,T;H)}|. \tag{4.3}$$

From [12, Lemma 1] and (3.4) it follows that

$$|\langle A_1(y), v_0\rangle_H| \leq \max\{\beta, 1\} \|v_0\|_H \|y\|_H^{p-1}.$$

Further from (3.7) and (2.18) we obtain similar estimate:

$$|\langle \mathcal{A}(y), v_0\rangle_{L^p(0,T;H)}| \leq \max\{\beta, 1\} \|v_0\|_{L^p(0,T;H)} \|y\|^{p-1}_{L^p(0,T;H)}. \tag{4.4}$$

Combining (4.3) and (4.4) we have the coercivity condition (2.6).

Taking into account the estimate (3.5) and the strict monotonicity of the term $|y|^{p-2}y$ we obtain:

$$\langle \mathcal{A}(y) - \mathcal{A}(v), y - v \rangle_{L^p(0,T;H)} \geq 2^{p-2} \|y - v\|_{L^p(0,T;L^p(\Omega))} > 0 \qquad (4.5)$$

$\forall y \neq v$ a.e. in $Q = \Omega \times (0, T)$. Thus we have the strict monotonicity of operator $\mathcal{A}$.

From the semicontinuity property of operator A_1 (see [12]) we obtain the similar property for operator $\mathcal{A}$. Taking into account (3.4) and the definition of operator $\mathcal{A}$ we obtain the boundedness property for $\mathcal{A}$. Hence, from the strict monotonicity, boundedness and semicontinuity we obtain that $\mathcal{A}$ is a pseudo-monotone operator (see for details [20, Proposition 2.5]).

If we consider $\mathcal{V} = L^p(0, T; H)$, $\mathcal{H} = L^2(0, T; L^2(\Omega))$, $\mathcal{V}^* = L^q(0, T; H^*)$ we obtain condition (2.3), for $\Lambda = \frac{d}{dt}$ the condition (2.4) is valid, for operator $\mathcal{A}$ properties (2.5) and (2.6) take place. Note that for considered set $\mathcal{K}$ we have that $0 \in K$, thus we have that $G(s)\mathcal{K} \subset \mathcal{K}$ $\forall s \geq 0$ (see [20]) and from Theorem 2.1 we have conditions (2.9).

Hence, for problem (3.2)-(3.3) all conditions of Theorems 2.2 and 2.3 hold true. Therefore for every control $U \in M_p^{\alpha,\beta}(\Omega)$ and every $f \in L^q(0, T; L^q(\Omega))$ the considered problem has a unique solution. □

Let us study the topological properties of the set of H-admisible solutions $\Xi_H \subset L^\infty(\Omega; \mathbb{R}^{N\times N}) \times L^p(0, T; H)$. Let τ be the topology on $L^\infty(\Omega; \mathbb{R}^{N\times N}) \times L^p(0, T; H)$ which we define as the product of the weak-$*$ topology of the space $L^\infty(\Omega; \mathbb{R}^{N\times N})$ and the weak topology of $L^p(0, T; H)$. In order to discuss further results we suggest that the following assumption is fulfilled:

Hypothesis A. Let for a sequence $\{u_n\}_{n\geq 1}$ that is weakly convergent in $L^p(0, T; H)$ we additionally have that $u_n \in L^\infty(0, T; L^2(\Omega))$ for all $n \in \mathbb{N}$.

Theorem 4.2. *Let $\rho(x) > 0$ be a degenerate weight function, let Hypothesis A hold true and let Hypothesis 2 from [12] hold true for $X = L^q(\Omega)$. Then for every $f \in L^q(0, T; L^q(\Omega))$ the set Ξ_H is sequentially τ-closed.*

Proof. Let $\{(U_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi_H$ be any τ-convergent sequence of admissible pairs to the problem (3.1)-(3.3), (3.8) (in view of Theorem 4.1 such choice is always possible). Let (U_0, y_0) be its τ-limit. Our aim is to prove that $(U_0, y_0) \in \Xi_H$.

Since $\{U_k = [\vec{a}_{1k}, \ldots, \vec{a}_{Nk}]\}_{k\in\mathbb{N}} \subset U_{ad}$, it follows that $|\ \mathrm{div}(\rho\vec{a}_{ik})| \leq \gamma_i$ a.e. in Ω $\forall i = 1, \ldots, N$ and $\forall k \in \mathbb{N}$. Let us show that $U_0 \in U_{ad}$.

Indeed, passing to the limit as $k \to \infty$ in the relations

$$\int_\Omega (\vec{a}_{ik}, \nabla\varphi)_{\mathbb{R}^N} \rho dx = -\int_\Omega \varphi\ \mathrm{div}(\rho\vec{a}_{ik})\, dx,\ \forall\varphi \in C_0^\infty(\Omega), \forall i = 1, \ldots, N,$$

$$-\gamma_i \int_\Omega \varphi \leq \int_\Omega \varphi\ \mathrm{div}(\rho\vec{a}_{ik}) dx \leq \gamma_i \int_\Omega \varphi\, dx, \quad \forall i = 1, \ldots, N, \forall\varphi \geq 0,$$

we may suppose that $|\ \mathrm{div}(\rho\vec{a}_i^0)| \leq \gamma_i$ a.e. in Ω $\forall i \in \{1, \ldots, N\}$ and

$$\mathrm{div}(\rho\vec{a}_{ik}) \rightharpoonup \mathrm{div}(\rho\vec{a}_i^0) \text{ in } L^q(\Omega) \quad \text{as } k \to \infty. \qquad (4.6)$$

Thus $U_k \rightharpoonup U_0 = [\vec{a}_1^0, \dots, \vec{a}_N^0]$ weakly-$*$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$, and $U_0 \in U_{ad}$.

It remains to show that the pair (U_0, y_0) satisfies variational inequality (3.9).

Since each of the pairs (U_k, y_k) is admissible to the OCP (3.1)-(3.3), (3.8), we have

$$\begin{aligned}\langle v', v - y_k\rangle_{L^p(0,T:H)} + \langle - \ &\mathrm{div}(U_k\rho(x)[(\nabla y_k)^{p-2}]\nabla y_k)\\ &+ |y_k|^{p-2}y_k, v - y_k\rangle_{L^p(0,T;H)}\\ &\geq \langle f, v - y_k\rangle_{L^p(0,T;H)}. \end{aligned}\tag{4.7}$$

Since $U_k \to U_0$ weakly-$*$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$ and $y_k \to y_0$ weakly in $L^p(0,T;H)$ as $k\to\infty$, one gets

$$\begin{aligned}\mathrm{div}(\rho\vec{a}_{ik}) &\rightharpoonup \ \mathrm{div}(\rho\vec{a}_i^0) \text{ in } L^q(\Omega),\ \forall i = 1,\dots,N,\\ y_k &\to y_0 \text{ strongly in } L^p(0,T;L^p(\Omega))(\text{see [22, Proposition 4.1]}),\\ \nabla y_k &\rightharpoonup \nabla y_0 \text{ in } L^p(0,T;L^p(\Omega,\rho dx)^N),\\ |y_k|^{p-2}y_k &\rightharpoonup |y_0|^{p-2}y_0 \text{ in } L^q(0,T;L^q(\Omega)) \text{ within a subsequence},\\ \{U_k[(\nabla y_k)^{p-2}]\nabla y_k\}_{k\in\mathbb{N}} &\text{ is bounded in } L^q(0,T;L^q(\Omega,\rho dx)^N).\end{aligned}$$

Then $U_k[(\nabla y_k)^{p-2}]\nabla y_k := \vec{\xi}_k \rightharpoonup \vec{\xi}$ in $L^q(0,T;L^q(\Omega,\rho dx)^N)$ within a subsequence. Similarly to [12, Theorem 5] we obtain that function $-\ \mathrm{div}(\rho\vec{\xi}_k) + |y_k|^{p-2}y_k \in L^q(0,T;L^q(\Omega))$ having that Hypothesis 1 from [12] holds true if we set $V = H$, $X = L^q(\Omega)$ and $f, v' \in L^q(0,T;L^q(\Omega))$ $\forall k\in\mathbb{N}$, and, obviously, $\mathrm{div}(\rho\vec{\xi}_k) \in L^q(0,T;L^q(\Omega))$ $\forall k \in \mathbb{N}$. Further, the relation

$$\begin{aligned}\int_0^T\int_\Omega \mathrm{div}(\rho\vec{\xi}_k)\varphi\,dxdt &= -\int_0^T\int_\Omega \vec{\xi}_k\cdot\nabla\varphi\rho\,dxdt\\ &\to -\int_0^T\int_\Omega \vec{\xi}\cdot\nabla\varphi\rho\,dxdt = \int_0^T\int_\Omega \mathrm{div}(\rho\vec{\xi})\varphi\,dxdt\\ &\qquad \forall\varphi\in C_0^\infty(\Omega\times[0,T]),\end{aligned}$$

means that $\mathrm{div}(\rho\vec{\xi}_k) \to \mathrm{div}(\rho\vec{\xi})$ weakly in $L^q(0,T;L^q(\Omega))$ implying $\{\vec{\xi}_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{X}$, where

$$\mathcal{X} = \{\vec{f}\in L^q(0,T;L^q(\Omega,\rho dx)^N)|\ \mathrm{div}(\rho\vec{f}) \in L^q(0,T;L^q(\Omega))\},$$

that is

$$\varlimsup_{k\to\infty}(\|\vec{\xi}_k\|^q_{L^q(0,T;L^q(\Omega,\rho dx)^N)} + \|\,\mathrm{div}(\rho\vec{\xi}_k)\|^q_{L^q(0,T;L^q(\Omega))})^{1/q} < +\infty.$$

Therefore, as a result, passing to the limit in (4.7) as $k\to\infty$, we obtain

$$\begin{aligned}&\langle v', v - y_0\rangle_{L^p(0,T;H)} + \langle -\,\mathrm{div}(\rho\vec{\xi}) + |y_0|^{p-2}y_0, v - y_0\rangle_{L^p(0,T;H)}\\ &\qquad \geq \langle f, v - y_0\rangle_{L^p(0,T;H)},\ \forall v\in\mathcal{K}, v'\in L^q(0,T;L^q(\Omega)), v(0,x) = 0.\end{aligned}\tag{4.8}$$

It remains to prove that $\vec{\xi} = U_0[(\nabla y_0)^{p-2}]\nabla y_0$.

To do this we apply the similar suggestions to [12, Theorem 5] and [22] and by initial assumptions (see (3.5)), we have

$$\int_0^T \int_\Omega (U_k([(\nabla y_k)^{p-2}]\nabla y_k - [\vec{z}^{p-2}]\vec{z})) \cdot (\nabla y_k - \vec{z})\varphi\rho\, dxdt \geq 0 \tag{4.9}$$

for a fixed element $\vec{z}$ of $\mathbb{R}^N$.

Let us show that the sequence $\{\operatorname{div}(\rho U_k[(\vec{z})^{p-2}]\vec{z})\}_{k\in\mathbb{N}}$ is weakly convergent in $L^q(0,T;L^q(\Omega))$. Taking into account the definition of the elements $\operatorname{div}(\rho U_k[\vec{z}^{p-2}]\vec{z})$ for all $k \in N$ (see [12]) and boundedness of $\{\operatorname{div}(\rho U_k[(\nabla y_k)^{p-2}]\nabla y_k)\}_{k\in\mathbb{N}}$ in $L^q(0,T;L^q(\Omega))$ we get

$$\begin{aligned} &\operatorname{div}(\rho U_k([(\nabla y_k)^{p-2}]\nabla y_k - [\vec{z}^{p-2}]\vec{z})) \\ &\qquad \rightharpoonup \operatorname{div}(\rho\vec{\xi}) - \operatorname{div}(\rho U_0[\vec{z}^{p-2}]\vec{z}) \text{ in } L^q(0,T;L^q(\Omega)). \end{aligned} \tag{4.10}$$

Combining the property (4.10), and the fact that

$$U_k[\vec{z}^{p-2}]\vec{z} \rightharpoonup U_0[\vec{z}^{p-2}]\vec{z} \text{ in } L^q(0,T;(L^q(\Omega,\rho dx))^N)$$

it is easy to see that all suppositions of Lemma 2.3 for the sequences $\{\rho^\varepsilon \equiv (\rho)\}_{\varepsilon>0}$ are fulfilled having put in the statement of this lemma $\varepsilon = k$, $\vec{f} = U_k([(\nabla y_k)^{p-2}]\nabla y_k - [\vec{z}^{p-2}]\vec{z})$ and $g_\varepsilon = y_k$ for all $k \in \mathbb{N}$. Hence, we get

$$\int_0^T \int_\Omega (\vec{\xi} - U_0[\vec{z}^{p-2}]\vec{z}) \cdot (\nabla y_0 - \vec{z})\varphi\rho\, dxdt \geq 0, \quad \forall \vec{z} \in R^N$$

for all positive $\varphi \in C_0^\infty(\Omega \times [0,T])$. After localization, we have

$$\rho(\vec{\xi} - U_0[\vec{z}^{p-2}]\vec{z}) \cdot (\nabla y_0 - \vec{z}) \geq 0. \tag{4.11}$$

Taking into account conditions (3.4)–(3.6) and suggestions from [22] we have that the identity $\xi = \hat{A}(U_0, \nabla y_0) = U_0(x)[(\nabla y_0)^{p-2}]\nabla y_0$ holds true a.e. in $\Omega \times (0,T)$.

Thus, the above inequality takes the form

$$\begin{aligned} &\langle v', v - y\rangle_{L^p(0,T;H)} + \langle -\operatorname{div}(\rho U_0[(\nabla y_0)^{p-2}]\nabla y_0) + |y_0|^{p-2}y_0, v - y_0\rangle_{L^p(0,T;H)} \\ &\qquad \geq \langle f, v - y_0\rangle_{L^p(0,T;H(\Omega,\rho dx))} \,\forall v \in \mathcal{K}, v' \in L^q(0,T;L^q(\Omega)), v(0,x) = 0. \end{aligned}$$

Thus τ-limit pair (U_0, y_0) is admissible to the problem (3.1)-(3.3), (3.8), hence, $(U_0, y_0) \in \Xi_H$. □

Theorem 4.3. *Let $\rho(x)$ be a degenerate weight function. Then the set of H-optimal solutions to the problem* (3.1)-(3.3), (3.8) *is non-empty for every $f \in L^q(0,T;L^q(\Omega))$.*

Proof. First of all we note that in virtue of Theorem 4.1 for the given function $f \in L^q(0,T;L^q(\Omega))$ and every admissible control $U \in U_{ad}$ there exists an H-solution $y = y(U,f) \in L^p(0,T;L^p(\Omega))$ to the problem (3.2)-(3.3). Let $\{(U_k, y_k) \in \Xi_H\}_{k\in\mathbb{N}}$ be an H-minimizing sequence to the problem (3.1)-(3.3), (3.8), that is,

$$\lim_{k\to\infty} I(U_k, y_k) = \inf_{(U,y)\in\Xi_H} I(U,y) < +\infty.$$

Hence, taking into account the Definition 3.1 of U_{ad} and Definition 3.5, we may suppose that within a subsequence, there exists $(U^*, y^*) \in L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;H)$, such that $U_k \to U^*$ weakly-$*$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$, $y_k \rightharpoonup y^*$ in $L^p(0,T;H)$. Since Ξ_H is sequentially τ-closed, the pair (U^*, y^*) is H-admissible to the problem (3.1)-(3.3), (3.8). In view of lower τ-semicontinuity of the cost functional we obtain that $I(U^*, y^*) \le \varliminf_{k\to\infty} I(U_k, y_k) = \inf_{(U,y)\in\Xi_H} I(U,y)$. Hence, (U^*, y^*) is an H-optimal pair. □

5. Attainability of H-Optimal Solutions

In this section we propose an appropriate non-degenerate perturbation for the original degenerate OCP (3.1)-(3.3), (3.8) and show that H-optimal solutions of (3.1)-(3.3), (3.8) can be attained by optimal solutions of perturbed problems. In view of results obtained in the previous section we assume that the set of H-optimal solutions to the considered problem is non-empty.

Let ρ be a degenerate weight function with properties (2.1), and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of ρ in the sense of Definition 2.1.

Definition 5.1. We say that a bounded sequence

$$\{(U_\varepsilon, y_\varepsilon) \in \mathbb{Y} = L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;H(\Omega,\rho^\varepsilon dx))\}_{\varepsilon>0}$$

w-converges to $(U,y) \in L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;W)$ in the variable space $\mathbb{Y}$ as $\varepsilon \to 0$, if $U_\varepsilon \to U$ weakly-$*$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$, $y_\varepsilon \rightharpoonup y$ in $L^p(0,T;L^p(\Omega))$ and $\nabla y_\varepsilon \rightharpoonup \nabla y$ in $L^p(0,T;L^p(\Omega,\rho^\varepsilon dx)^N)$

Similarly to [17, Definition 8] we consider the next concept.

Definition 5.2. We say that a minimization problem

$$\left\langle \inf_{(U,y)\in\Xi_H} I(U,y) \right\rangle \tag{5.1}$$

is a weak variational limit (or variational w-limit) of the sequence

$$\left\{ \left\langle \inf_{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(U_\varepsilon,y_\varepsilon) \right\rangle ;\, \Xi_\varepsilon \in \mathbb{Y},\, \varepsilon > 0 \right\}, \tag{5.2}$$

with respect to w-convergence in the variable space $\mathbb{Y}$, if the following conditions are satisfied:

(1) if $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, and a sequence $\{(U_k, y_k) \in \Xi_{\varepsilon_k}\}_{\varepsilon>0}$ – w-converges to a pair (U, y), then

$$(U, y) \in \Xi_H : I(U, y) \le \varliminf_{k\to\infty} I_{\varepsilon_k}(U_k, y_k); \tag{5.3}$$

(2) for every pair $(U, y) \in \Xi_H$ and any value $\delta > 0$ there exists a realizing sequence $\{(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}\}_{\varepsilon>0}$ such that

$$(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon \ \forall \varepsilon > 0, \ (\hat{U}_\varepsilon, \hat{y}_\varepsilon) \xrightarrow{w} (\hat{U}, \hat{y}), \tag{5.4}$$

$$\|U - \hat{U}\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \left(\int_0^T \|y - \hat{y}\|_\rho^p dt\right)^{1/p} \le \delta, \tag{5.5}$$

$$I(U, y) \ge \varlimsup_{\varepsilon\to 0} I_\varepsilon(\hat{U}_\varepsilon, \hat{y}_\varepsilon) - \delta. \tag{5.6}$$

Similarly to [23] we can assume that Definition 5.2 is motivated by the following property of variational w-limits.

Theorem 5.1. *Assume that* (5.1) *is a weak variational limit of the sequence* (5.2)*, and the constrained minimization problem* (5.1) *has a solution. Suppose* $\{(U^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *is a sequence of optimal pairs to* (5.2)*. Then there exists a pair* $(U^0, y^0) \in \Xi_H$ *such that* $(U^0_\varepsilon, y^0_\varepsilon)$ *w-converges to* (U^0, y^0)*, and*

$$\inf_{(U,y)\in\Xi_H} I(U, y) = I(U^0, y^0) = \lim_{\varepsilon\to 0} \inf_{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(U_\varepsilon, y_\varepsilon).$$

Remark 5.1. Let us recall that sequential K-upper and K-lower limits of a sequence of sets $\{E_k\}_{k\in\mathbb{N}}$ are defined as follows, respectively:

$$K_s - \varlimsup E_k = \{y \in X : \exists\sigma(k) \to \infty, \exists y_k \to y, \forall k \in \mathbb{N} : y_k \in E_{\sigma(k)}\},$$
$$K_s - \varliminf E_k = \{y \in X : \exists y_k \to y, \exists k \ge k_0 \in \mathbb{N} : y_k \in E_k\}.$$

The sequence $\{E_k\}_{k\in\mathbb{N}}$ sequentially converges in the sense of Kuratovski to the set E (shortly, K_s-converges), if $E = K_s - \varliminf E_k = K_s - \varlimsup E_k$.

Let us consider the sequence $\{\mathcal{K}_\varepsilon\}_{\varepsilon>0}$ of non-empty closed and convex subsets, which sequentially converges to the set $\mathcal{K}$ in the sense of Kuratovski as $\varepsilon \to 0$ with respect to weak topology of the space $L^p(0, T; H(\Omega, \rho^\varepsilon dx))$ and the sequence $\{\tilde{\mathcal{K}}_\varepsilon\}_{\varepsilon>0}$ of non-empty closed and convex subsets, which sequentially converges to the set $\tilde{\mathcal{K}} = \{v \in L^p(0, T; H) | v' \in L^q(0, T; L^q(\Omega)), v(0, x) = 0\}$ in the sense of Kuratovski as $\varepsilon \to 0$ with respect to the topology τ_1:

$$v_\varepsilon \rightharpoonup v \text{ in } L^p(0, T; H(\Omega, \rho^\varepsilon dx)), v'_\varepsilon \rightharpoonup v' \text{ in } L^q(0, T; L^q(\Omega)), v_\varepsilon(0, x) = 0.$$

Let Hypothesis 2 from [17] hold true for $X = L^q(\Omega)$ and $V = H(\Omega, \rho^\varepsilon dx)$ $\forall \varepsilon > 0$. Taking into account Theorem 5.1, we consider the following collection of perturbed OCPs in coefficients for non-degenerate parabolic variational inequalities:

$$\text{Minimize } \left\{ I_\varepsilon(U, y) = \int_0^T \int_\Omega |y(x) - z_\partial(x)|^p \, dxdt \right\} \tag{5.7}$$

$$U \in U^\varepsilon_{ad}, y \in \mathcal{K}_\varepsilon, \tag{5.8}$$

$$\begin{aligned}\langle v', v-y\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\\ +\langle -\operatorname{div}(\rho^\varepsilon U[(\nabla y)^{p-2}]\nabla y)+|y|^{p-2}y, v-y\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\\ \geq \langle f, v-y\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\ \forall v\in\tilde{\mathcal{K}}_\varepsilon,\end{aligned} \quad (5.9)$$

$$\begin{aligned}U^\varepsilon_{ad}=\{U=[\vec{a}_1,\dots,\vec{a}_N]\in M_p^{\alpha,\beta}(\Omega):\\ |\operatorname{div}(\rho^\varepsilon\vec{a}_i)|\leq\gamma_i,\ \text{a.e. in }\Omega,\ \forall i=1,\dots,N\},\end{aligned} \quad (5.10)$$

where the elements $z_\partial\in L^p(0,T;L^p(\Omega))$, $f\in L^q(0,T;L^q(\Omega))$ and $\gamma=(\gamma_1,\dots,\gamma_N)\in\mathbb{R}^N$ are the same as for the original problem (3.1)-(3.3), (3.8). For every $\varepsilon>0$ we define Ξ_ε as a set of all admissible pairs to the problem (5.7)-(5.10), namely $(U,y)\in\Xi_\varepsilon$ if and only if the pair (U,y) satisfies (5.8)-(5.10).

Note that each of perturbed OCPs (5.7)-(5.10) is solvable provided $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a non-degenerate perturbation of $\rho\geq 0$ (see [20]).

Lemma 5.1. *Let $\{\rho^\varepsilon=(\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho(x)\geq 0$. Let $\{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible pairs to the problem* (5.7)-(5.10) *and let Hypothesis A hold true for weakly convergent sequences in $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$. Then there exists a pair (U^*,y^*) and a subsequence*

$$\{(U_{\varepsilon_k},y_{\varepsilon_k})\}_{k\in\mathbb{N}}\subset\{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$$

such that $(U_{\varepsilon_k},y_{\varepsilon_k})$ w-converges to (U^*,y^*) as $k\to\infty$ and $(U^*,y^*)\in\Xi_H$.

Proof. Let us consider the variational inequality

$$\begin{aligned}\langle v'_\varepsilon, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\\ +\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon), v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\\ +\langle |y_\varepsilon|^{p-2}y_\varepsilon, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\\ \geq\langle f, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))},\ \forall v_\varepsilon\in\tilde{\mathcal{K}}_\varepsilon.\end{aligned} \quad (5.11)$$

As follows from (5.10) that the sequence $\{U_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega;\mathbb{R}^{N\times N})$.

Let us prove the boundedness of $\{y_\varepsilon\}_{\varepsilon>0}$ in the space $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$ by contradiction. Namely, suppose that $\|y_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\to\infty$, $\varepsilon\to 0$. Then on the one hand

$$\begin{aligned}\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon)+|y_\varepsilon|^{p-2}y_\varepsilon, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\\ \leq\langle -v'_\varepsilon, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}+\langle f, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\\ \leq(\|v'_\varepsilon\|_{L^q(0,T;L^q(\Omega))}+\|f\|_{L^q(0,T;L^q(\Omega))})\|y_\varepsilon-v_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))},\end{aligned} \quad (5.12)$$

$\forall v_\varepsilon\in\tilde{\mathcal{K}}_\varepsilon$ and $\forall\varepsilon>0$.

On the other hand, for arbitrary fixed element $v\in\tilde{\mathcal{K}}$ let us consider the sequence $\{v_\varepsilon\in\tilde{\mathcal{K}}_\varepsilon\}_{\varepsilon>0}$ such that $v_\varepsilon\to v$ in τ_1-topology (such sequence always exists provided $\tilde{\mathcal{K}} = K_s - \lim \tilde{\mathcal{K}}_\varepsilon$) and then, using the estimate (see Theorem 4.1)

$$\begin{aligned}\langle \mathcal{A}(U,y), y-v\rangle_{L^p(0,T;H)} &\geq \min\{\alpha,1\}\|y\|^p_{L^p(0,T;H)}\\ &\quad - \max\{\beta,1\}\|v\|^p_{L^p(0,T;H)}\|y\|^{p-1}_{L^p(0,T;H)},\ v\in L^p(0,T;H),\end{aligned}$$

we obtain the following relations:

$$\frac{\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon) + |y_\varepsilon|^{p-2}y_\varepsilon, v_\varepsilon - y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon - v_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}$$
$$\geq \|y_\varepsilon\|^{p-1}_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\frac{\left(\min\{\alpha,1\} - \frac{\max\{\beta,1\}\|v_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}\right)}{\left(1+\frac{\|v_\varepsilon\|_{L^p(0,T:H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}\right)} \to \infty \text{ as } \varepsilon\to 0,$$

since the sequence $\{v_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$. The obtained contradiction with (5.12) implies that $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$.

Hence, there exists a subsequence $\{\varepsilon_k\}$ of the sequence $\{\varepsilon\}$ converging to 0 and elements $U^*\in M_p^{\alpha,\beta}(\Omega)$, $y^*\in L^p(0,T;L^p(\Omega))$, $\vec{v}\in L^p(0,T;L^p(\Omega,\rho dx)^N)$ and $\vec{\xi}\in L^q(0,T;L^q(\Omega,\rho dx)^N)$ such that

$$\begin{aligned} U_{\varepsilon_k} &\to U^* \quad \text{weakly-}* \text{ in } L^\infty(\Omega;\mathbb{R}^{N\times N}),\\ y_{\varepsilon_k} &\rightharpoonup y^* \quad \text{in } L^p(0,T;L^p(\Omega)),\\ \nabla y_{\varepsilon_k} &\rightharpoonup \vec{v} \quad \text{in } L^p(0,T;L^p(\rho^{\varepsilon_k}dx)^N),\\ U_{\varepsilon_k}[(\nabla y_{\varepsilon_k})^{p-2}]\nabla y_{\varepsilon_k} := \vec{\xi}_{\varepsilon_k} &\rightharpoonup \vec{\xi} \text{ in } L^q(0,T;L^q(\Omega^{\varepsilon_k}dx)^N). \end{aligned} \tag{5.13}$$

By Theorem 2.4, taking into account properties of the Bochner integral and definitions of equivalent functions (see [21, Definition 1.6]), we have that $y^*\in L^p(0,T;H)$ and $\vec{v}=\nabla y^*$ and moreover, we have $y^*\in\mathcal{K}$.

Following arguments of the proof of [17, Lemma 11] we obtain that $U^*\in U_{ad}$.

In what follows, we consider the relation (5.11) for $(U_{\varepsilon_k}, y_{\varepsilon_k})$ and pass to the limit in it as $k\to\infty$ using the property of the strong convergence and the following relations:

$$|y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k} \rightharpoonup |y^*|^{p-2}y^* \text{ in } L^q(0,T;L^q(\Omega)) \text{ within a subsequence,} \tag{5.14}$$

$$\langle -\operatorname{div}(\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k}), y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} \to \langle -\operatorname{div}(\rho\vec{\xi}), y^*\rangle_{L^p(0,T;H)}. \tag{5.15}$$

The latter is valid in view of Lemma 2.3 and boundedness of the sequence $\{\vec{\xi}_{\varepsilon_k}\}\subset X(\Omega,\rho^{\varepsilon_k}dx)$, which we can obtain by the similar manner as in Theorem 4.2 for

$$X(\Omega,\rho^{\varepsilon_k}dx) = \{\vec{f}\in L^q(0,T;L^q(\Omega,\rho^{\varepsilon_k}dx)^N) \mid \operatorname{div}(\rho^{\varepsilon_k}\vec{f})\in L^q(0,T;L^q(\Omega))\},$$

with the norm

$$\|\vec{f}\|_{X(\Omega,\rho^{\varepsilon_k}dx)} = (\|\vec{f}\|^q_{L^q(0,T;L^q(\Omega,\rho^{\varepsilon_k}dx)^N)} + \|\operatorname{div}(\rho^{\varepsilon_k}\vec{f})\|^q_{L^q(0,T;L^q(\Omega))})^{1/q}.$$

Let us prove relation (5.14). We have that $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^p(0,T;L^p(\Omega))$, $\nabla y_{\varepsilon_k} \rightharpoonup \nabla y^*$ in $L^p(0,T;L^p(\Omega,\rho^{\varepsilon_k}dx)^N)$ and from [22, Proposition 4.1] we obtain that there exists an element $\tilde{y}$ such that $y_{\varepsilon_k} \to \tilde{y}$ strongly in $L^1(0,T;L^1(\Omega))$. However, it is easy to see that $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^1(0,T;L^1(\Omega))$. Hence, $y^* = \tilde{y}$ a.e. on $(0,T)\times\Omega$. It means that up to a subsequence $y_{\varepsilon_k} \to y^*$ a.e. in $(0,T)\times\Omega$ and together with boundedness of $\{|y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k}\}_{k\in\mathbb{N}}$ in $L^q(0,T;L^q(\Omega))$ we have $|y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k} \rightharpoonup |y^*|^{p-2}y^*$ in $L^q(0,T;L^q(\Omega))$ (within a subsequence).

Since $v'_{\varepsilon_k} \rightharpoonup v'$ in $L^q(0,T;L^q(\Omega))$ we can obtain that

$$\langle v'_{\varepsilon_k}, v_{\varepsilon_k} - y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} \to \langle v', v-y^*\rangle_{L^p(0,T;H)} \text{ as } k\to\infty. \tag{5.16}$$

Therefore, as a result of limit passage in (5.11), taking into account (5.14), (5.15) and (5.16), we obtain

$$\begin{aligned}\langle v', v-y^*\rangle_{L^p(0,T;H)} &+ \langle -\operatorname{div}(\rho\vec{\xi}), v-y^*\rangle_{L^p(0,T;H)} + \langle |y^*|^{p-2}y^*, v\rangle_{L^p(0,T;H)}\\ &- \varlimsup_{k\to\infty}\langle |y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k}, y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))}\\ &\geq \langle f, v-y^*\rangle_{L^p(0,T;H)},\ \forall v\in\tilde{\mathcal{K}}.\end{aligned} \tag{5.17}$$

In order to prove the lemma, it is left to show that $\vec{\xi} = U^*[(\nabla y^*)^{p-2}]\nabla y^*$. However it can be done in a similar manner as we did it proving Theorem 4.2.

Now, let us show that

$$\lim_{k\to\infty}\langle |y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k}, y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} = \lim_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt = \int_0^T\int_\Omega |y^*|^p\,dxdt.$$

On the one hand, in view of property of lower semicontinuity, weak convergence $y_{\varepsilon_k} \to y^*$ in $L^p(0,T;L^p(\Omega))$ as $k\to\infty$, implies that:

$$\int_0^T\int_\Omega |y^*|^p\,dxdt \leq \varliminf_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt.$$

On the other hand, from (5.17), taking into account the representation of the vector-function ξ, we obtain:

$$\begin{aligned}\varlimsup_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt &\leq \langle v' - \operatorname{div}(U^*(x)\rho(x)[(\nabla y^*)^{p-2}]\nabla y^*) - f, v-y^*\rangle_{L^p(0,T;H)}\\ &+\langle |y^*|^{p-2}y^*, v\rangle_{L^p(0,T;H)},\ \forall v\in\tilde{K}.\end{aligned}$$

Having put in the last inequality $v = y^*$, we get

$$\varlimsup_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt \leq \int_0^T\int_\Omega |y^*|^p\,dxdt.$$

Hence, summing up, the chain of inequalities

$$\int_0^T \int_\Omega |y^*|^p \, dxdt \le \varliminf_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^p \, dxdt \\ \le \varlimsup_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^p \, dxdt \le \int_0^T \int_\Omega |y^*|^p \, dxdt$$

turns into equality

$$\lim_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^p \, dxdt = \int_0^T \int_\Omega |y^*|^p \, dxdt$$

which implies, in view of criterium of strong convergence that $y_{\varepsilon_k} \to y^*$ strongly in $L^p(0,T;L^p(\Omega))$ as $k \to \infty$.

Therefore, variational inequality (5.17) can be represented in the form

$$\langle v', v - y^*\rangle_{L^p(0,T;H)} + \langle -\operatorname{div}(U^*(x)\rho(x)[(\nabla y^*)^{p-2}]\nabla y^*) \\ + |y^*|^{p-2}y^*, v - y^*\rangle_{L^p(0,T;H)} \ge \langle f, v - y^*\rangle_{L^p(0,T;H)}, \ \forall v \in \tilde{\mathcal{K}}. \quad (5.18)$$

Thus, w-limit pair (U^*, y^*) is admissible to the problem (3.1)-(3.3), (3.8), hence, $(U^*, y^*) \in \Xi_H$. □

Theorem 5.2. *Let $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho(x) > 0$. Then the minimization problem* (3.1)-(3.3), (3.8) *is a weak variational limit of the sequence* (5.7)-(5.10) *as $\varepsilon \to 0$ with respect to the w-convergencs in the variable space $\mathbb{Y}$.*

Proof. As an evident consequence of the previous lemma and the lower semi-continuity property of the cost functional (5.7) with respect to w-convergence in variable space $\mathbb{Y}$, we have the following conclusion: if $\{\varepsilon_k\}$ be a subsequence of indices $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $\{(U_k, y_k) \in \Xi_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is a sequence of admissible solutions to corresponding perturbed problems (5.7)-(5.10) such that $(U_k, y_k) \to (U, y)$ with respect to w-convergence, then properties (5.3) are valid.

To discuss properties (5.4)-(5.6) similarly to suggestions from [17] and [19] we can obtain that for an admissible pair $(U, y) \in \Xi_H$ there exists a realizing sequence $\{(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}\}_{\varepsilon>0}$ such that

$$(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon \ \forall \varepsilon > 0, \ \hat{U}_\varepsilon \to U \ *-\text{weakly in } L^\infty(\Omega; \mathbb{R}^{N\times N}); \\ \operatorname{div}(\rho^\varepsilon \vec{a}_{i\varepsilon}) \rightharpoonup \operatorname{div}(\rho \vec{a}_i) \text{ in } L^q(0,T;L^q(\Omega)) \ \forall i \in \{1,\dots,N\}, \\ \hat{y}_\varepsilon \to y \text{ strongly in } L^p(0,T;L^p(\Omega)), \ \nabla y_\varepsilon \rightharpoonup \nabla y \text{ in } L^p(0,T;L^p(\Omega;\rho^\varepsilon dx)^N).$$

From these suggestions the equality $I(U,y) = \lim_{\varepsilon\to 0} I_\varepsilon(\hat{U}_\varepsilon, \hat{y}_\varepsilon)$ follows.

Taking into account Definition 5.2 and previous suggestions of this proof we obtain the statement of the theorem. □

References

1. P.I. Kogut, G. Leugering, *Optimal L1 -control in coefficients for Dirichlet elliptic problems: H-optimal solutions*, Z. Anal. Anwend., **31** (1) (2012), 31–53.
2. P.I. Kogut, G. Leugering, *Optimal L1 -control in coefficients for Dirichlet elliptic problems: W-optimal solutions*, J. Optim. Theory Appl., **150** (2011), 205–232.
3. G. Buttazzo, G. Dal Maso, A. Garroni, A. Malusa, *On the relaxed formulation of some shape optimization problems*, Adv. Math. Sci. Appl., **1** (7) (1997), 1–24.
4. J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer Verlag, New York, 1971.
5. F. Murat F, *CompacitBЈ'e par compensation*, Ann. Sc. Norm. Sup. Pisa, **5** (1978), 489–507.
6. G. Butazzo, P.I. Kogut, *Weak optimal controls in coefficients for linear elliptic problems*, Revista Matematica Complutense, **24** (2011), 83–94.
7. F. Murat, *Un contre-exemple pour le probleme du controle dans les coefficients.* C. R. Acad. Sci. Paris, Ser. A-B, **273** (1971), A708–A711.
8. V.Ye. Kapustjan, O.P. Kogut, *Solenoidal controls in coefficients of nonlinear elliptic boundary value problems*, Computer Mathematics, **12** (2010), 138–143.
9. O.P. Kupenko, R. Manzo, *Approximation of an optimal control problem in coefficient for variational inequality with anisotropic p-Laplacian*, Nonlinear Differential Equations and Applications, **23** (2016)
10. O.P. Kupenko, R. Manzo, *On Optimal Controls in Coefficients for Ill-Posed NonLinear Elliptic Dirichlet Boundary Value Problems*, DCDS, Series B, **23(4)** (2018), 1363–1393.
11. P. Kogut, R. Manzo, A. Putchenko, *On approximate solutions to the Neumann elliptic boundary value problem with non-linearity of exponential type*, Boundary Value Problems, **208** (2016)
12. O.P. Kupenko, *Optimal Control Problems in Coefficients for Degenerate Variational inequalities of Monotone Type. I. Existence of Optimal Solutions*, Journal of Computational and Applied Mathematics, **106** (2011), 88–104.
13. N. Zadoianchuk, O. Kupenko, *On Solvability for One Class of Optimal Control Problems for Degenerate Elliptic Variation Inequalities*, Journal of Computational and Applied Mathematics, **114** (4) (2013), 10–23.
14. N.V. Zadoianchuk, *Optimal Control Problem for Degenerate Parabolic Variation Inequality: Solvability Theorem*, Journal of Computational and Applied Mathematics **1(115)** (2014), 10–23
15. N.V. Zadoyanchuk, *On the Existence of Strong Solutions for a Degenerate Parabolic Inequality with Mixed Boundary Conditions*, Journal of Mathematical Sciences **217** (2016), 441–455.
16. N.V. Kasimova, *Solvability Issue for Optimal Control Problem in Coefficients for Degenerate Parabolic Variational Inequality*, Part of the Understanding Complex Systems book series (UCS). Contemporary Approaches and Methods in Fundamental Mathematics and Mechanics (2021), 457–473
17. O.P. Kupenko, *Optimal control problems in coefficients for degenerate variational inequalities of monotone type. II. Attainability problem*, J. Comput. Appl. Math., **1** (107) (2012), 15–34.
18. N. Kasimova, *Optimal Control Problem for Some Degenerate Variation Inequality: Attainability Problem*, Journal of Optimization, Differential Equations and Their Applications, **26** (2) (2018), 37–54. doi:10.15421/141809

19. N. Kasimova, *Attainability Issue for the Optimal Control Problem in Coefficients for a Degenerate Parabolic Variational Inequality*, Journal of Mathematical Sciences, **258** (2021), 636–654.
20. J.-L. Lions *Some methods of Solving Non-Linear Boundary Value Problems*, Dunod-Gauthier-Villars, Paris, 1969.
21. H. Gajewski, K. Gröger, K. Zacharias *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Academie-Varlar, Berlin, 1974.
22. V.V. Zhikov, S.E. Pastukhova, *Lemmas on Compensated Compactness in Elliptic and Parabolic Equations*, Proceedings of the Steklov Institute of Mathematics, **270** (2010), 104–131.
23. C. D'Apice, U. De Maio, P.I. Kogut, *Suboptimal boundary control for elliptic equations in critically perforated domains*, Ann. Inst. H. Poincare' Anal. Non Line'aire, **25** (2008), 1073–1101.

Received 01.05.2023

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 22–52, DOI 10.15421/142302

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

SINGULAR DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS FOR MODELING STRONGLY OSCILLATING PROCESSES

Vasiliy Ye. Belozyorov[*], Svetlana A. Volkova[†], Vadym G. Zaytsev[‡]

Abstract. The normal system of ordinary differential equations, whose right-hand sides are the ratios of linear and nonlinear positive functions, is considered. A feature of these ratios is that some of their denominators can take on arbitrarily small nonzero values. (Thus, the modules of the corresponding derivatives can take arbitrarily large value.) In the sequel, the constructed system of differential equations is used to model strongly oscillating processes (for example, processes determined by the rhythms of electroencephalograms measured at certain points in the cerebral cortex). The obtained results can be used to diagnose human brain diseases.

Key words: system of differential equations, Lyapunov exponents, fractal dimension.

2010 Mathematics Subject Classification: 34A34, 34D20, 37D45, 93A30.

Communicated by Prof. G. Sklyar

1. Introduction

In this article a dynamic process determined by observed electroencephalogram (EEG) rhythms measured at a certain point in the cerebral cortex is investigated [1]– [5]. The main tool of such research is recurrent analysis [6]– [15].

The application of recurrent analysis to the study of EEGs was considered in many scientific articles [2]– [5]. In our opinion, the most fundamental approach to revealing the hidden laws that determine the behavior of the mentioned EEGs was demonstrated in [1].

Let

$$x_0 = x(t_0), x_1 = x(t_1), ..., x_N = x(t_N) \tag{1.1}$$

be a finite sequence (time series) of numerical values of some scalar dynamical variable $x(t)$ measured with the constant time step Δt in the moments $t_i = t_0 + i\Delta t$; $x_i = x(t_i)$; $i = 0, 1, ..., N$ (thus, $\Delta t = t_N/N$) [16–18].

Using the methods of Recurrence Quantification Analysis (RQA), the dimension m of the embedding space and the optimal time delay τ of the mentioned time series are determined.

[*]Department of Applied Mathematics, Oles Honchar Dnipro National University, 72, Gagarin's Avenue, 49010, Dnipro, Ukraine, belozvye2017@gmail.com

[†]Department of Computer Information Technologies, Ukrainian State University of Science and Technologies, Academican Lazaryan's Street, 2, 49010, Dnipro, Ukraine, svolkovav2017@gmail.com

[‡]Department of Applied Mathematics, Oles Honchar Dnipro National University, 72, Gagarin's Avenue, 49010, Dnipro, Ukraine, vadymzaytsev65@gmail.com

It must be said that the quantities m and τ must be determined very precisely. The fact is that if the dimension m is less than the real dimension of the space in which the process takes place, then there is no need to talk about high-quality modeling. With the help of these characteristics, the hidden variables $x(t), y(t), z(t), ...$, which determine a system of rational differential equations simulating electrical signals in the cerebral cortex, are restored.

The properties of this system are the main subject of study in this work.

Note that in the problem of studying brain diseases, the time series (1.1) has a chaotic behavior. A common practice in chaotic time series analysis has been to reconstruct the phase space by utilizing the delay-coordinate embedding technique, and then to compute the dynamical invariant magnitudes such as unstable periodic orbits, a fractal dimension of the underlying chaotic set, and its Lyapunov spectrum. As a large body of literature exists on applying of the technique of the time series to study chaotic attractors [19]– [23], a relatively unexplored issue is its applicability to dynamical systems of differential equations depending on parameters. Our focus will be concentrated on the analysis of influence of parameters of found dynamic system on the behavior of its solutions. These parameters are determined by the structure of series (1.1) and by choice of approximating functions in right sides of the got system of differential equations.

To create a model by measuring the variables characterizing any dynamic process, it is necessary to solve the following three main problems.

Usually, a continuous dynamic process is described using a system of differential equations. This remark leads to the first problem.

Problem 1. It is necessary to establish the type of functions on the right side of the differential equations, which most correspond to the description of the processes presented on the electroencephalograms.

It is known that any dynamic process depends on many variables. Most of these variables are functions of some small number of independent variables. Identifying these independent variables leads to the second problem.

Problem 2. Determine the dimension of phase space in which the explored process takes place.

Problem 3. After the structure of the differential equations describing the dynamic process is established, it is necessary to determine the numerical value of coefficients of these equations.

After that, it remains only to check how the resulting model (solutions of the resulting system of differential equations) is adequate to the real process.

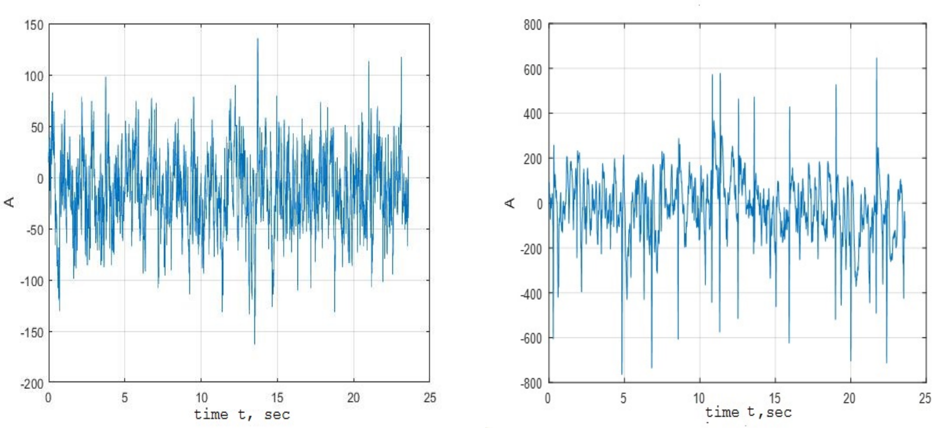
2. Mathematical preliminaries

2.1. Model design

We will begin this section by studying **Problem 1**. For this study, we will consider EEGs obtained for healthy and sick patients (see Fig.2.1, Fig.2.2).

Let's note several features inherent in these EEGs.

1. The diagrams have a pronounced oscillating character with a frequency of 400 - 500 hertz.
2. The oscillation amplitudes in the diagram of the sick patient are several times greater than the amplitudes of oscillations in the diagram of the healthy patient.
3. Both diagrams contain a large number of spontaneous bursts of amplitudes, which indicates the chaotic nature of the processes.
4. The presence of times t_i, at which a spontaneous increase in the amplitude of oscillations is observed, indicates that at points t_i there is a sharp increase in the derivative of the process under study; $i = 1, 2, ...$
5. The oscillating process takes place in some ball centered at the point $\mathbf{0}$.



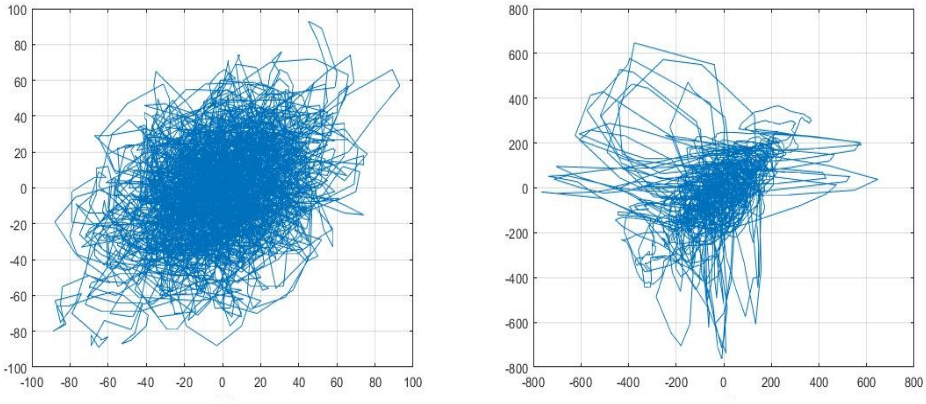
(a1) (a2)

Fig. 2.1. The electroencephalogram taken from a certain point in the cerebral cortex: (a1) a healthy patient, (a2) a patient with an epileptic disease (see [24]).

Let's return to modeling the process $x(t)$, which is generated by the time series (1.1).

We introduce the following real singular function depending on parameters a, b, γ, ω, f and e (or $\delta, \gamma, \beta, \omega, \alpha$ and ε):

$$h(t) = \frac{a \cdot \sin(\gamma t) + b \cdot \cos(\gamma t)}{1 - f \cdot \sin(\omega t) - e \cdot \cos(\omega t)} = \frac{\delta \sin(\gamma t + \beta)}{1 + \varepsilon \cos(\omega t + \alpha)}, \tag{2.1}$$



(b1) (b2)

Fig. 2.2. Graphs of the same processes as on Fig.2.1, but in coordinates $(x(t), x(t+\tau))$: (b1) the healthy patient, (b2) the patient with an epileptic disease (see [24]).

where $\gamma, \omega \in \mathbb{R}; \delta = \sqrt{a^2+b^2}$, $\sin(\beta) = \dfrac{b}{\delta}, \cos(\beta) = \dfrac{a}{\delta}$; $|\varepsilon| = \sqrt{f^2+e^2} < 1$, $\sin(\alpha) = \dfrac{f}{\varepsilon}, \cos(\alpha) = -\dfrac{e}{\varepsilon}$ (see Fig.2.3).

Taking into account the form of the function $h(t)$, we can assume that the simplest description of the derivative $\dot{x}(t)$ satisfying items 1) - 5) should look like this:

$$\dot{x}(t) \sim \frac{c \cdot x(t)}{1+\varepsilon\cos(x(t)+\alpha)} + \dots + \frac{\delta\sin(\omega t+\beta)}{1+\varepsilon\cos(\omega t+\mu)},$$

where the last term takes into account the possibility of the influence of external perturbations on the formation of the structure of differential equations; here $c, \alpha, \varepsilon, \delta, \omega, \beta, \mu$ are real constants.

Using the function $h(t)$, we construct the following system of ordinary differential equations:

$$\begin{cases} \dot{x}_1(t) = \dfrac{a_{11}x_1}{1-f_{11}\sin(x_1)-e_{11}\cos(x_1)} + \dots \\ \qquad + \dfrac{a_{1,n-1}x_{n-1}}{1-f_{1,n-1}\sin(x_{n-1})-e_{1,n-1}\cos(x_{n-1})} \\ \qquad + \dfrac{b_1\sin(x_n)+c_1\cos(x_n)}{1-f_{1n}\sin(x_n)-e_{1n}\cos(x_n)}, \\ \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x}_{n-1}(t) = \dfrac{a_{n-1,1}x_1}{1-f_{n-1,1}\sin(x_1)-e_{n-1,1}\cos(x_1)} + \dots \\ \qquad + \dfrac{a_{n-1,n-1}x_{n-1}}{1-f_{n-1,n-1}\sin(x_{n-1})-e_{n-1,n-1}\cos(x_{n-1})} \\ \qquad + \dfrac{b_{n-1}\sin(x_n)+c_{n-1}\cos(x_n)}{1-f_{n-1,n}\sin(x_n)-e_{n-1,n}\cos(x_n)}, \\ \dot{x}_n(t) = \omega. \end{cases} \tag{2.2}$$

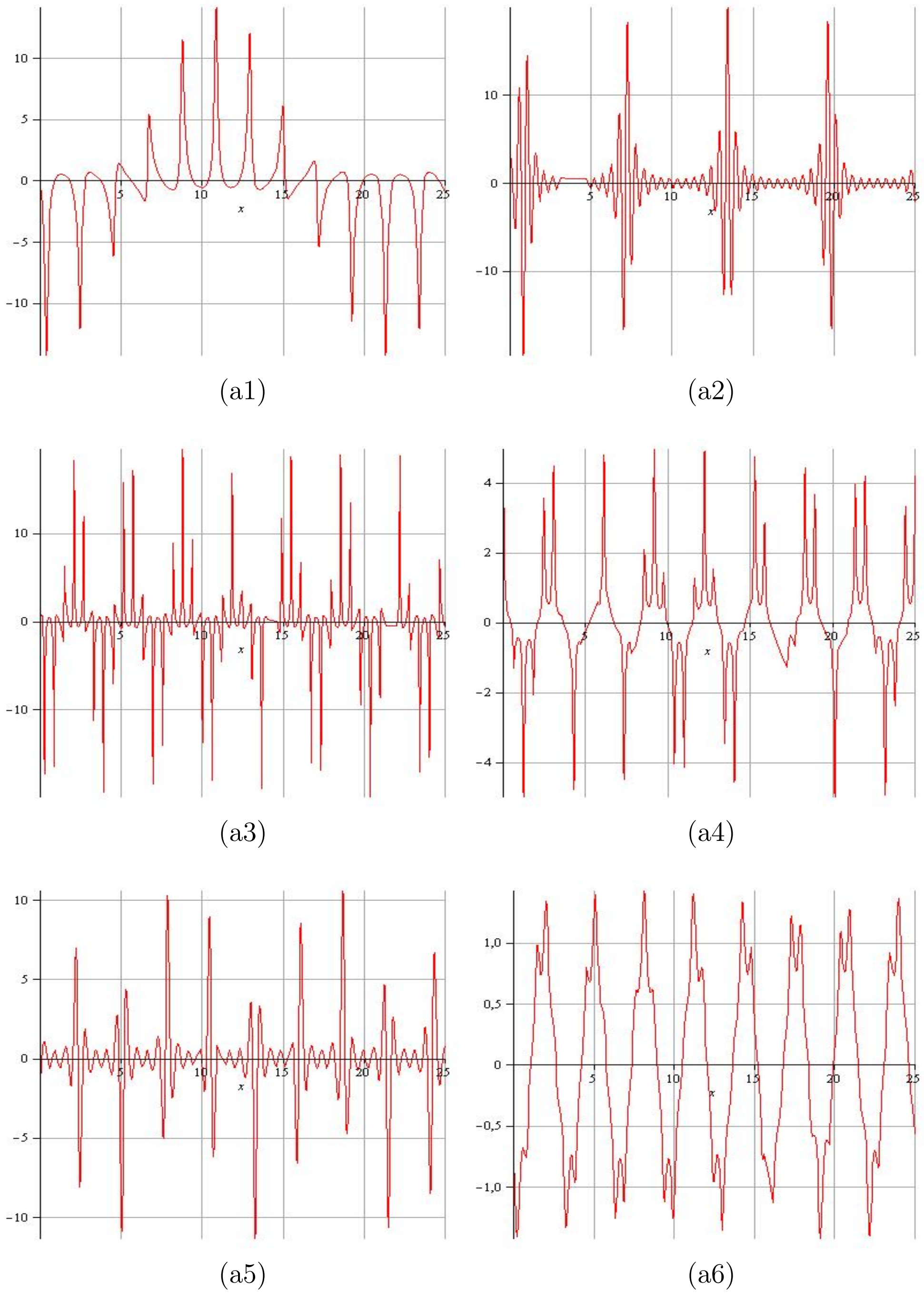


Fig. 2.3. Graphs of function (2.1) for different parameter values: (a1) $\varepsilon = 0.93$, $\gamma = -3.3$, $\beta = 6, \omega = -3, \alpha = -2$; (a2) $\varepsilon = 0.95, \gamma = -12$, $\beta = 2, \omega = 1, \alpha = -4$; (a3) $\varepsilon = 0.95, \gamma = -2$, $\beta = -1, \omega = 10$, $\alpha = 10$; (a4) $\varepsilon = 0.8, \gamma = -10$, $\beta = -10, \omega = -10.3, \alpha = -3$; (a5) $\varepsilon = 0.9, \gamma = 9.9$, $\beta = -1, \omega = 2.3, \alpha = -52$; (a6) $\varepsilon = 0.3, \gamma = -2$, $\beta = -1, \omega = 10, \alpha = 1$; $\delta = 1$.

Here $a_{ij}, f_{ij}, e_{ij}, b_i, c_i, \omega > 0$ are real parameters; $\sqrt{f_{ij}^2 + e_{ij}^2} < 1; i = 1, ..., n - 1; j = 1, ..., n$. (Thus, system (2.2) depends on $(n-1)^2 + 2n(n-1) + 2(n-1) + 1 = n(3n-2)$ parameters, and all of them are rationally included in this system.)

Note that the denominators in the right-hand sides of system (2.2) can take on arbitrarily small positive values.

Definition 2.1. System (2.2) will be called singular.

The inclusion of external perturbations in equations (2.2) cannot always be correctly described. Therefore, sometimes instead of model (2.2), it is necessary to consider the following model

$$\begin{cases} \dot{x}_1(t) = \dfrac{a_{11}x_1}{1 - f_1 \sin(x_1) - e_1 \cos(x_1)} + \cdots + \dfrac{a_{1,n}x_n}{1 - f_n \sin(x_n) - e_n \cos(x_n)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_n(t) = \dfrac{a_{n,1}x_1}{1 - f_1 \sin(x_1) - e_1 \cos(x_1)} + \cdots + \dfrac{a_{n,n}x_n}{1 - f_n \sin(x_n) - e_n \cos(x_n)}. \end{cases} \tag{2.3}$$

Now, if we put $\delta = 1, \gamma = 0, \beta = \pi/2$ in formula (2.1), then after a linear change of variables $\mathbf{x} \to A\mathbf{x}$, system (2.3), can be transformed into the following system:

$$\begin{cases} \dot{x}_1(t) = h_1(a_{11}x_1 + \cdots + a_{1n}x_n), \\ \dot{x}_2(t) = h_2(a_{21}x_1 + \cdots + a_{2n}x_n), \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ \dot{x}_n(t) = h_n(a_{n1}x_1 + \cdots + a_{nn}x_n). \end{cases} \tag{2.4}$$

Here

$$h_i(a_{i1}x_1 + ... + a_{in}x_n) = \frac{a_{i1}x_1 + \cdots + a_{in}x_n}{1 \pm \varepsilon_i \cos(a_{i1}x_1 + \cdots + a_{in}x_n + \alpha_i)},$$

and $\varepsilon_i = \sqrt{f_i^2 + e_i^2} < 1; i = 1, ..., n$.

System (2.4) can be considered as a system of neural ODEs, which makes it possible to use neural network methods for its study [15, 22, 23, 25].

2.2. On boundedness of solutions of singular system (2.2)

We now recall several well-known results from the theory of differential equations [26, 27].

Let's define the norm of the vector $\mathbf{w} = (w_1, ..., w_k)^T \in \mathbb{R}^k$ by the formula $\|\mathbf{w}\| = |w_1| + \cdots + |w_k|$. The norm of matrix $C \in \mathbb{R}^{k \times k}$ is defined similarly: $\|C\| = \sum_{i=1}^k \sum_{j=1}^k |c_{ij}|$.

Consider the system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = (A + B(t))\mathbf{x}(t) + \mathbf{g}(t) \in \mathbb{R}^k, \tag{2.5}$$

where $\mathbf{x}(t) = (x_1(t), ..., x_k(t))^T, \mathbf{g}(t) = (g_1(t), ..., g_k(t))^T \in \mathbb{R}^k, A = \{a_{ij}\}, B(t) = \{b_{ij}(t)\} \in \mathbb{R}^{k \times k}; i, j = 1, ..., k$.

Theorem 2.1. *[26] Assume that for a homogeneous* ($\mathbf{g}(t) \equiv \mathbf{0}$) *system (2.5) the following conditions are fulfilled:*

(a1) the matrix A is constant and such that its eigenvalues λ_i satisfy the condition $\Re e(\lambda_i) \leq 0, i = 1, ..., k$;

(a2) the variable continuous matrix $B(t)$ depends on time and such that

$$\int_{t_0}^{\infty} \Big\| B(t) \Big\| \, dt < \infty.$$

Then for any vector of initial conditions $\mathbf{x}_0$ the solution $\mathbf{x}(t, \mathbf{x}_0)$ of system (2.5) is bounded at $t \to \infty$.

Theorem 2.2. *[26] Let us assume that under the conditions of Theorem 2.1 for an inhomogeneous* ($\mathbf{g}(t) \not\equiv \mathbf{0}$) *system (2.5) the following conditions also fulfilled:*

$$\int_{t_0}^{\infty} \mathrm{tr}(A + B(t)) \, dt > -\infty \;\; and \;\; \left\| \int_{t_0}^{\infty} \mathbf{g}(t) \, dt \right\| < \infty.$$

Then for any vector of initial conditions $\mathbf{x}_0$ the solution $\mathbf{x}(t, \mathbf{x}_0)$ of system (2.5) is bounded at $t \to \infty$.

Theorem 2.3. *[26] If the function $\phi(t)$ tends monotonically to zero ($\lim_{t\to\infty} \phi(t) = 0$) and the function $\psi(t)$ has a bounded antiderivative ($\int_{t_0}^{\infty} \psi(t) \, dt < \infty$), then the integral $\int_{t_0}^{\infty} \phi(t)\psi(t) \, dt$ converges.*

Theorem 2.4. *[27] (Global Existence and Uniqueness) Suppose that the function $\mathbf{F}(t, \mathbf{x}) \in \mathbb{R}^k$ is piecewise continuous in t and $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^k, \forall t \in [t_0, \infty)$ satisfies the conditions*

$$\|\mathbf{F}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \;\; and \;\; \|\mathbf{F}(t, \mathbf{x}_0)\| \leq P,$$

where $L > 0, P > 0$ are constants. Then, the state equation $\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x})$ with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ has a unique solution over $[t_0, \infty)$.

Let $\mathbf{x}_1 = (x_1, ..., x_{n-1})^T \in \mathbb{R}^{n-1}$. We introduce the following square matrices

$$A_1 = \{a_{ij}\}, \;\; B_1(\mathbf{x}_1) = \left\{ a_{ij} \frac{f_{ij} \sin(x_j) + e_{ij} \cos(x_j)}{1 - f_{ij} \sin(x_j) - e_{ij} \cos(x_j)} \right\} \in \mathbb{R}^{(n-1)\times(n-1)};$$

$i, j = 1, ..., n - 1$.

Let us also introduce the real $(n - 1)$-vector

$$\mathbf{g}_1(t) = \left(\frac{b_1 \sin(\omega t) + c_1 \cos(\omega t)}{1 - f_{1,n} \sin(\omega t) - e_{1,n} \cos(\omega t)}, ..., \frac{b_{n-1} \sin(\omega t) + c_{n-1} \cos(\omega t)}{1 - f_{n-1,n} \sin(\omega t) - e_{n-1,n} \cos(\omega t)} \right)^T.$$

In this case, instead of system (2.2), we can consider the following system

$$\dot{\mathbf{x}}_1(t) = (A_1 + B_1(\mathbf{x}_1))\mathbf{x}_1 + \mathbf{g}_1(t) \in \mathbb{R}^{n-1} \tag{2.6}$$

with the initial condition $\mathbf{x}_1(t_0) = \mathbf{x}_{10}$.

The following theorem is the main one in this paper.

Theorem 2.5. *Let the matrix* $A_1 = \{a_{ij}\}; i,j = 1,...,n-1$, *in singular system (2.6) be Hurwitz. If* $\sqrt{f_{ij}^2 + e_{ij}^2} < 1; i = 1,...,n-1; j = 1,...,n$, *then for any vector of initial conditions* $\mathbf{x}_{10}$ *the solution* $\mathbf{x}_1(t,\mathbf{x}_{10})$ *of system (2.6) is bounded at* $t \to \infty$.

Proof. (c1) Let us estimate the norm of the matrix $(A_1 + B_1(\mathbf{x}_1))$. Since $\sqrt{f_{ij}^2 + e_{ij}^2} < 1$, then we have

$$\begin{aligned}\|A_1 + B_1(\mathbf{x}_1)\| &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \frac{|a_{ij}|}{1 - f_{ij}\sin(x_j) - e_{ij}\cos(x_j)} \\ &\le \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \frac{|a_{ij}|}{1 - \sqrt{f_{ij}^2 + e_{ij}^2}} = K > 0.\end{aligned}$$

(c2) Let $\mathbf{g}_1(t) \equiv 0$. Now suppose that a solution $\mathbf{w}(t) = \mathbf{x}_1(t,\mathbf{x}_{10})$ of system (2.6) exists. Then we can estimate its norm $\|\mathbf{w}(t)\|$.

We have

$$\mathbf{w}(t) = \exp(A_1 t_0)\mathbf{w}(t_0) + \int_{t_0}^{\infty} \exp(A_1(t-\tau))B_1(\mathbf{w}(\tau))\mathbf{w}(\tau)\,d\tau$$

and

$$\|\mathbf{w}(t)\| \le \|\exp(A_1 t_0)\|\|\mathbf{w}(t_0)\| + \int_{t_0}^{\infty} \|\exp(A_1(t-\tau))B_1(\mathbf{w}(\tau))\|\|\mathbf{w}(\tau)\|\,d\tau$$

Since the matrix A_1 is Hurwitz, then we have $\|\exp(A_1 t)\| < c\cdot\exp(-\Lambda t) < c\cdot\exp(-\Lambda t_0) = N_1$ and according to the Bellman-Gronwall Lemma [27], we have

$$\|\mathbf{w}(t)\| \le N_1\|\mathbf{w}(t_0)\| \exp\Big(\int_{t_0}^{\infty} \|\exp(A_1(\tau)B_1(\mathbf{w}(\tau))\|\,d\tau\Big), \tag{2.7}$$

where $c > 0$, $-\Lambda = \max(\Re e(\lambda_1),...,\Re e(\lambda_n)) < 0$, and $\lambda_1,...,\lambda_n$ are eigenvalues of matrix A_1.

A rougher estimate of the norm $\|\mathbf{w}(t)\|$ can be obtained as follows:

$$\|\dot{\mathbf{w}}(t)\| \le \|A_1 + B_1(\mathbf{w}_1(t))\| \cdot \|\mathbf{w}(t)\| \le K\|\mathbf{w}(t)\|.$$

From here it follows that $\|\mathbf{w}(t)\| \le \exp(Kt)\|\mathbf{w}(t_0)\|$.

(c3) Now, let's estimate the integral

$$\int_{t_0}^{\infty} \exp(A_1(\tau)B_1(\mathbf{w}(\tau))\,d\tau, \tag{2.8}$$

where the elements of the matrix B_1 are

$$b_{ij} = a_{ij}\frac{f_{ij}\sin(w_j(t)) + e_{ij}\cos(w_j(t))}{1 - f_{ij}\sin(w_j(t)) - e_{ij}\cos(w_j(t))}.$$

Let t_k be the root of the equation $w_j(t_{j,k+1}) = w_j(t_{j,k})+2\pi; j = 1, ..., n-1; k = 0, 1, ...$ In this case, we can get the following estimate:

$$\left\| \int_{t_0}^{\infty} \exp(A_1(\tau)B_1(\mathbf{w}(\tau))\,d\tau \right\| \le \int_{t_0}^{\infty} \left\| \exp(A_1(\tau)B_1(\mathbf{w}(\tau)) \right\| d\tau$$

$$\le \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \frac{c|a_{ij}|}{1-\sqrt{f_{ij}^2+e_{ij}^2}}$$

$$\times \left[\lim_{l\to\infty} \sum_{k=0}^{l} \underbrace{\int_{t_{j,k}}^{t_{j,k+1}} \exp(-\Lambda t)|f_{ij}\sin(w_j(t)) + e_{ij}\cos(w_j(t))|\,dt}_{\hookrightarrow 0} \right.$$

$$\left. + \int_{t_{j,l+1}}^{t_{j,\xi(l)}} \exp(-\Lambda t)|f_{ij}\sin(w_j(t)) + e_{ij}\cos(w_j(t))|\,dt \right]$$

$$\le \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \frac{c|a_{ij}|}{1-\sqrt{f_{ij}^2+e_{ij}^2}} \int_{t_{j,l+1}}^{t_{j,\xi(l)}} \exp(-\Lambda t)|f_{ij}\sin(w_j(t)) + e_{ij}\cos(w_j(t))|\,dt$$

$$\le 2c_0 \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \frac{|a_{ij}|\sqrt{f_{ij}^2+e_{ij}^2}}{1-\sqrt{f_{ij}^2+e_{ij}^2}} \int_{t_{j,l+1}}^{t_{j,\xi(l)}} |\sin(w_j(t)) + \cos(w_j(t))|\,dt$$

$$\le 2c_0 \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \frac{|a_{ij}|\sqrt{f_{ij}^2+e_{ij}^2}}{1-\sqrt{f_{ij}^2+e_{ij}^2}} < \infty.$$

Here $c_0 = c\cdot(\Lambda)^{-1}\cdot\exp(-\Lambda t_0); w_j(t_{j,\xi(l)}) < w_j(t_{j,l+1}) + 2\pi; j = 1, ..., n-1.$

Thus, integral (2.8) is bounded.

As follows from item (c2), the solution $\mathbf{w}(t)$ of system (2.6) has an order of growth no higher than the function $\exp(N_1 t)$:

$$\lim_{t\to\infty} \frac{\|\mathbf{w}(t)\|}{\exp(Kt)} = l\ (0 \le l < \infty). \tag{2.9}$$

Taking into account (2.9) ($\|\mathbf{w}(t)\| \sim l\exp(Kt)$), we estimate the integral

$$H_{ij} = \int_{t_0}^{\infty} \frac{f_{ij}\sin(w_j(t)) + e_{ij}\cos(w_j(t))}{1 - f_{ij}\sin(w_j(t)) - e_{ij}\cos(w_j(t))}\,dt.$$

Since $\varepsilon_{ij} = \sqrt{f_{ij}^2+e_{ij}^2} < 1$, it is obvious that

$$H_{ij} \sim Q_{ij} = \int_{t_0}^{\infty} \frac{f_{ij}\sin(l\exp(Kt)) + e_{ij}\cos(l\exp(Kt))}{1 - f_{ij}\sin(l\exp(Kt)) - e_{ij}\cos(l\exp(Kt))}\,dt.$$

Let's change the variable $s(t) = l\exp(N_1 t)$. Then we will have

$$H_{ij} \sim Q_{ij} = \int_{s_0}^{\infty} \frac{\delta_{ij}\sin(s+\beta_{ij})}{s(1+\varepsilon_{ij}\cos(s+\alpha_{ij}))}\,ds = \int_{s_0}^{\infty} \frac{h_{ij}(s)\,ds}{s},$$

where the function $h_{ij}(s)$ is defined by formula (2.1) and $s_0 = s(t_0) = l\exp(Kt_0)$.

The improper integral Q_{ij} can be written as follows

$$Q_{ij} = \int_{s_0}^{\infty} \phi(s)\psi(s)\,ds,$$

where the functions $\phi(s)$ and $\psi(s)$ satisfy the conditions

$$\lim_{s\to\infty}\phi(s) = \lim_{s\to\infty}\frac{1}{s} = 0,\ \lim_{s\to\infty}\int_{s_0}^{\infty}\psi(s)\,ds = \lim_{s\to\infty}\int_{s_0}^{s}\frac{\delta_{ij}\sin(s+\beta_{ij})}{1+\varepsilon_{ij}\cos(s+\alpha_{ij})}\,ds$$
$$\le \lim_{s\to\infty}\frac{|\delta_{ij}|}{\sqrt{1-\varepsilon_{ij}^2}}\left|\int_{s_0}^{s}\sin(s+\beta_{ij})\,ds\right| \le \frac{2|\delta_{ij}|}{\sqrt{1-\varepsilon_{ij}^2}} < \infty.$$

Thus, the indicated functions $\phi(s)$ and $\psi(s)$ satisfy the conditions of Theorem 2.3. Consequently, the integral Q_{ij} and also the integral H_{ij} converge. Therefore, integral (2.8) also converges. Then according to (2.7), we get $\|\mathbf{w}(t)\| < \infty$. All conditions of Theorem 2.1 are satisfied.

(c4) Now let $\mathbf{g}_1(t) \not\equiv 0$. It is necessary to estimate the integral

$$\int_{t_0}^{\infty}\mathbf{g}_1(t)\,dt.$$

Any component of the vector $\mathbf{g}_1(t)$ has the form

$$g_{in} = \frac{b_i\sin(\omega t) + c_i\cos(\omega t)}{1 - f_{in}\sin(\omega t) - e_{in}\cos(\omega t)}.$$

Further reasoning repeats the reasoning of item (c3).

Let us introduce a number $t_{i,k}$ such that $\omega t_{i,k+1} = \omega t_{i,k} + 2\pi$. In this case, we can get the following estimate:

$$\left\|\int_{t_0}^{\infty}\mathbf{g}_1(t)\,dt\right\| = \sum_{i=1}^{n-1}\left|\int_{t_0}^{\infty} g_{in}(t)\,dt\right| \le \sum_{i=1}^{n-1}\frac{1}{\sqrt{1-\varepsilon_{in}^2}}$$
$$\times\left[\lim_{l\to\infty}\sum_{k=0}^{l}\underbrace{\left|\int_{t_{i,k}}^{t_{i,k+1}}(b_i\sin(\omega t)+c_i\cos(\omega t))\,dt\right|}_{\hookrightarrow 0}\right.$$
$$\left. + \left|\int_{t_{i,l+1}}^{t_{i,\xi(l)}}(b_i\sin(\omega t)+c_i\cos(\omega t))\,dt\right|\right]$$
$$\le \sum_{i=1}^{n-1}\frac{\sqrt{b_i^2+c_i^2}}{\sqrt{1-\varepsilon_{in}^2}} \times \left[\lim_{l\to\infty}\sum_{k=0}^{l}\underbrace{\left|\int_{t_{i,k}}^{t_{i,k+1}}(\sin(\omega t)+\cos(\omega t))\,dt\right|}_{\hookrightarrow 0}\right.$$
$$\left. + \left|\int_{t_{i,l+1}}^{t_{i,\xi(l)}}(\sin(\omega t)+\cos(\omega t))\,dt\right|\right] \le 2\sum_{i=1}^{n-1}\frac{\sqrt{b_i^2+c_i^2}}{\sqrt{1-\varepsilon_{in}^2}} = N_2 < \infty,$$

where $N_2 > 0$ is a constant.

Further, using the technique of proving that the integral H_{ij} is bounded (see item (c3)) and condition $a_{11} + \cdots + a_{nn} < 0$ (see [26]), we obtain

$$\int_{t_0}^{\infty} \operatorname{tr}(A_1 + B_1(\mathbf{w}_1(t))\, dt = \sum_{i=1}^{n-1} \int_{t_0}^{\infty} \frac{a_{ii}}{1 - f_{ii}\sin(w_i(t)) - e_{ii}\cos(w_i(t))}\, dt$$
$$= \sum_{i=1}^{n-1} a_{ii} \int_{s_0}^{\infty} \frac{ds}{s(1 + \varepsilon_{ii}\cos(s + \alpha_{ii}))}. \tag{2.10}$$

For the integral in formula (2.10), we have the following estimate:

$$\lim_{s\to\infty} \int_{s_0}^{s} \frac{ds}{1 + \varepsilon_{ii}\cos(s + \alpha_{ii})}$$
$$= \lim_{s\to\infty} \frac{2}{\sqrt{1-\varepsilon_{ii}^2}} \arctan\left(\sqrt{\frac{1-\varepsilon_{ii}}{1+\varepsilon_{ii}}} \tan\frac{s+\alpha_{ii}}{2}\right)$$
$$- \frac{2}{\sqrt{1-\varepsilon_{ii}^2}} \arctan\left(\sqrt{\frac{1-\varepsilon_{ii}}{1+\varepsilon_{ii}}} \tan\frac{s_0+\alpha_{ii}}{2}\right) < \frac{2\pi}{\sqrt{1-\varepsilon_{ii}^2}}.$$

Now the last inequality and Theorem 2.3 allow us to obtain such an estimate for integral (4.2):

$$\int_{t_0}^{\infty} \operatorname{tr}(A_1 + B_1(\mathbf{w}_1(t))\, dt > 2\pi(a_{11} + \cdots + a_{n-1,n-1}) \sum_{i=1}^{n-1} \frac{1}{\sqrt{1-\varepsilon_{ii}^2}} > -\infty.$$

Now, to prove the boundedness of the solutions of system (2.6), it only remains to apply Theorem 2.2.

(c5) Consider the linear system

$$\dot{\mathbf{x}}_1(t) = (A_1 + B_1(\mathbf{w}(t)))\mathbf{x}_1 + \mathbf{g}_1(t) \in \mathbb{R}^{n-1}. \tag{2.11}$$

where $B_1(\mathbf{w}(t))$ is piecewise continuous functions of t.

Now we check the fulfillment of the conditions of Theorem 2.4 for system (2.6). Over any finite interval of time $[t_0, \infty)$, the elements of $A_1 + B_1(\mathbf{w}(t))$ are bounded. Therefore, we have

$$\|\mathbf{F}(t, \mathbf{x}_1) - \mathbf{F}(t, \mathbf{y}_1)\| = \|(A_1 + B_1(\mathbf{x}_1))\mathbf{x}_1 - (A_1 + B_1(\mathbf{y}_1))\mathbf{y}_1\| \le N_1\|\mathbf{x}_1 - \mathbf{y}_1\|,$$

$$\|\mathbf{F}(t, \mathbf{x}_{10})\| = \|(A_1 + B_1(\mathbf{x}_{10}))\mathbf{x}_{10} + \mathbf{g}_1(t)\| \le N_1\|\mathbf{x}_{10}\| + N_2 \le P.$$

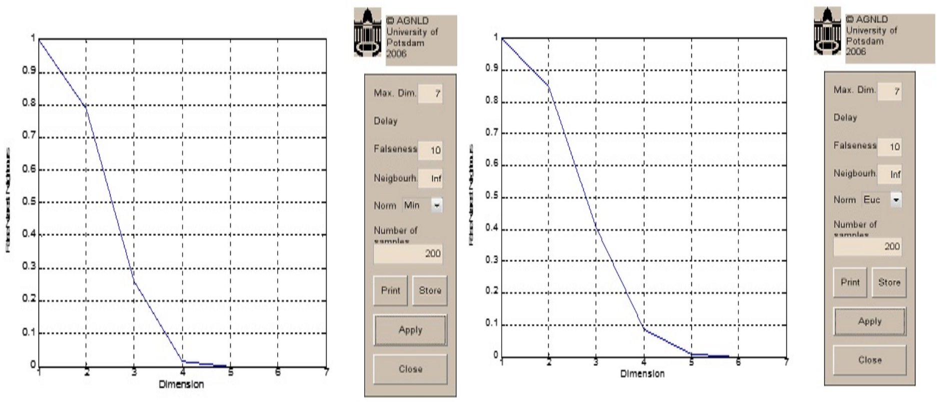
Thus, if $L = N_1$, $P > N_2$, and $\|\mathbf{x}_{10}\| \le (P - N_2)/N_1$, then the conditions of Theorem 2.4 are satisfied for any $t \in [t_0, \infty)$. This means that under these conditions a solution to system (2.6) exists and is unique.

The proof of Theorem 2.5 is complete. □

Note that systems (2.3) and (2.4) are particular cases of system (2.6) (Only it should be remembered that in these systems $A \in \mathbb{R}^{n\times n}$.) Therefore, Theorem 2.5 is also true for systems (2.3) and (2.4).

3. Rationale for using equations (2.2) to model EEG rhythms

In the study of dynamic processes, as a rule, only a few variables describing the process are available for direct measurement. The remaining variables (the so-called hidden variables) are inaccessible to observation. This raises the problem of reconstructing these unobserved variables from known observable variables. The first step towards solving this problem is to establish the minimum number of all variables (measured and hidden) on which the dynamic process depends (**Problem 2**).



(a1) (a2)

Fig. 3.1. Dimension of the embedding space for time series (1.1): healthy (a1) and sick (a2) patients.

Consider the time series (1.1). Using the recurrent analysis [2, 17, 18, 25, 28], we calculate the dimension m of the embedding space and the optimal time delay τ. Using these characteristics, we construct m time series

$$\left\{\begin{array}{ll} x_1(t) = \{x_1(t_0) = x(t_0), & x_2(t) = \{x_2(t_0) = x(t_0+\tau), \\ \qquad x_1(t_1) = x(t_1), & \qquad x_2(t_1) = x(t_1+\tau), \\ \qquad \vdots & \qquad \vdots \\ \qquad x_1(t_k) = x(t_k)\}, & \qquad x_2(t_k) = x(t_k+\tau)\}, \\ \multicolumn{2}{c}{\dots\dots\dots\dots\dots\dots} \\ \multicolumn{2}{c}{x_m(t) = \{x_m(t_0) = x(t_0+(m-1)\tau),} \\ \multicolumn{2}{c}{\qquad x_m(t_1) = x(t_1+(m-1)\tau),} \\ \multicolumn{2}{c}{\qquad \vdots} \\ \multicolumn{2}{c}{\qquad x_m(t_k) = x(t_k+(m-1)\tau)\},} \end{array}\right. \tag{3.1}$$

defining the behavior of a real dynamical system. (Here $t_k + (m-1)\tau < t_N$.)

As experimental studies show, the processes presented in Fig.2.1 can be embedded in the phase space, the dimension of which is $4, 5$, or 6 [28]. Therefore, in the future, we will assume that $m = 5$ (see Fig.3.1).

In addition, at the next stage of modeling (see Section 4), a model will be built that depends on a small number of parameters and adequately describes the processes presented in Fig.2.1.

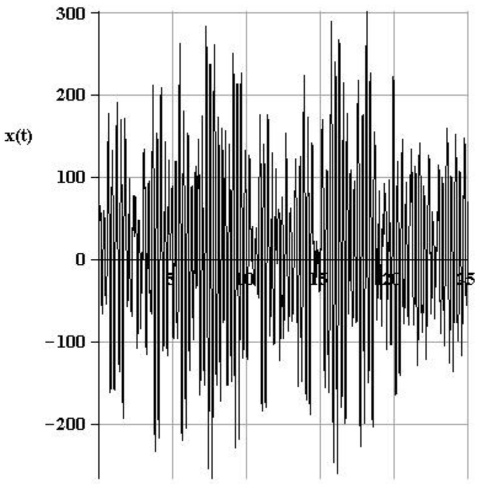
In the case $m = 5$, to simplify system (2.2), some of the parameters f_{ij}, e_{ij} in the denominators will be omitted. As a result, instead of $(m-1)^2 + 2m(m-1) + 2(m-1) + 1 = 16 + 40 + 8 + 1 = 65$ parameters, the newly obtained system will contain only 16+10+8+1=35 parameters $a_{ij}, f_i, e_i, b_i, c_i, \omega$. An example of such system is given below:

$$\begin{cases} \dot{x}(t) = \dfrac{0 \cdot x}{1 - 0.23\sin(x) + 0.85\cos(x)} + \dfrac{1 \cdot y}{1 + 0.72\sin(y) - 0.37\cos(y)} \\ \quad + \dfrac{-10.1 \cdot z}{1 - 0.67\sin(z) + 0.67\cos(z)} + \dfrac{0 \cdot u}{1 + 0.68\sin(u) - 0.68\cos(u)} \\ \quad + \dfrac{0 \cdot \sin(v) + 0 \cdot \cos(v)}{1 - 0.69\sin(v) + 0.70\cos(v)}, \\ \dot{y}(t) = \dfrac{-11 \cdot x}{1 - 0.23\sin(x) + 0.85\cos(x)} + \dfrac{-0.1 \cdot y}{1 + 0.72\sin(y) - 0.37\cos(y)} \\ \quad + \dfrac{0 \cdot z}{1 - 0.67\sin(z) + 0.67\cos(z)} + \dfrac{10.9 \cdot u}{1 + 0.68\sin(u) - 0.68\cos(u)} \\ \quad + \dfrac{0 \cdot \sin(v) + 0 \cdot \cos(v)}{1 - 0.69\sin(v) + 0.70\cos(v)}, \\ \dot{z}(t) = \dfrac{10 \cdot x}{1 - 0.23\sin(x) + 0.85\cos(x)} + \dfrac{0 \cdot y}{1 + 0.72\sin(y) - 0.37\cos(y)} \\ \quad + \dfrac{0 \cdot z}{1 - 0.67\sin(z) + 0.67\cos(z)} + \dfrac{1 \cdot u}{1 + 0.68\sin(u) - 0.68\cos(u)} \\ \quad + \dfrac{0 \cdot \sin(v) + 110 \cdot \cos(v)}{1 - 0.69\sin(v) + 0.70\cos(v)}, \\ \dot{u}(t) = \dfrac{0 \cdot x}{1 - 0.23\sin(x) + 0.85\cos(x)} + \dfrac{-20 \cdot y}{1 + 0.72\sin(y) - 0.37\cos(y)} \\ \quad + \dfrac{-110.4 \cdot z}{1 - 0.67\sin(z) + 0.67\cos(z)} + \dfrac{-0.1 \cdot u}{1 + 0.68\sin(u) - 0.68\cos(u)} \\ \quad + \dfrac{-100 \cdot \sin(v) - 110 \cdot \cos(v)}{1 - 0.69\sin(v) + 0.70\cos(v)}, \\ \dot{v}(t) = \omega. \end{cases} \tag{3.2}$$

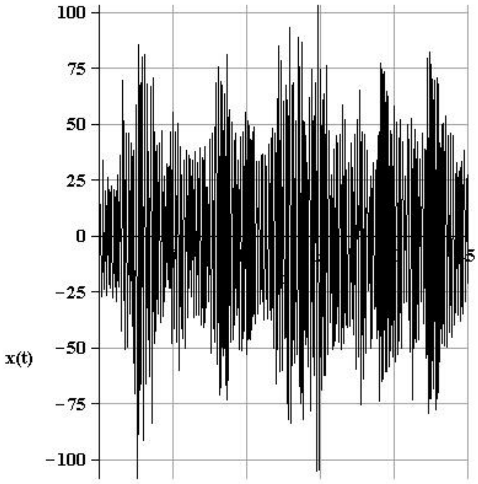
The following Fig.3.2 shows the application of system (3.2) (at certain values of the coefficients) for modeling the processes shown in Fig.2.1.

Thus, model (3.2), with the help of appropriate parameter settings, can correctly describe the dynamics of rhythms (see Fig.3.2) in the cerebral cortex, shown in Fig.2.1.

In the case of $m = 5$, system (3.2) can be transformed into system (2.3). For this, it is necessary:



(a1)



(a2)

Fig. 3.2. Simulation of the process shown in Fig.2.1 with the help of system (3.2): (a1) $\omega = 10.5$; (a2) $\omega = 12$.

1) In the first four equations of system (3.2), make substitutions $b_i \sin(v) + c_i \cos(v) \to a_{i5}v; i = 1, ..., 4$;

2) Replace the fifth equation of system system (3.2) with equation

$$
\begin{aligned}
\dot{v}(t) = &\frac{a_{51}x}{1 - f_1 \sin(x) - e_1 \cos(x)} + \frac{a_{52}y}{1 - f_2 \sin(y) - e_2 \cos(y)} \\
&+ \frac{a_{53}z}{1 - f_3 \sin(z) - e_3 \cos(z)} + \frac{a_{54}u}{1 - f_4 \sin(u) - e_4 \cos(u)} \\
&+ \frac{a_{55}v}{1 - f_5 \sin(v) - e_5 \cos(v)}.
\end{aligned}
$$

Note that the number of parameters in the newly obtained system at $m = 5$ will remain the same: 35.

4. Simplified identification of processes described by system (2.2)

In this section, we will begin to solve **Problem 3**. In the future, the number of variables in systems of equations, we will denote by n, where $n \geq m = 5$.

Note that the bounded variables $x_2(t), ..., x_n(t)$ derived from the measured variable $x_1(t) = x(t)$. Therefore, for models built using EEG, the equation $\dot{x}_n(t) = \omega$ must be replaced by the equation $\dot{x}_n(t) = a_{nn}x_n(t) + \phi(x_1(t), ..., x_{n-1}(t))$, where $a_{nn} < 0$. A possible form of such model can be as follows:

$$\begin{cases} \dot{x}_1(t) & = & \dfrac{a_{10} + \cdots + a_{1,n-1}x_{n-1} + b_1 \sin(x_n) + c_1 \cos(x_n)}{1 - f_1 \sin(x_n) - e_1 \cos(x_n)}, \\ & = & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x}_{n-1}(t) & = & \dfrac{a_{n-1,0} + \cdots + a_{n-1,n-1}x_{n-1} + b_{n-1} \sin(x_n) + c_{n-1} \cos(x_n)}{1 - f_{n-1} \sin(x_n) - e_{n-1} \cos(x_n)}, \\ \dot{x}_n(t) & = & \omega_0 + \omega_1 x_1 + \cdots + \omega_n x_n. \end{cases} \tag{4.1}$$

Here $a_{ij}, f_i, e_i, b_i, c_i, \omega_i$ are real parameters; $\sqrt{f_i^2 + e_i^2} < 1; i = 1, ..., n-1; j = 0, ..., n$. (Thus, system (4.1) depends on $(n-1)n + 4(n-1) + n + 1 = n^2 + 4n - 3$ parameters, and all of them are rationally included in this system.)

Note that by replacing

$$x_n(t) = \omega_0 t + \int_{t_0}^{t} (\omega_1 x_1(t) + \cdots + \omega_n x_n(t))\, dt \to \omega t,$$

system (4.1) can be reduced to system (2.6) . This means that under the conditions of Theorem 2.5 the solutions of system (4.1) will be bounded. (To prove Theorem 2.5 for system (4.1), it is necessary to slightly change item (c4) in its proof. In the presence of item (c3) in the same proof, such changes are quite obvious.)

Model (4.1) is still difficult to study. Therefore, in the future we will focus on the study of the following model:

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{1 - f_1 \sin(x_i) - e_1 \cos(x_i)} \displaystyle\sum_{j=0, j\neq i}^{n} a_{1j} x_j, \\ \cdot\ , \\ \dot{x}_i(t) = \dfrac{1}{1 - f_i \sin(x_i) - e_i \cos(x_i)} \displaystyle\sum_{j=0}^{n} a_{ij} x_j, \\ \cdot\ , \\ \dot{x}_n(t) = \dfrac{1}{1 - f_n \sin(x_i) - e_n \cos(x_i)} \displaystyle\sum_{j=0, j\neq i}^{n} a_{nj} x_j. \end{cases} \tag{4.2}$$

Here $f_i^2 + e_i^2 < 1; i \in \{1, ..., n\}$. (In total, n different models of type (4.2) can be designed in this way.)

Let us introduce the following matrix

$$A_i = \begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,i-1} & a_{i-1,i+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,i-1} & a_{i+1,i+1} & \cdots & a_{+1,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & a_{n,i+1} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}.$$

Theorem 4.1. *Suppose that for some $i \in \{1, ..., n\}$ the matrix A_i for the singular system (4.2) be Hurwitz. If $\forall i \in \{1, ..., n\}$, we have $\sqrt{f_i^2 + e_i^2} < 1$, then for any vector of initial conditions $\mathbf{x}_0$ the solutions*

$$x_1(\mathbf{x}_0)(t), ..., x_{i-1}(\mathbf{x}_0)(t), x_{i+1}(\mathbf{x}_0)(t), ..., x_n(\mathbf{x}_0)(t)$$

of system (4.2) is bounded at $t \to \infty$. If, in addition, $a_{ii} < 0$, then the solution $x_i(\mathbf{x}_0)(t)$ is also bounded.

Proof. Let the vector $\mathbf{b} = (b_1, ..., b_i, b_{i+1}, ..., b_n)^T$ be the solution of the linear equation $\mathbf{a} + A_i\mathbf{b} = 0$, where $\mathbf{a} = (a_{10}, ..., a_{i-1,0}, a_{i+1,0}, ..., a_{n0})^T$.

Without loss of generality, we can assume that $\mathbf{a} = 0$. Indeed, if this is not true, then with the help of the change of variables $y_j = x_j + b_j, j \neq i$, we will pass from system (4.2) to a new system in which condition $\mathbf{a} = 0$ is already satisfied.

Now it remains to apply Theorem 2.5 to system (4.2).

Let $\alpha_j(t) = 1 - f_j \sin(x_i(t)) - e_j \cos(x_i(t)); j = 1, ..., n$.

We have that the obtained solutions $x_1(t), ..., x_{i-1}(t), x_{i+1}(t), ..., x_n(t)$ of system (4.2) are bounded. In this case, in the i-th equation

$$\dot{x}_i(t) = \frac{a_{ii}x_i(t)}{1 - f_i \sin(x_i(t)) - e_i \cos(x_i(t))} + \phi(x_1(t), ..., x_n(t))$$

of system (4.2) the function $\phi(x_1(t), ..., x_n(t))$ is also bounded. Finally, by virtue of conditions $a_{ii} < 0$ and $\alpha_i(t) > 0$, we obtain that the solution $x_i(t)$ and all solutions $x_1(t), ..., x_n(t)$ of system (4.2) are bounded. □

Note that Theorem 4.1 admits the following obvious generalization.

Theorem 4.2. *If, under the conditions of Theorem 4.1, the matrix A_i is replaced by a matrix $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ such that A is Hurwitz, then the assertion of Theorem 4.1 remains valid.*

In this paper, Theorem 4.2 will not be required. It is presented in order to show how you can expand the modeling capabilities for time series (1.1).

4.1. Algorithm for constructing model (4.2) from known time series

Let us write the equations of system (4.2) in the following form

$$\dot{x}_i(t) = \frac{a_{i1}x_i + \cdots + a_{in}x_n}{1 - f_i \sin(x_i) - e_i \cos(x_i)} = \phi_i(x_1, ..., x_n); i = 1, ..., n. \tag{4.3}$$

Now we rewrite the equations of system (4.3) as follows

$$\begin{aligned} \dot{x}_i(t) &= a_{i1}x_i + \cdots + a_{in}x_n + \dot{x}_i f_i \sin(x_i) + \dot{x}_i e_i \cos(x_i) \\ &= \psi_i(x_1, ..., x_n); i = 1, ..., n. \end{aligned} \tag{4.4}$$

From the point of view of the theory of differential equations, systems (4.3) and (4.4) describe the same dynamics. However, from the point of view of approximation theory (determining the coefficients $a_{i1}, ..., a_{in}, f_i, e_i$ from the known values of the functions $x_i(t), i = 1, ..., n$), these are different problems for systems (4.3) and (4.4).

Indeed, in case of system (4.3) it is necessary to minimize by $a_{11}, ..., e_n$ the loss function $\sum_{i=1}^{n}|\dot{x}_i - \phi_i(x_1, ..., x_n, a_{11}, ..., e_n)|$, and in case of system (4.4) it is necessary to minimize by $a_{11}, ..., e_n$ the loss function $\sum_{i=1}^{n}|\dot{x}_i - \psi_i(x_1, ..., x_n, a_{11}, ..., e_n)|$, where the equations (4.3) are rational and the equations (4.4) are linear.

It is clear that in the case of system (4.4), the approximation problem will be simpler than in the case of system (4.3). That is why we chose system (4.4) for solving the approximation problem. (It should be remembered that the approximation results for system (4.4) may be worse than for system (4.3).)

To simplify the notation, we can assume that in model (4.2) $i = n$.

1. Based on the known time series $\mathbf{x}(t) = \{x_0, x_1, ..., x_N\}$, determine the dimension of the embedding space m and the delay time τ.

2. Based on the known m (here $m = 5$) and τ, construct five time series

$$\mathbf{x}(t) = \{x_0, x_1, x_2, ..., x_L\}, \mathbf{x}(t+\tau) = \{y_0, y_1, y_2, ..., y_L\},$$
$$\mathbf{x}(t+2\tau) = \{z_0, z_1, z_2, ..., z_L\}, \mathbf{x}(t+3\tau) = \{u_0, u_1, u_2, ..., u_L\},$$
$$\mathbf{x}(t+4\tau) = \{v_0, v_1, v_2, ..., v_L\}$$

   that are given on the same time interval $T_L \le t_0 + (m-1)\tau \le T$ in equally spaced $L \le N$ nodes: $0, \Delta t, ..., k\Delta t, ..., L\Delta t = T_L \le T$. Thus, $\Delta t = T_L/L$.

3. Fix a learning selections

$$x_0, x_1, ..., x_k; y_0, y_1, ..., y_k; z_0, z_1, ..., z_k; u_0, u_1, ..., u_k; v_0, v_1, ..., v_k,$$

   where $36 \le k \le L$.

4. Construct the columns of numerical derivatives D_x, D_y, D_z, D_u, D_v, where

$$D_x = \frac{1}{\Delta t}\begin{pmatrix} x_1 - x_0 \\ \vdots \\ x_k - x_{k-1} \end{pmatrix} \in \mathbb{R}^k, ..., D_v = \frac{1}{\Delta t}\begin{pmatrix} v_1 - v_0 \\ \vdots \\ v_k - v_{k-1} \end{pmatrix} \in \mathbb{R}^k.$$

5. Construct five Jacobi matrices J_x, J_y, J_z, J_u, J_v:

$$J_x = \begin{pmatrix} 1 & x_0 & y_0 & z_0 & u_0 & D_{x1}\sin(v_0) & D_{x1}\cos(v_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{k-1} & y_{k-1} & z_{k-1} & u_{k-1} & D_{xk}\sin(v_{k-1}) & D_{xk}\cos(v_{k-1}) \end{pmatrix} \in \mathbb{R}^{k\times 7},$$

....

$$J_u = \begin{pmatrix} 1 & x_0 & y_0 & z_0 & u_0 & D_{u1}\sin(v_0) & D_{u1}\cos(v_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{k-1} & y_{k-1} & z_{k-1} & u_{k-1} & D_{uk}\sin(v_{k-1}) & D_{uk}\cos(v_{k-1}) \end{pmatrix} \in \mathbb{R}^{k\times 7},$$

$$J_v = \left(\begin{array}{cccccc} 1 & x_0 & y_0 & z_0 & u_0 & v_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{k-1} & y_{k-1} & z_{k-1} & u_{k-1} & v_{k-1} \end{array} \right| \rightarrow$$

$$\left| \begin{array}{cc} D_{v1}\sin(v_0) & D_{v1}\cos(v_0) \\ \vdots & \vdots \\ D_{vk}\sin(v_{k-1}) & D_{vk}\cos(v_{k-1}) \end{array} \right) \in \mathbb{R}^{k\times 8}.$$

6. Introduce a vector of unknown parameters

$$\mathbf{p} = (\mathbf{p}_x, ..., \mathbf{p}_v)^T$$

$$= (\underbrace{a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, f_1, e_1}_{\mathbf{p}_x}, ..., \underbrace{a_{50}, a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, f_5, e_5}_{\mathbf{p}_v})^T \in \mathbb{R}^{36}$$

7. Fix the parameters $0 < \varepsilon \leq 1$ and $\lambda > 0$, and minimize the five loss functions (see [29]):

$$\|J_x\mathbf{p}_x - D_x\|_2^2 + \lambda\|\mathbf{p}_x\|_1 \text{ with restriction } \sqrt{f_1^2 + e_1^2} \leq 1 - \varepsilon,$$

$$..., \|J_v\mathbf{p}_v - D_v\|_2^2 + \lambda\|\mathbf{p}_v\|_1 \text{ with restriction } \sqrt{f_5^2 + e_5^2} \leq 1 - \varepsilon.$$

8. Using any search optimization method, calculate the vector $\mathbf{p} = \mathbf{p}^* = (\mathbf{p}_x^*, ..., \mathbf{p}_v^*)^T$ whose subvectors $\mathbf{p}_x^*, ..., \mathbf{p}_v^*$ minimize the introduced loss functions.

The results of the operation of this algorithm are presented by the following examples.

$$\begin{cases} \dot{x}(t) = \dfrac{-180.15 - 3.03x + 8.10y + 1.73z + 0.57u}{1 - 0.80\sin(v) - 0.56\cos(v)}, \\ \dot{y}(t) = \dfrac{-58.21 - 10.23x - 3.96y + 13.09z - 2.71u}{1 - 0.70\sin(v) - 0.66\cos(v)}, \\ \dot{z}(t) = \dfrac{22.01 + 0.05x - 12.17y + 1.94z + 9.61u}{1 - 0.90\sin(v) - 0.40\cos(v)}, \\ \dot{u}(t) = \dfrac{219.68 - 2.11x - 0.01y - 10.98z + 4.24u}{1 - 0.71\sin(v) - 0.63\cos(v)}, \\ \dot{v}(t) = \dfrac{199.01 + 3.27x - 4.23y - 1.31z - 7.52u}{1 - 0.40\sin(v) + 0.55\cos(v)}. \end{cases} \tag{4.5}$$

Here initial values are: $x_0 = 91.70, y_0 = -11.01, z_0 = 10.02, u_0 = -10.74, v_0 = 0$.

$$\begin{cases} \dot{x}(t) = \dfrac{138.74 - 7.44x + 15.11y - 0.09z + 2.16u}{1 + 0.80\sin(v) - 0.50\cos(v)}, \\ \dot{y}(t) = \dfrac{80.68 - 12.12x - 1.48y + 13.22z + 0.50u}{1 - 0.70\sin(v) - 0.65\cos(v)}, \\ \dot{z}(t) = \dfrac{-52.62 - 0.52x - 12.38y + 2.88z + 9.01u}{1 - 0.80\sin(v) + 0.40\cos(v)}, \\ \dot{u}(t) = \dfrac{-66.91 - 0.96x - 2.44y - 12.64z + 2.22u}{1 - 0.3\sin(v) + 0.88\cos(v)}, \\ \dot{v}(t) = \dfrac{10 + 2.92x - 3.79y - 1.16z - 10.79u}{1 + 0.30\sin(v) - 0.10\cos(v)}. \end{cases} \tag{4.6}$$

Here initial values are: $x_0 = 11.96, y_0 = -11.01, z_0 = 10.02, u_0 = -10.74, v_0 = 0$.

$$\begin{cases} \dot{x}(t) = \dfrac{145.70 - 7.50x + 15.11y - 0.10z + 2.13u}{1 + 0.80\sin(v) - 0.55\cos(v)}, \\ \dot{y}(t) = \dfrac{81.25 - 12.12x - 2.52y + 13.25z + 0.50u}{1 - 0.71\sin(v) + 0.65\cos(v)}, \\ \dot{z}(t) = \dfrac{-54.11 - 0.53x - 12.38y + 2.88z + 9.02u}{1 - 0.89\sin(v) + 0.42\cos(v)}, \\ \dot{u}(t) = \dfrac{232.55 - 2.34x + 0.34y - 10.71z + 2.97u}{1 - 0.33\sin(v) - 0.88\cos(v)}, \\ \dot{v}(t) = \dfrac{10.12 + 2.91x - 3.79y - 1.16z - 10.77u}{1 - 0.10\sin(v) - 0.30\cos(v)}. \end{cases} \tag{4.7}$$

Here initial values are: $x_0 = 18.70, y_0 = -10.00, z_0 = 85.01, u_0 = -15.00, v_0 = 0$.
(Examples (4.5)-(4.7) of modeling EEG for a sick patient.)

$$\begin{cases} \dot{x}(t) = \dfrac{-1.00 + 0.22x - 7.23y - 5.78z - 8.15u}{1 - 0.78\sin(v) - 0.56\cos(v)}, \\ \dot{y}(t) = \dfrac{-0.00 + 4.19x - 0.57y - 2.96z - 5.75u}{1 - 0.77\sin(v) + 0.56\cos(v)}, \\ \dot{z}(t) = \dfrac{1.22 + 2.73x + 3.26y - 0.14z - 2.44u}{1 - 0.29\sin(v) + 0.14\cos(v)}, \\ \dot{u}(t) = \dfrac{0.34 + 7.00x + 6.91y + 5.10z - 0.30u}{1 - 0.67\sin(v) - 0.43\cos(v)}, \\ \dot{v}(t) = \dfrac{17.32 - 4.82x + 0.56y + 0.87z + 3.58u}{1 + 0.87\sin(v) + 0.20\cos(v)}. \end{cases} \tag{4.8}$$

Here initial values are: $x_0 = 1.70, y_0 = 6.06, z_0 = 11.41, u_0 = -10.21, v_0 = 0$.

$$\begin{cases} \dot{x}(t) = \dfrac{9.61 - 0.74x + 2.74y + 2.51z - 3.73u}{1 + 0.78\sin(v) + 0.46\cos(v)}, \\ \dot{y}(t) = \dfrac{17.22 - 3.04x - 0.187y + 2.61z + 2.05u}{1 - 0.64\sin(v) + 0.36\cos(v)}, \\ \dot{z}(t) = \dfrac{-0.04 - 2.52x - 2.05y + 0.36z + 3.49u}{1 - 0.79\sin(v) + 0.14\cos(v)}, \\ \dot{u}(t) = \dfrac{-9.04 + 3.43x - 2.06y - 2.90z - 0.04u}{1 - 0.82\sin(v) - 0.43\cos(v)}, \\ \dot{v}(t) = \dfrac{-2.18 - 4.05x + 4.08y - 1.91z - 2.27u}{1 + 0.30\sin(v) - 0.40\cos(v)}. \end{cases} \tag{4.9}$$

Here initial values are: $x_0 = -21.70, y_0 = -1.26, z_0 = 10.07, u_0 = -10.21, v_0 = 0$.

$$\begin{cases} \dot{x}(t) = \dfrac{24.51 - 0.92x + 2.45y + 2.93z - 4.37u}{1 - 0.48\sin(v) - 0.56\cos(v)}, \\ \dot{y}(t) = \dfrac{31.81 - 2.80x - 1.35y + 2.50z + 2.06u}{1 - 0.47\sin(v) + 0.56\cos(v)}, \\ \dot{z}(t) = \dfrac{7.74 - 3.70x - 0.53y + 0.58z + 3.50u}{1 - 0.90\sin(v) + 0.40\cos(v)}, \\ \dot{u}(t) = \dfrac{-3.40 + 3.44x - 1.02y - 2.70z - 3.86u}{1 - 0.71\sin(v) - 0.63\cos(v)}, \\ \dot{v}(t) = \dfrac{-23.18 - 4.12x + 3.22y - 1.70z - 1.99u}{1 - 0.0\sin(v) - 0.0\cos(v)}. \end{cases} \tag{4.10}$$

Here initial values are: $x_0 = 40.02, y_0 = 33.43, z_0 = -11.21, u_0 = 7.62, v_0 = 0$. (Examples (4.8)-(4.10) of modeling the EEG for a healthy patient.)

It should be said that in all the above examples, the solution $v(t)$ is unbounded (it oscillates around the straight line $v = a_{50}t$). In this case, we get a contradiction with the time series $v_0, v_1, v_2, ...$, which is built from the bounded series $x_0, x_1, x_2, ...$, and therefore must also be bounded.

This contradiction can be removed in the following way. For example, let's replace the fifth equation of system (4.9) with the equation

$$\dot{v}(t) = \frac{-2.18 - 4.05x + 4.08y - 1.91z - 2.27u + a_{55}v}{1 + 0.30\sin(v) - 0.40\cos(v)}. \tag{4.11}$$

Consider the graphs of the trajectory $v(t)$ at $a_{55} = 0$ and $a_{55} \neq 0$:

Comparing graphs Fig.4.2(b2) and Fig.4.3(a4) , it can be seen that the behavior of the curves presented in these graphs is similar. We add that the behavior of the curves $x(t), y(t), z(t)$ and $u(t)$ obtained from the system (4.9) and the behavior of the same curves obtained from the system (4.9), taking into account (4.11), is also similar. As for quantitative differences, we note that the variable $v(t)$ is included in the first four equations of system (4.9) (and the same system, but with equation (4.11)) only in complex $\alpha_i(t) = 1 - f_i\sin(v(t)) - e_i\cos(v(t))$, where $0 < \alpha_i(t) < 2; i = 1, ..., 5$. The last restriction gives quantitative differences in the solutions of system (4.9) (and the same system, but with equation (4.11)).

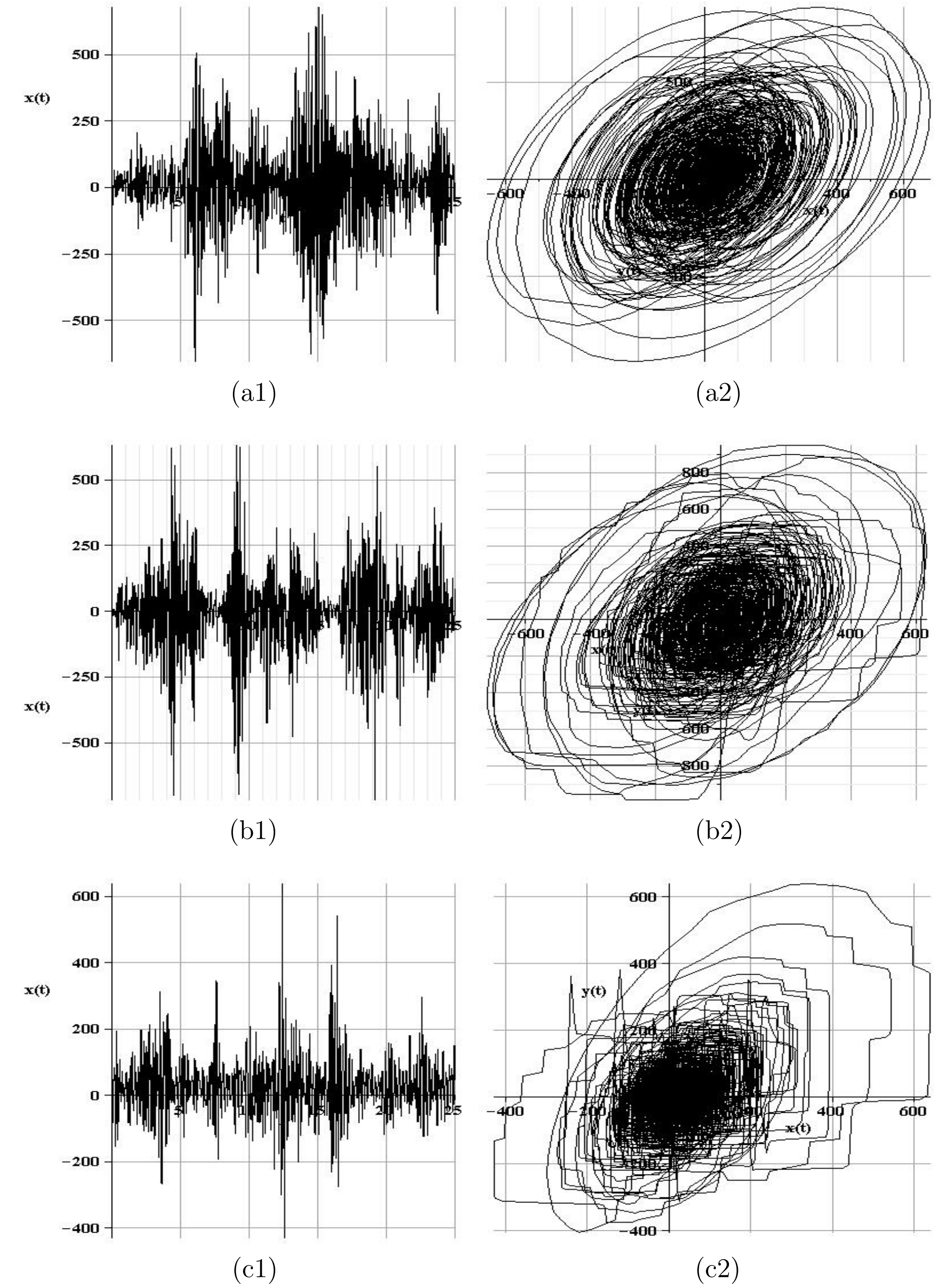


Fig. 4.1. The electroencephalogram taken from a specific point in the cerebral cortex of the patient with an epileptic disease: at points 1-500 of time series (1.1), (a1) in coordinates $(x(t), t)$ and (a2) in coordinates $(x(t), x(t+\tau))$; at points 501-1000 of time series (1.1), (b1) in coordinates $(x(t), t)$ and (b2) in coordinates $(x(t), x(t+\tau))$; at points 2001-2500 of time series (1.1), (c1) in coordinates $(x(t), t)$ and (c2) in coordinates $(x(t), x(t+\tau))$.

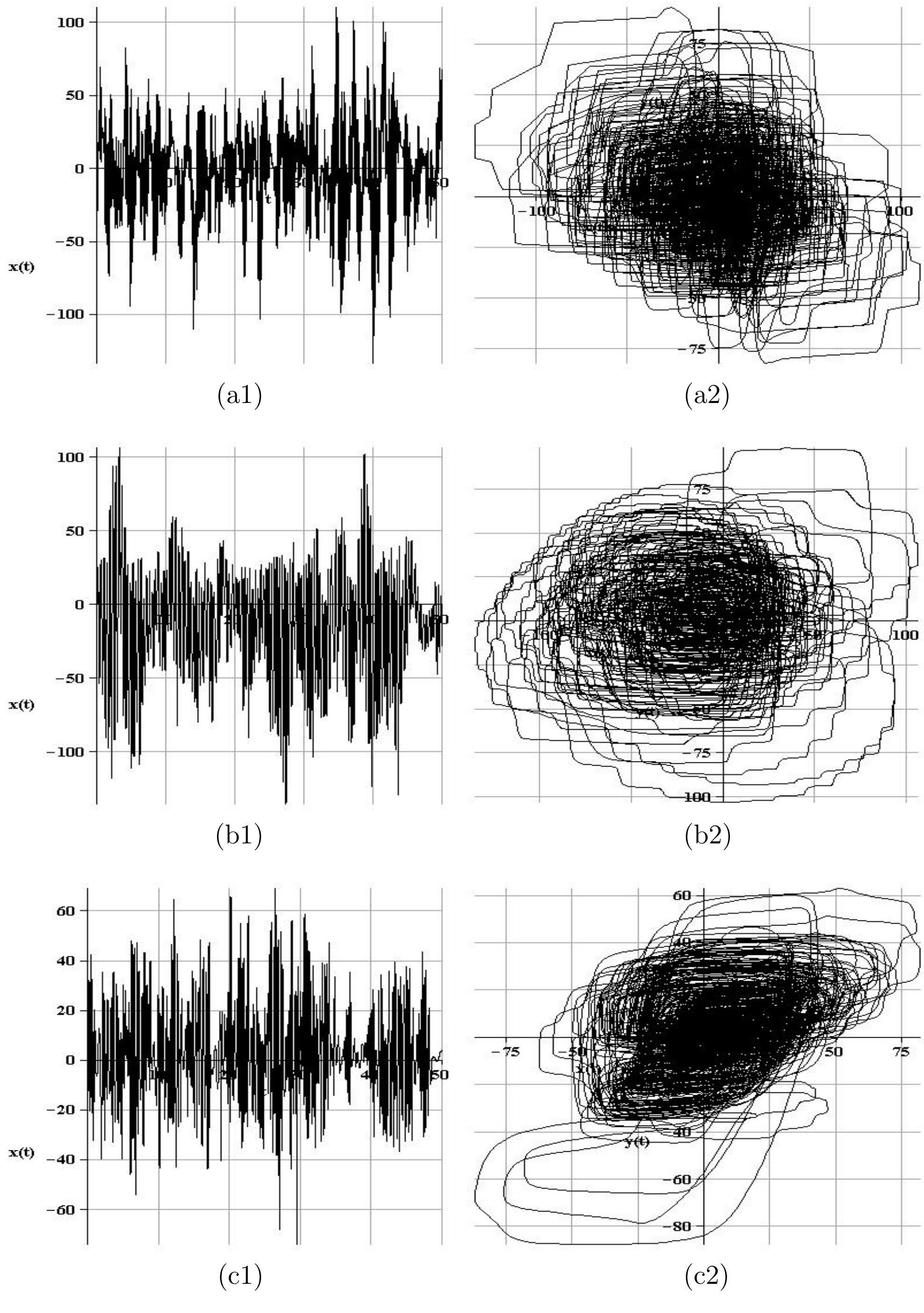


Fig. 4.2. The electroencephalogram taken from a specific point in the cerebral cortex of a healthy patient: at points 1-500 of time series (1.1), (a1) in coordinates $(x(t), t)$ and (a2) in coordinates $(x(t), x(t+\tau))$; at points 501-1000 of time series (1.1), (b1) in coordinates $(x(t), t)$ and (b2) in coordinates $(x(t), x(t+\tau))$; at points 1-4065 of time series (1.1), (c1) in coordinates $(x(t), t)$ and (c2) in coordinates $(x(t), x(t+\tau))$.

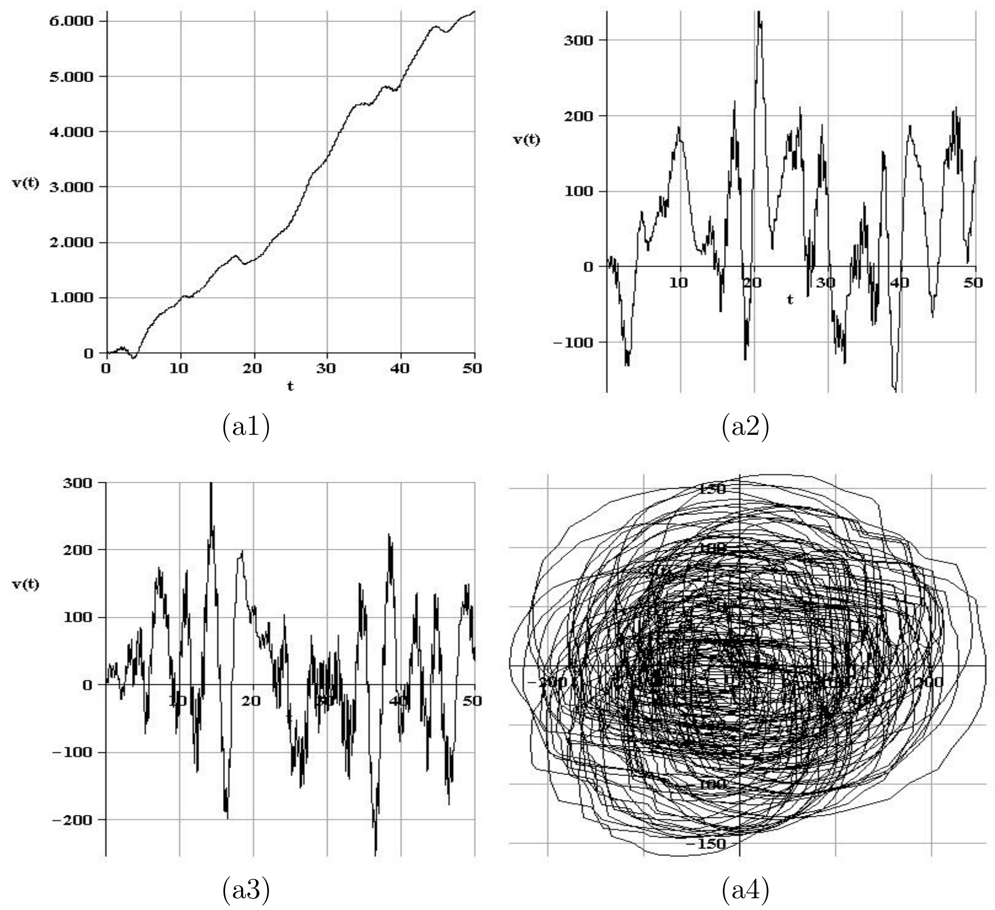


Fig. 4.3. Graphs of the variable $v(t)$ from equation (4.11): (a1) $a_{55} = 0$; (a2) $a_{55} = -1$; (a3) $a_{55} = -2$. Graph of the projection of the phase trajectory of system (4.9) with equation (4.11) onto plane (x, y) at $a_{55} = -2$ (a4).

To complete the simulation, it is necessary to analyze the Lyapunov exponents for the time series presented in Fig.2.1(a1,a2) [21].

Before starting the calculations of Lyapunov exponents, sectioning of each of the time series shown in Fig.2.1(a1) and Fig.2.1(a2) was carried out.

Each time series consists of 4065 points. This set of points was divided into 4 disjoint subsets (≈ 1000 points each). Thus, we get the following differences.

1. The Lyapunov exponents for a sick patient have greater modulo values than the same exponents for a healthy patient. (The number of positive Lyapunov exponents for a sick patient is 2. For a healthy patient, the same number varies from 2 to 3. Thus, both processes shown in Fig.2.1 are hyperchaotic.)

2. The dimension of the embedding space of a sick patient for each section is $m = 4$ or $m = 5$. At the same time, the dimension of the embedding space of a healthy patient for similar sections varies from $m = 4$ to $m = 6$.

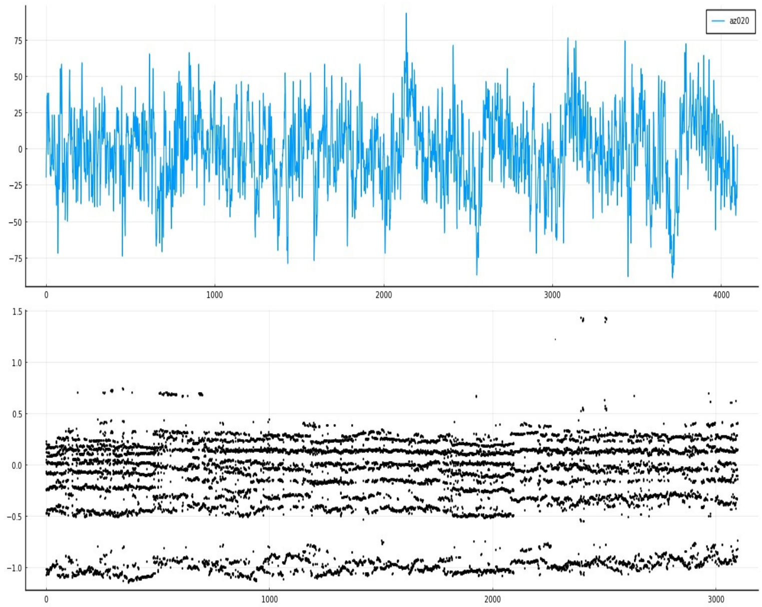


Fig. 4.4. The electroencephalogram Fig.2.1(a1) and the distribution of its Lyapunov exponents for time series (1.1)

3. Now let's compare the experimental Lyapunov exponents (Fig.4.4, Fig.4.5) and the Lyapunov exponents of dynamic systems (4.6) and (4.9) simulating the corresponding time series (see Fig.4.6).

Let $\Lambda_1 \geq ... \geq \Lambda_n$ are the Lyapunov exponents for a dynamical system in $\mathbb{R}^n$. Assume that j is the largest integer for which $\Lambda_1 + \cdots + \Lambda_j \geq 0$. The Kaplan-Yorke dimension is given by the formula [30, 31]:

$$d_{KL} = j + \frac{\Lambda_1 + \cdots + \Lambda_j}{|\Lambda_{j+1}|}. \tag{4.12}$$

In the equations (4.5)-(4.10), for coordinate $v(t)$, we have $v(t) \to \infty$ or $v(t) \to -\infty$ as $t \to \infty$ (see Fig.4.3). This means that systems (4.5)-(4.10) behave as nonstationary. Therefore, it is necessary to consider the attractors of these systems as projections along the axis v onto a 4-dimensional subspace $(x, y, z, u) \in \mathbb{R}^5$. In this case, the dimension of the attractor in this case will be less than 4.

Let's compare the fractal dimensions of the attractors presented in Fig.2.2 and Fig.4.1, Fig.4.2 calculated by formula (4.12).

Taking into account Fig.4.4, we have two situations: 1) $\Lambda_1 \approx 0.3, \Lambda_2 \approx 0.25, \Lambda_3 \approx 0.2, \Lambda_4 \approx 0.0, \Lambda_5 \approx -1.0$; from here it follows that $d_{KL} \approx 4.75$. 2) $\Lambda_1 \approx 0.25, \Lambda_2 \approx 0.15, \Lambda_3 \approx 0.0, \Lambda_4 \approx -0.25, \Lambda_5 \approx -0.5, \Lambda_6 \approx -1.0$; here we have $d_{KL} \approx 4.6$.

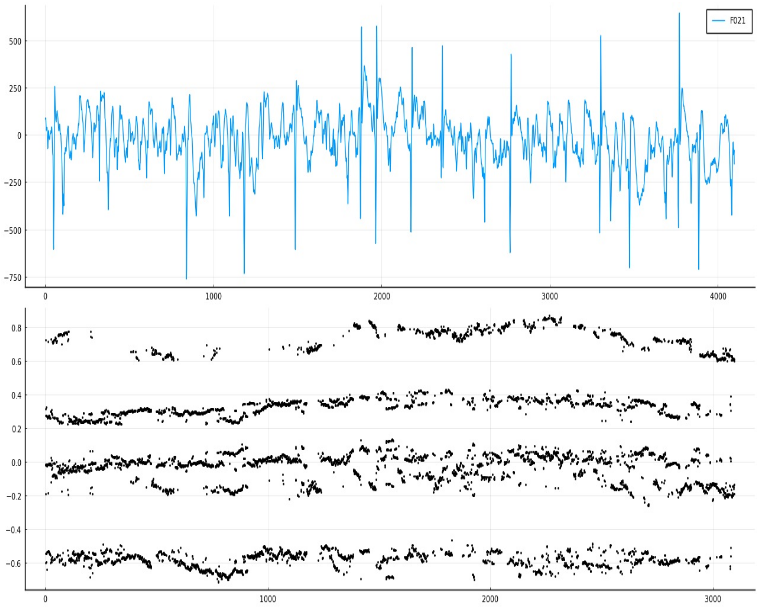


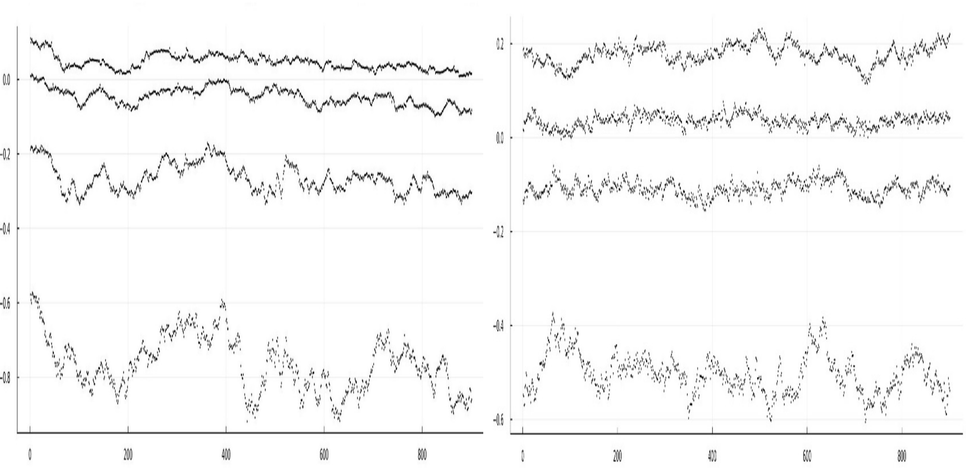
Fig. 4.5. The electroencephalogram Fig.2.1(a2) and the distribution of its Lyapunov exponents for time series (1.1)

Taking into account Fig.4.5, we have: 1) $\Lambda_1 \approx 0.6, \Lambda_2 \approx 0.03, \Lambda_3 \approx 0.0, \Lambda_4 \approx -0.7$; from here it follows that $d_{KL} \approx 3.9$ or 2) $\Lambda_1 \approx 0.6, \Lambda_2 \approx 0.2, \Lambda_3 \approx 0.0, \Lambda_4 \approx -0.2, \Lambda_5 \approx -0.7$; from here it follows that $d_{KL} \approx 7.0$.

Thus, the fractal dimension of the healthy patient attractor is greater than that of the sick one. This statement also holds for the attractors of model system (4.6) (for Fig.4.6(a1), we have $\Lambda_1 \approx 0.11$, $\Lambda_2 \approx -0.07, \Lambda_3 \approx -0.31, \Lambda_4 \approx -0.82$; here $d_{KL} \approx 2.8$) and model system (4.9) (for Fig.4.6 (a2), we have $\Lambda_1 \approx 0.19, \Lambda_2 \approx 0.03$, $\Lambda_3 = -0.13, \Lambda_4 = -0.51$; here $d_{KL} \approx 3.7$). The only question is: why is the dimension of the sick patient attractor greater than $m = 5$? The fact is that when calculating the Lyapunov exponents for time series, the noise component plays an important role. Its presence gives an overestimated value of the fractal dimension (especially for the attractor of a sick patient).

This suggests that the chaos generated by the signals of the cerebral cortex of a healthy patient has a more complex structure than the chaos generated by the signals of the cerebral cortex of a sick patient. The same statement can be confirmed by comparing Fig.4.1(a2),(b2),(c2) and Fig.4.2(a2),(b2),(c2).

In addition, we note that attractors are not explicitly represented in these figures, but can be constructed from trajectories (see [2]). In both cases, the normal activity is the internal trajectory (solid black area; see Fig.4.1 and Fig.4.2), and the seizure is the external trajectory (higher amplitude activity; sparse area formed by a single trajectory; see Fig.4.1).



(a1) (a2)

Fig. 4.6. Distribution of Lyapunov exponents for model (4.6) (a1) and model (4.9) (a2). Here $m = 4$ is less than the dimension $m = 5$ of the real embedding space (see Fig.4.5). This means that different models (see Fig.4.3) must be used for different measurement intervals.

5. On the existence of limit cycles in system (4.2)

It is known that chaotic processes in dynamical systems usually begin with a cascade of bifurcations of limit cycles (Feigenbaum's scenario of doubling the period [32]). Since the processes presented on any EEG are clearly chaotic (see the figures of this article), it is necessary to show that model (4.2) , at certain values of the coefficients, generates a limit cycle.

Consider the following simplest version of system (4.2) for $n = 2$:

$$\dot{x}_1(t) = \frac{a_{11}x_1 + a_{12}x_2}{1 - \varepsilon_1 \cos(x_1)}, \dot{x}_2(t) = \frac{a_{21}x_1 + a_{22}x_2}{1 - \varepsilon_2 \cos(x_2)}. \tag{5.1}$$

We introduce a matrix $S = \{s_{ij}\} \in \mathbb{R}^{2\times 2}$ such that either

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } S^{-1}AS = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \tag{5.2}$$

where $A = \{a_{ij}\} \in \mathbb{R}^{2\times 2}$ and $\det A \neq 0$.

Now we introduce a change of variables $x_1 = s_{11}y_1 + s_{12}y_2, x_2 = s_{21}y_1 + s_{22}y_2$ in system (5.1) and construct the following function

$$V = \frac{y_1^2 + y_2^2}{2}.$$

Then, taking into account (5.2), we get

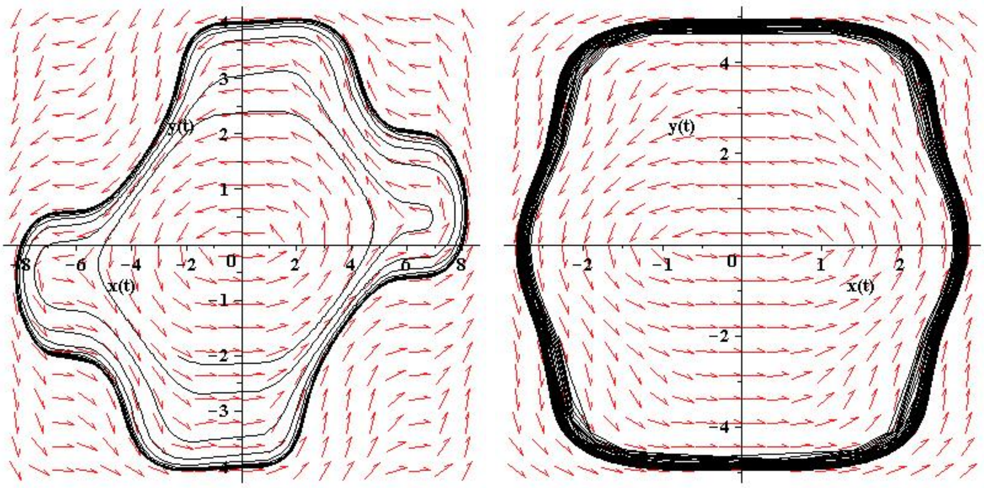
$$\dot{V}_t = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \frac{\phi_1(y_1, y_2)\cos(s_{11}y_1 + s_{12}y_2) + \phi_2(y_1, y_2)\cos(s_{21}y_1 + s_{22}y_2)}{|\varepsilon_1|^{-1} - \cos(s_{11}y_1 + s_{12}y_2)}$$

$$+\frac{\psi_1(y_1,y_2)\cos(s_{11}y_1+s_{12}y_2)+\psi_2(y_1,y_2)\cos(s_{21}y_1+s_{22}y_2)}{|\varepsilon_2|^{-1}-\cos(s_{21}y_1+s_{22}y_2)}.$$

(Here $\phi_i(y_1,y_2), \psi_i(y_1,y_2)$ are quadratic forms, $|\varepsilon_i|<1$; $i=1,2$. The situation $\lambda_1=\lambda_2=\lambda$ is not excluded.)

Let us introduce the set $\mathbb{H}=\{(y_1,y_2)^T\in\mathbb{R}^2|\dot{V}_t\geq 0\}$ and the boundary $\mathbb{L}=\{(y_1,y_2)^T\in\mathbb{R}^2|\dot{V}_t=0\}$ of this set. Now we use Theorem 2.5. Since the matrix A is Hurwitz, then $\lambda_1<0,\lambda_2<0$. Consider the behavior of the function $V(y_1,y_2)$ on the line $s_{11}y_1+s_{12}y_2=s_{21}y_1+s_{22}y_2=(2k+1)\pi/2$, where $k=0\pm 1,\pm 2,\ldots$. Obviously, if $k\to\infty$, then $s_{11}y_1+s_{12}y_2=s_{21}y_1+s_{22}y_2=(2k+1)\pi/2\to\infty$. In this case $\dot{V}_t(y_1,y_2)\to\lambda_1y_1^2+\lambda_2y_2^2<0$ and the nonnegative function $V(y_1,y_2)$ is decreasing along the mentioned straight line.

It is clear that there must be a moment $t_c>0$ such that $\dot{V}_t(y_1(t_c),y_2(t_c))=0$. From here it follows that the boundary $\mathbb{L}$ of the set $\mathbb{H}$ is closed and the set itself is a compact positively invariant set with respect to (5.1). Thus, all the conditions of Theorem 2.2 [25] are satisfied and the set $\mathbb{L}$ contains a stable limit cycle (see Fig.5.1). (If $(y_1,y_2)^T\in\mathbb{H}$, then $\dot{V}_t(y_1,y_2)\geq 0$ and the unique equilibrium point $(0,0)^T\in\mathbb{H}$ is a repeller. Therefore, the trajectory $(y_1(t),y_2(t))^T$ is attracted to some set $\mathbb{C}\subset\mathbb{L}$, which must be the limit cycle.)



(a1) (a2)

Fig. 5.1. Limit cycles of system (5.1) at the following parameter values: (a1) $a_{11}=0.52, a_{12}=-7.23, a_{21}=1.19, a_{22}=-0.57, \varepsilon_1=0.99, \varepsilon_2=-0.85$; (a2) $a_{11}=0.36, a_{12}=-10.21, a_{21}=15.22, a_{22}=-0.37, \varepsilon_1=0.97, \varepsilon_2=-0.87$. Here $x(t)=x_1(t), y(t)=x_2(t)$.

6. Conclusion

The paper presents new models (2.2), (2.3), (2.4), (4.1), and (4.2) describing strongly oscillating processes. As an application of one of these models (this is system (4.2)), the problem of modeling signals arising in the cerebral cortex, in particular, signals arising in epilepsy, was considered. It is shown that the constructed model distinguishes quite well the signals generated by the brain of a healthy patient and a patient with epilepsy.

However, the question of using model (4.2) for the diagnosis of epilepsy remains open. The fact is that the model (4.2) does not give accurate quantitative characteristics of epileptic seizures occurring in the cerebral cortex of a sick patient. In our opinion, this is due to the fact that rather crude computational tools are used to tune the model parameters (4.2): least squares method, methods LASSO, SIND [14, 29], and so on. Therefore, to improve the quality of modeling, it is necessary to use a more powerful tool. This tool is recurrent neural networks.

The use of neural networks will bring the quality of modeling to such a state in which it will be possible to take bifurcation analysis [9, 16] to study system (4.2). In this case, we will be able to connect the values of the coefficients of model (4.2) with the parameters of EEG, and hence with the real state of the sick patient.

At the moment, model (4.2) makes it possible to distinguish between healthy and sick patients only (without detailing their states): in a sick patient, the amplitude of signal oscillations is several times greater than in a healthy patient.

Why model (4.2) is presented in this form? There are three main reasons:

1. The electroencephalogram $x(t)$ shows that the electrical processes occurring in the cerebral cortex have a strongly oscillating, almost periodic nature. This means that there is a sequence of times $t_1, t_2, ...$ such that the modules of the derivatives $|\dot{x}(t_1)|, |\dot{x}(t_2)|, ...$ increase sharply. It is this fact that is taken into account in the proposed form of denominators in systems (2.2) and (4.2).

2. The calculated variables $x_2(t), x_3(t), ...$ are obtained from the experimental dependence $x_i(t) = x(t)$ using the delay method [6, 9] (here $i = 1$). This means that the jump moments of the derivatives $\dot{x}(t_1), \dot{x}(t_2)$, of the function $x_1(t)$ must be shifted for the functions $x_2(t) = x_1(t+\tau), x_3(t) = x_1(t+2\tau), ...$ by τ: $t_1 + \tau, t_1 + 2\tau, ..., t_2 + \tau, t_2 + 2\tau, ...$ That is why the terms in the denominators of the equations of system (4.2) are linear combinations of the terms of only one denominator $1 - f_1 \sin(x_1(t)) - e_1 \cos(x_1(t))$. (In system (4.2), any function $x_i(t)$ can be taken as an experimental variable; $i \in \{1, ..., n\}$. In examples (4.5)–(4.10) $i = 5$.)

3. In all denominators of equations (2.4) in the role of the function $\cos(t)$ any periodic function can be taken. Let us assume that the amplitude of oscillations of this function is A. Then in all equations (2.4) parameter ε_i must be replaced by parameter $\varepsilon_i/A; i = 1, ..., n$.

In conclusion, let us say a few words about future studies of epilepsy models.

First of all, we note one important circumstance. All real EEGs are usually very noisy. Therefore, before modeling, it will be necessary to filter the data obtained from these EEGs.

The standard filtering process is to cut large amplitudes (jumps). However, in our situation, such a process must be carried out very carefully: filtering can remove amplitudes that are critical for diagnosis. (In this case, the introduction of functions (2.1) would be unjustified.)

As previous studies have shown, it is impossible to build one model that would approximate the entire time series. Therefore, along with filtering, the question of sectioning the time series also arises. This sectioning should be done in such a way that only one model is used to model each section of the series.

Let i be any number from the set $\{1, ..., n\}$. In the present work, model

$$\begin{cases} \dot{x}_1(t) = \dfrac{a_{10} + a_{11}x_1 + \cdots + a_{1n}x_n}{1 + \varepsilon_1 \cdot \cos(x_i + \alpha_1)}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ \dot{x}_i(t) = \dfrac{a_{i0} + a_{i1}x_1 + \cdots + a_{in}x_n}{1 + \varepsilon_i \cdot \cos(x_i + \alpha_i)}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ \dot{x}_n(t) = \dfrac{a_{n0} + a_{n1}x_1 + \cdots + a_{nn}x_n}{1 + \varepsilon_n \cdot \cos(x_i + \alpha_n)} \end{cases} \tag{6.1}$$

was investigated. (There are $n^2 + 3n$ parameters $a_{10}, a_{1n}..., a_{nn}, \varepsilon_1, \alpha_1, ..., \varepsilon_n, \alpha_n$.)

The next step is to explore the universal model

$$\begin{cases} \dot{x}_1(t) = \dfrac{a_{10} + a_{11}x_1 + \cdots + a_{1n}x_n}{1 + \varepsilon_1 \cdot \cos(b_{11}x_1 + \cdots + b_{1n}x_n + \gamma_1)}, \\ \cdot\ , \\ \dot{x}_n(t) = \dfrac{a_{n0} + a_{n1}x_1 + \cdots + a_{nn}x_n}{1 + \varepsilon_n \cdot \cos(b_{n1}x_n + \cdots + b_{nn}x_n + \gamma_n)} \end{cases} \tag{6.2}$$

(There are $2n^2 + 3n$ parameters $a_{10}, a_{1n}..., a_{nn}, b_{11}, ..., b_{nn}, \varepsilon_1, \gamma_1, ..., \varepsilon_n, \gamma_n$.)

Finally, we note that it is possible to propose a model that generalizes models (6.1) and (6.2):

$$\begin{cases} \dot{x}_1(t) = \dfrac{a_{10} + a_{11}x_1 + \cdots + a_{1n}x_n}{1 + \varepsilon_1 \cdot \cos(h_1(x_1, ..., x_n))}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ \dot{x}_n(t) = \dfrac{a_{n0} + a_{n1}x_1 + \cdots + a_{nn}x_n}{1 + \varepsilon_n \cdot \cos(h_n(x_n, ..., x_n))}, \end{cases} \tag{6.3}$$

where $h_i(x_n, ..., x_n); i = 1, ..., n$, are continuous functions of their arguments.

In order to guarantee the boundedness of solutions of systems (6.1), (6.2), and (6.3) the following conditions:

1. The matrix $A = \{a_{ij}\}; i, j = 1, ..., n$, is Hurwitz;

2. The parameters $|\varepsilon_i| < 1; i = 1, ..., n$,

must be satisfied. (The proof of the last assertion almost completely repeats the proof of Theorem 2.5. Therefore, there is no need to give it again.)

The fulfillment of these conditions makes it possible to vary the parameters of systems (6.1), (6.2), and (6.3) within a very wide range, which guarantees the absence of unbounded solutions (a mandatory condition for any simulation).

We hope that the use of recurrent neural networks to adjust the coefficients of systems (6.1) and (6.2) will lead to more adequate models of epilepsy than the models discussed in this article.

References

1. V. Ye. Belozyorov, O. V. Pohorielov, V. N. Serdiuk, V. G. Zaytsev, *New approach to problem of diagnostics of cerebral cortex diseases using chaotic dynamics methods*, In: The collection of 7 th The international conference "Social Science and Humanity", September 23–29, London, **2**(2017), 7–28.
2. W. W. Lytton, *Computer modeling of epilepsy*, Nature Reviews Neuroscience, **9**(8)(2008), 626–637.
3. D. A. Torse, R. Khanai, V. V. Desai, *Classification of epileptic seizures using recurrence plots and machine learning techniques*, Proceedings of the 2019 IEEE International Conference on Communication and Signal Processing (ICCSP), April (2019), India, 1-6, https://ieeexplore.ieee.org/document/8697965.
4. D. A. Torse, R. Khanai, K. Pai, S. Iyer, *Hardware implementation of automated seizure detection system using EEG signals and edge computing*, Proceedings of the IEEE Sixth International Conference on Trends in Electronics and Informatics (ICOEI 2022), April (2022), India, https://ieeexplore.ieee.org/xpl/conhome /9776594/proceeding, 470–475.
5. V. Raju, V. Malsoru, K. Srinivas, B. K. Rani, G. Madhukar, *Nonlinear dynamical systems to study epileptic seizures and extract average amount of mutual information from encephalographs* , IP Indian Journal of Neurosciences, **8**(4)(2022), 254–259.
6. C. L. Webber, N. Marwan (eds), *Reccurence Quantification Analysis. Theory and Best Practice*, Springer, NY, Dordrecht, Heidelberg, London, 2015.
7. R. V. Donner, Y. Yong Zou, Y. F. Donges, N. Marwan, J. Kurths, *Recurrence networks - a novel paradigm for nonlinear time series analysis*, New Journal of Physics, **12**(2010), Article ID 033025, 40 pages.
8. J. -P. Eckmann, S. Kamphorst, D. Ruelle, *Recurrence plots of dynamical systems*, Europhysics Letters, **4**(1987), 973–977.
9. H. Kantz, T. Schreiber, *Nonlinear Time Series Analysis - 2nd Edition*, Cambridge University Press, Cambridge, 2004.
10. N. Marwan, M.Romano, M. Thiel, J. Kurths, *Recurrence plots for the analysis of complex systems*, Physics Reports, **438**(2007), 237–329.
11. N. Marwan, *How to avoid pitfalls in recurrence plot based data analysis*, International Journal of Bifurcation and Chaos, **21**(2011), 1003–1017.
12. A. P. Shultz, Y. Zou, N. Marwan, M. T. Turvey, *Local minima-based recurrence plots for continuous dynamical systems*, International Journal of Bifurcation and Chaos, **21**(2011), 1065–1075.
13. J. P. Zbilut, C. I. Webber, *Embeddings and delays as derived from quantification of recurrence plots*, Physics Letters A, **171**(1992), 199–203.
14. N. M. Mangan, S. L. Brunton, J. L. Proctor, J. N. Kutz, *Inferring biological networks by sparse identification of nonlinear dynamics*, IEEE Transactions on Molecular, Biological, and Multi-Scale Communications, **2**(1)(2016), 52–63.

15. C. K. Ye, *Recurrence analysis of time series*, *A project report submitted in partial fulfilment of the requirements for the award of Bachelor of Science (Hons.) Applied Mathematics With Computing*, Faculty of Engineering and Science Universiti Tunku Abdul Rahman, April (2019), Malaysia, http://eprints.utar.edu.my/3383/, 1–59.
16. Y. -C. Lai, N. Ye, *Recent developments in chaotic time series analysis*, *Int. J. Bifurcation and Chaos*,**13**(2003), 1383–1422.
17. M. Li, *Fractal time series – a tutorial review*, *Mathematical Problems in Engineering*, **2010**(2019), Article ID 157264, 26 pages.
18. Z. Liu, *Chaotic time serias analysis*, *Mathematical Problems in Engineering*, **2010**(2010), Article ID 720190, 31 pages.
19. F. K. Fuss, *A robust algorithm for optimization and customization of fractal dimensions of time series modified by nonlinearly scaling their time derivatives: mathematical theory and practical applications*, *Computational and Mathematical Methods in Medicine*, **2013**(2913), Article ID 178476, 19 pages.
20. Y. H. Li, X. J. Tian, X. Q. Li, *Nonlinear second-order model and equilibrium point characteristic analysis of DC traction power supply system*, *ISRN Applied Mathematics*, **2013**(2013), Article ID 981692, 8 pages.
21. V. Ye. Belozyorov, D. V. Dantsev, Y. V. Koshel, *On the equivalence of real dynamic process and its models*, *Journal of Optimization, Differential Equations and Their Applications (JODEA)*, **29**(2)(2021), 76–91.
22. V. Ye. Belozyorov, D. V. Dantsev, *Modeling of chaotic processes by means of antisymmetric neural ODEs*, *Journal of Optimization, Differential Equations and Their Applications (JODEA)*, **30**(1)(2022), 1–41.
23. V. Ye. Belozyorov, Ye. M. Kosariev, M. M. Pulin, V. G. Sychenko, V. G. Zaytsev. *A new mathematical model of dynamic process in direct current traction power supply system*, *Journal of Optimization, Differential Equations and Their Applications (JODEA)*, **27**(1) (2019), 21–55.
24. Toolboxes for Complex Systems, *Potsdam Institute for Climate Impact Research (PIK)*, Germany, http://tocsy.pik-potsdam.de.
25. V. Ye. Belozyorov, D. V. Dantsev, *Stability of neural ordinary differential equations with power nonlinearities*, *Journal of Optimization, Differential Equations and Their Applications (JODEA)*, **28**(2)(2020), 21–46.
26. B. P. Demidovich, *Lectures on the mathematical stability theory*, Nauka, Moscow, 1967 (in Russian).
27. H. K. Khalil, *Nonlinear Systems – 2nd Edition*, Prentice Hall, New-Jersy, 1996.
28. F. Takens, *Detecting strange attractors in turbulence*, *Lecture Notes in Mathematics* , Springer-Verlag, NY, Berlin, **898**(1981), 366–381.
29. Q. Qu, J. Sun, *Finding a sparse vector in a subspace: linear sparsity using alternating directions*, *arXiv preprint arXiv:1412.4659v3[cs.IT]*, (2016), 1–39.
30. S. P. Banks, *Analysis and structure of a new chaotic system*, *International Journal of Bifurcation and Chaos*, **9**(8) (1999), 1571–1583.
31. V. Ye. Belozyorov, S. A. Volkova, *Odd and even functions in the design problem of new chaotic attractors*, *International Journal of Bifurcation and Chaos*, **32**, (2022), Article ID 2250218, 26 pages.
32. V. Ye. Belozyorov, *Universal approach to the problem of emergence of chaos in autonomous dynamical systems*, *Nonlinear Dynamics*, **95**(1), (2019), 579–595.

Received 5.05.2023

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 53–73, DOI 10.15421/142303

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

OPTIMAL DISTRIBUTED GLOBALLY BOUNDED CONTROL FOR PARABOLIC – HYPERBOLIC EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS AND A LINEAR QUALITY CRITERION

Volodymyr O. Kapustyan [*], Nataliia V. Gorban [†],
Anna V. Sukretna [‡], Ivan D. Fartushnyi [§]

Abstract. For the problem of optimal control of a parabolic-hyperbolic process with nonlocal point boundary conditions, an explicit form of the solution is obtained in the form of formal series according to the system of eigenfunctions, which are generated by the spatial differential operator and boundary conditions. At the same time, the unequivocal solvability of the intermediate problems is established for each iteration. In addition, sufficient conditions for the convergence of the series are established, which determine the obtained formal solution of the optimal control problem, which justifies its correctness

Key words: Optimal control problem, parabolic-hyperbolic equations, Fourier coefficients, approximation.

2010 Mathematics Subject Classification: 49J20, 49K20, 58J37.

Communicated by Prof. O. Kiselyova

1. Introduction

It is known [1] that problems of optimal globally bounded distributed control for linear stationary parabolic or hyperbolic equations with local boundary conditions and a linear quality criterion have unique solutions in the form convergent series in a complete system of eigenfunctions generated by a spatial differential operator and boundary conditions. Problems of construction of approximate controls for of nonlinear parabolic equations were considered in [2, 3] and for wave equations were studied in [4]

In this paper, we substantiate the possibility of generalizing the above approach to non-self-adjoint optimal control problems related to parabolic-hyperbolic equations with non-local boundary conditions and a linear quality criterion. A

[*]Department of Economic Cybernetics, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", 12, Politechnical str., build. 17, Kyiv, 03056, Ukraine, v.kapustyan@kpi.ua

[†]Department of the Mathematical Methods of System Analysis, Institute of Applied System Analysis, National Academy of Sciences and Ministry of Education and Science of Ukraine, 37/35, Peremogy av., build. 35, IASA, 03056 Kyiv, Ukraine, nata_gorban@i.ua

[‡]Integral and Differential Equations Department, Taras Shevchenko National University of Kyiv, 64, Volodymyrska str., Kyiv, 01601, Ukraine, sukretna.a.v@knu.ua

[§]Department of Economic Cybernetics, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", 12, Politechnical str., build. 17, Kyiv, 03056, Ukraine, i.fartushny@kpi.ua

formal solution of the problem is constructed. The unique solvability of intermediate problems at each iteration is established. Sufficient conditions for the convergence of the series defining found a formal solution.

Some types of optimal control problems for such systems were considered in [5]. The properties of solutions such problems that had no analogues in the self-adjoint case were set there.

When implementing this generalization, a number of features should be borne in mind. First, under a fixed control, we consider the classical solution of the corresponding boundary value problem with absolutely continuous control. Secondly, as a control norm, which is globally bounded, we consider a special equivalent norm generated by some positive definite operator. Thirdly, at intermediate iterations, the Fourier coefficients of controls are determined at different time intervals.

2. Statement of the problem and its preliminary analysis

Let the controlled process be described by function $y(x,t)$, which satisfies the equation

$$L\,y(x,t) = u(x,t),\ \ (x,t) \in D, \tag{2.1}$$

initial condition

$$y(x,-\alpha) = \varphi(x) \tag{2.2}$$

and boundary conditions

$$y(0,t) = 0, \quad y'(0,t) = y'(1,t), \quad -\alpha \le t \le T, \tag{2.3}$$

where $D = \{(x,t):\ 0 < x < 1,\ -\alpha < t \le T,\ \ \alpha, T > 0\}$, control u and function φ will be assumed to be given, and their smoothness properties will be refined below,

$$L\,y \;=\; \begin{cases} y_t - y_{xx}, & t \;>\; 0, \\ y_{tt} - y_{xx}, & t \;<\; 0. \end{cases}$$

The formal solution of the problem (2.1) – (2.3) can be represented [6] as

$$y(x,t) = X_0(x)y_0(t) + \sum_{k=1}^{\infty}(X_{2k-1}(x)y_{2k-1}(t) + X_{2k}(x)y_{2k}(t)), \tag{2.4}$$

where the functions $y_i(t)$ are defined as solutions to the Cauchy problems

$$\begin{gathered} \frac{dy_0(t)}{dt} = u_0(t),\ \ t > 0, \\ \frac{d^2y_0(t)}{dt^2} = u_0(t),\ \ t < 0, \\ y_0(-\alpha) = \varphi_0; \end{gathered} \tag{2.5}$$

$$\frac{dy_{2k-1}(t)}{dt}+\lambda_k^2 y_{2k-1}(t)=u_{2k-1}(t),\;\; t>0$$

$$\frac{d^2y_{2k-1}(t)}{dt^2}+\lambda_k^2 y_{2k-1}(t)=u_{2k-1}(t),\;\; t<0, \tag{2.6}$$

$$y_{2k-1}(-\alpha)=\varphi_{2k-1},\;\; \lambda_k=2k\pi;$$

$$\frac{dy_{2k}(t)}{dt}+\lambda_k^2 y_{2k}(t)=-2\lambda_k y_{2k-1}(t)+u_{2k}(t),\;\; t>0,$$

$$\frac{d^2y_{2k}(t)}{dt^2}+\lambda_k^2 y_{2k}(t)=-2\lambda_k y_{2k-1}(t)+u_{2k}(t),\;\; t<0, \tag{2.7}$$

$$y_{2k}(-\alpha)=\varphi_{2k},\;\; k=1,2,...,$$

and here $y_i(t)=(y(\cdot,t),Y_i(\cdot))_{L_2(0,1)}\in C^1(-\alpha,T),\;\; u_i(t)=(u(\cdot,t),Y_i(\cdot))_{L_2(0,1)}$, $i\geq 0$; functions $X_i(x)$ and $Y_i(x)$ belong to the Riesz bases

$$W_0=\{X_0(x)=x,\;\; X_{2k-1}(x)=x\cos(2\pi kx),\;\; X_{2k}(x)=\sin(2\pi kx),\;\; k=1,\ldots\},$$

$$\begin{aligned} R_0 \;\; = \;\; & \{Y_0(x)=2,\; Y_{2k-1}(x)=4\cos(2\pi kx), \\ & Y_{2k}(x)=4(1-x)\sin(2\pi kx),\; k=1,\ldots\}; \end{aligned}$$

$$(X_i,Y_j)_{L_2(0,1)}\;=\delta_{ij}=\begin{cases} 1, & i\;=\;j, \\ 0, & i\;\neq\;j,\; i,j\;=\;0,1,...; \end{cases}$$

the sequence of numbers $\{\varphi_k\}$ is taken from the representation functions $\varphi(x)$ in the Riesz basis W_0.

For any function $\phi(x)\in L_2(0,1)$ fair assessment [7]

$$r||\phi||^2_{L_2(0,1)}\leq\sum_{k=0}^{\infty}\phi_k^2\leq R||\phi||^2_{L_2(0,1)}, \tag{2.8}$$

где $r=3/4,\;\; R=16,\;\; \phi_k=(\phi,Y_k)_{L_2(0,1)}$.

Moreover, in [8] it is proved that in the space $L_2(0,1)$ we can introduce an equivalent norm according to the rule

$$||\phi||^2_D=(D\phi,\phi)=\sum_{k=0}^{\infty}\phi_k^2, \tag{2.9}$$

where $D:L_2(0,1)\to L_2(0,1)$ – some positive definite operator.

In [6] the problem (2.1) – (2.3) was studied for $u(x,t)=0$ and in [9] for $u(x,t)\neq 0$ it is proved that series (2.4) is the unique solution to the problem (2.1) – (2.3) and $y(x,t)\in C^1(\bar{D})\cap C^2(D_-)\cap C^{2,1}(D_+)$ if α is a rational number, $\varphi\in C(0,1),\;\; u\in C(D)$ and the conditions

$$\sum_{k=1}^{\infty}\lambda_k^2\Big(|\varphi_{2k-1}|+|\varphi_{2k}|\Big)<\infty, \tag{2.10}$$

$$\sum_{k=1}^{\infty}\lambda_k\Big(||u_{2k-1}||_{C(-\alpha,T)}+||u_{2k}||_{C(-\alpha,T)}\Big)<\infty, \tag{2.11}$$

where $D_-=\{(x,t):\ 0<x<1,\ -\alpha<t\le 0\}$, $D_+=\{(x,t):\ 0<x<1,\ 0<t\le T\}$.

In the paper [9] for a fixed continuous control an integral representation for solution of problems (2.5) – (2.7) is obtained

$$y_0(t)=\Phi^0_{0,+}(t)\varphi_0+\int_{-\alpha}^{0}V^0_{0,+}(t,\tau)u_0(\tau)d\tau$$
$$+U^0_{0,+}(t)u_0(0)+\int_0^t\mathcal{U}^0_{0,+}(t,\tau)u_0(\tau)d\tau,\quad t\ >\ 0,$$

$$y_0(t)=\Phi^0_{0,-}(t)\varphi_0+\int_{-\alpha}^{0}V^0_{0,-}(t,\tau)u_0(\tau)d\tau$$
$$+U^0_{0,-}(t)u_0(0)+\int_{-\alpha}^t\mathcal{V}^0_{0,-}(t,\tau)u_0(\tau)d\tau,\quad t\ <\ 0, \tag{2.12}$$

where

$$\Phi^0_{0,+}(t)=1,\quad V^0_{0,+}(t,\tau)=-(\alpha+\tau),\quad U^0_{0,+}(t)=\alpha,\qquad \mathcal{U}^0_{0,+}(t,\tau)=1,$$
$$\Phi^0_{0,-}(t)=1,\quad V^0_{0,-}(t,\tau)=-(\alpha+t),\quad U^0_{0,-}(t)=\alpha+t,\quad \mathcal{V}^0_{0,-}(t,\tau)=t-\tau;$$

$$y_{2k-1}(t)=\Phi^{2k-1}_{2k-1,+}(t)\varphi_{2k-1}+\int_{-\alpha}^{0}V^{2k-1}_{2k-1,+}(t,\tau)v_{2k-1}(\tau)d\tau$$
$$+U^{2k-1}_{2k-1,+}(t)u_{2k-1}(0)+\int_0^t\mathcal{U}^{2k-1}_{2k-1,+}(t,\tau)u_{2k-1}(\tau)d\tau,\quad t>0;$$

$$y_{2k-1}(t)=\Phi^{2k-1}_{2k-1,-}(t)\varphi_{2k-1}+\int_{-\alpha}^{0}V^{2k-1}_{2k-1,-}(t,\tau)u_{2k-1}(\tau)d\tau$$
$$+U^{2k-1}_{2k-1,-}(t)u_{2k-1}(0)+\int_{-\alpha}^t\mathcal{V}^{2k-1}_{2k-1,-}(t,\tau)u_{2k-1}(\tau)d\tau,\quad t<0, \tag{2.13}$$

here the following notations are used

$$\Phi^{2k-1}_{2k-1,+}(t)=\frac{\exp(-\lambda_k^2t)}{\delta_k(\alpha)},\qquad V^{2k-1}_{2k-1,+}(t,\tau)=-\frac{\exp(-\lambda_k^2t)}{\delta_k(\alpha)\lambda_k}\sin\lambda_k(\alpha+\tau),$$
$$U^{2k-1}_{2k-1,+}(t)=\frac{\exp(-\lambda_k^2t)\sin(\lambda_k\alpha)}{\delta_k(\alpha)\lambda_k},\quad \mathcal{U}^{2k-1}_{2k-1,+}(t,\tau)=\exp(-\lambda_k^2(t-\tau)),$$
$$\Phi^{2k-1}_{2k-1,-}(t)=\frac{\delta_k(|t|)}{\delta_k(\alpha)},\qquad V^{2k-1}_{2k-1,-}(t,\tau)=-\frac{\sin\lambda_k(t+\alpha)}{\lambda_k\delta_k(\alpha)}\delta_k(|\tau|),$$
$$U^{2k-1}_{2k-1,-}(t)=\frac{\sin\lambda_k(t+\alpha)}{\lambda_k\delta_k(\alpha)},\qquad \mathcal{V}^{2k-1}_{2k-1,-}(t,\tau)=\frac{1}{\lambda_k}\sin\lambda_k(t-\tau),$$

$$
\begin{aligned}
y_{2k}(t) =& \Phi^{2k}_{2k-1,+}(t)\varphi_{2k-1} + \Phi^{2k}_{2k,+}(t)\varphi_{2k} \\
&+ \int_{-\alpha}^{0} \Big(V^{2k}_{2k-1,+}(t,\tau)u_{2k-1}(\tau) + V^{2k}_{2k,+}(t,\tau)u_{2k}(\tau)\Big)\,d\tau \\
&+ U^{2k}_{2k-1,+}(t)u_{2k-1}(0) + U^{2k}_{2k,+}(t)u_{2k}(0) \\
&+ \int_{0}^{t} \Big(\mathcal{U}^{2k}_{2k-1,+}(t,\tau)u_{2k-1}(\tau) + \mathcal{U}^{2k}_{2k,+}(t,\tau)u_{2k}(\tau)\Big)\,d\tau, \quad t > 0,
\end{aligned}
$$

$$
\begin{aligned}
y_{2k}(t) =& \Phi^{2k}_{2k-1,-}(t)\varphi_{2k-1} + \Phi^{2k}_{2k,-}(t)\varphi_{2k} \\
&+ \int_{-\alpha}^{0} \Big(V^{2k}_{2k-1,-}(t,\tau)u_{2k-1}(\tau) + V^{2k}_{2k,-}(t,\tau)u_{2k}(\tau)\Big)\,d\tau \\
&+ U^{2k}_{2k-1,-}(t)u_{2k-1}(0) + U^{2k}_{2k,-}(t)u_{2k}(0) \\
&+ \int_{-\alpha}^{t} \Big(\mathcal{V}^{2k}_{2k-1,-}(t,\tau)u_{2k-1}(\tau) + \mathcal{V}^{2k}_{2k,-}(t,\tau)u_{2k}(\tau)\Big)\,d\tau, \quad t < 0; \qquad (2.14)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi^{2k}_{2k-1,+}(t) =& \frac{\exp(-\lambda_k^2 t)}{\delta_k^2(\alpha)}\Big((\alpha-1)\sin\lambda_k\alpha - \alpha\lambda_k\cos\lambda_k\alpha - 2\lambda_k t\delta_k(\alpha)\Big), \\
\Phi^{2k}_{2k,+}(t) =& \frac{\exp(-\lambda_k^2 t)}{\delta_k(\alpha)}, \\
V^{2k}_{2k-1,+}(t,\tau) =& \frac{\exp(-\lambda_k^2 t)}{\lambda_k\delta_k(\alpha)}\Bigg(\frac{\cos\lambda_k\alpha\sin\lambda_k\tau}{\lambda_k} - \tau\cos\lambda_k(\alpha+\tau) \\
&- \frac{\delta_k(|\tau|)}{\delta_k(\alpha)}\Big(\alpha - \frac{\sin 2\lambda_k\alpha}{2\lambda_k}\Big) + 2\sin\lambda_k(\alpha+\tau)\Big(\frac{\sin\lambda_k\alpha}{\delta_k(\alpha)} + \lambda_k t\Big)\Bigg), \\
V^{2k}_{2k,+}(t,\tau) =& -\frac{\exp(-\lambda_k^2 t)}{\lambda_k\delta_k(\alpha)}\sin\lambda_k(\tau+\alpha), \\
U^{2k}_{2k-1,+}(t) =& \frac{\exp(-\lambda_k^2 t)}{\lambda_k\delta_k^2(\alpha)}\Bigg(-2\sin^2\lambda_k\alpha + \alpha - \frac{\sin 2\lambda_k\alpha}{2\lambda_k} - 2t\lambda_k\sin\lambda_k\alpha\delta_k(\alpha)\Bigg), \\
U^{2k}_{2k,+}(t) =& \frac{\exp(-\lambda_k^2 t)}{\delta_k(\alpha)}\frac{\sin\lambda_k\alpha}{\lambda_k}, \\
\mathcal{U}^{2k}_{2k-1,+}(t,\tau) =& -2\lambda_k(t-\tau)\exp(-\lambda_k^2(t-\tau)), \\
\mathcal{U}^{2k}_{2k,+}(t,\tau) =& \exp(-\lambda_k^2(t-\tau)), \\
\Phi^{2k}_{2k-1,-}(t) =& \frac{1}{\delta_k^2(\alpha)}\Big(-2\sin\lambda_k(t+\alpha) + \delta_k(|t|)(\sin\lambda_k\alpha - \lambda_k\alpha\cos\lambda_k\alpha) \\
&+ \delta_k(\alpha)(\sin\lambda_k t - \lambda_k t\cos\lambda_k t) + \alpha\delta_k(|t|)\sin\lambda_k\alpha - t\delta_k(\alpha)\sin\lambda_k t\Big),
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{2k}_{2k,-}(t) &= \frac{\delta_k(|t|)}{\delta_k(\alpha)},\\
V^{2k}_{2k-1,-}(t,\tau) &= -\ \frac{\sin\lambda_k(t+\alpha)}{\delta_k(\alpha)}\left(\frac{(1-\tau)\sin\lambda_k\tau}{\lambda_k} - \tau\cos\lambda_k\tau - \frac{2\sin\lambda_k(\alpha+\tau)}{\lambda_k\delta_k(\alpha)}\right)\\
&+\ \frac{\cos\lambda_k\alpha\ \delta_k(|\tau|)}{\lambda_k}\left(\frac{\delta_k(|t|)}{\delta_k^2(\alpha)}\Big(\frac{\sin\lambda_k\alpha}{\lambda_k} - \alpha\cos\lambda_k\alpha\Big) + \frac{1}{\delta_k^2(\alpha)}\Big(\frac{\sin\lambda_k t}{\lambda_k}\right.\\
&\left.- t\cos\lambda_k t\Big)\right) -\ \frac{\sin\lambda_k\alpha\delta_k(|\tau|)}{\lambda_k\delta_k^2(\alpha)}\Big(\alpha\delta_k(|t|)\sin\lambda_k\alpha - t\delta_k(\alpha)\sin\lambda_k t\Big),\\
V^{2k}_{2k,-}(t,\tau) &= -\ \frac{\sin\lambda_k(t+\alpha)\delta_k(|\tau|)}{\lambda_k\delta_k(\alpha)},\\
U^{2k}_{2k-1,-}(t) &= -\ \frac{2\sin\lambda_k(t+\alpha)\sin\lambda_k\alpha}{\lambda_k\delta_k^2(\alpha)} -\ \frac{\cos\lambda_k\alpha}{\lambda_k}\left(\frac{\delta_k(|t|)}{\delta_k^2(\alpha)}\Big(\frac{\sin\lambda_k\alpha}{\lambda_k}\right.\\
&\left.- \alpha\cos\lambda_k\alpha\Big) + \frac{1}{\delta_k(\alpha)}\Big(\frac{\sin\lambda_k t}{\lambda_k} - t\cos\lambda_k t\Big)\right)\\
&+\ \frac{\sin\lambda_k\alpha}{\lambda_k\delta_k^2(\alpha)}\Big(\alpha\delta_k(|t|)\sin\lambda_k\alpha - t\delta_k(\alpha)\sin\lambda_k t\Big),\\
U^{2k}_{2k,-}(t) &= \frac{\sin\lambda_k(t+\alpha)}{\lambda_k\delta_k(\alpha)},\\
\mathcal{V}^{2k}_{2k-1,-}(t,\tau) &= \frac{1}{\lambda_k}\left((t-\tau)\cos\lambda_k(t-\tau) - \frac{\sin\lambda_k(t-\tau)}{\lambda_k}\right),\\
\mathcal{V}^{2k}_{2k,-}(t,\tau) &= \frac{\sin\lambda_k(t-\tau)}{\lambda_k}\ .
\end{aligned}
$$

The above integral representation of the solution to the problem (2.5) – (2.7) contains a function

$$\delta_k(\alpha) = \cos\lambda_k\alpha + \lambda_k\sin\lambda_k\alpha \tag{2.15}$$

in the denominator. For sufficiently large k and rational positive α, it satisfies the estimates

$$
\begin{aligned}
|\delta_k(\alpha)| &= \sqrt{1+\lambda_k^2}\ \Big|\sin(\lambda_k\alpha+\gamma_k)\Big| \le \sqrt{1+\lambda_k^2},\\
|\delta_k(\alpha)| &= \sqrt{1+\lambda_k^2}\ \Big|\sin(\lambda_k\alpha+\gamma_k)\Big| \ge \sqrt{1+\lambda_k^2}\sin\frac{\pi}{2q}\ ,
\end{aligned}
\tag{2.16}
$$

here $\gamma_k = \arcsin\left(\sqrt{1+\lambda_k^2}\right)^{-1}$, $\alpha = p/q$, $2kp = sq+r$, p, q are positive integers, r is a non-negative integer not exceeding $q-1$, s is a non-negative integer.

Consider the following optimal control problem: we need to find the function $u(x,t)$, which returns an extreme value to the quality criterion

$$J(u) = \int_{-\alpha}^{T}\int_0^1 q(x,t)y(x,t)dxdt, \tag{2.17}$$

when restrictions are met

$$\sum_{i=0}^{\infty}\left(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^{T} v_i^2(t)dt\right)\le \nu^2, \tag{2.18}$$

$$y(x,T)=\psi(x), \tag{2.19}$$

where $q(x,t)\in L_2(D)$, $\psi(x)\in L_2(0,1)$ are fixed functions whose properties will be specified below.

$$u_i(t)=\hat{u}_i(-\alpha)+\int_{-\alpha}^{t} v_i(\tau)d\tau, \quad t\ge -\alpha,\ i=0,1,\dots. \tag{2.20}$$

If the condition (2.19) is satisfied, then the solution to the problem (2.17) – (2.20) exists and is reached on boundary of the domain (2.18): the linear functional reaches extreme values in control when the last is located on the boundary of a closed convex region.

3. Formal solution of an extremal problem

The functions $q(x,t)$, $\psi(x)$ have series expansions

$$q(x,t)=Y_0(x)q_0(t)+\sum_{k=1}^{\infty}\Big(Y_{2k-1}(x)q_{2k-1}(t)+Y_{2k}(x)q_{2k}(t)\Big), \tag{3.1}$$

$$\psi(x)=X_0(x)\psi_0+\sum_{k=1}^{\infty}\Big(X_{2k-1}(x)\psi_{2k-1}+X_{2k}(x)\psi_{2k}\Big), \tag{3.2}$$

where $q_i(t)=\big(q(\cdot,t),X_i(\cdot)\big)_{L_2(0,1)}$, $\psi_i=\big(\psi(\cdot),Y_i(\cdot)\big)_{L_2(0,1)}$, $i=0,1,\dots$.

Then the problem (2.17) – (2.19) becomes

$$J(u)=\sum_{i=0}^{\infty}\left(\int_{-\alpha}^{T} q_i(t)y_i(t)dt\right)\to\ extr, \tag{3.3}$$

if

$$\sum_{i=0}^{\infty}\left(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^{T} v_i^2(t)dt\right)= \nu^2, \tag{3.4}$$

$$y_i(T)=\psi_i(x),\ \ i=0,1,\dots. \tag{3.5}$$

The Lagrange functional for the problem (3.3) – (3.5) will have the form

$$\begin{aligned} L=&\sum_{i=0}^{\infty}\left(\int_{-\alpha}^{T} q_i(t)y_i(t)dt\right)+\sum_{i=0}^{\infty}\mu_i\big(y_i(T)-\psi_i(x)\big)\\ &+\ \frac{m}{2}\left(\nu^2-\sum_{i=0}^{\infty}\left(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^{T} v_i^2(t)dt\right)\right), \end{aligned} \tag{3.6}$$

where $m,\ \mu_i,\ i=0,1,\dots$ are Lagrange multipliers.

The Lagrange functional (3.6) can be represented as

$$L = L_0 + \sum_{k=1}^{\infty} L_{2k-1,2k} + \frac{m}{2}\,\nu^2, \tag{3.7}$$

where

$$L_0 = \int_{-\alpha}^{T} q_0(t)y_0(t)dt + \mu_0\big(y_0(T) - \psi_0(x)\big) - \frac{m}{2}\left(\hat{u}_0^2(-\alpha) + \int_{-\alpha}^{T} v_0^2(t)dt\right), \tag{3.8}$$

$$\begin{aligned} L_{2k-1,2k} = \sum_{j=2k-1}^{2k} \Bigg(\int_{-\alpha}^{T} & q_j(t)y_j(t) + \mu_j(y_j(T) - \psi_j(x)) \\ & - \frac{m}{2}\left(\hat{u}_j^2(-\alpha) + \int_{-\alpha}^{T} v_j^2(t)dt\right)\Bigg). \end{aligned} \tag{3.9}$$

Let us obtain optimality conditions for the above problems, taking into account integral representations (2.12) – (2.14), (2.20). For the problem (3.8) we get

$$\frac{\partial L_0}{\partial \hat{u}_0(-\alpha)} = a_0^0 + \mu_0 b_0^0 - m\hat{u}_0(-\alpha) = 0,$$

$$0 = \frac{\partial L_0}{\partial v_0}\bigg|(t) = \begin{cases} a_{0,-}^0(t) + \mu_0 b_{0,-}^0(t) - m v_{0,-}(t), & t \in (-\alpha, 0), \\ a_{0,+}^0(t) + \mu_0 b_{0,+}^0(t) - m v_{0,+}(t), & t \in [0, T], \end{cases} \tag{3.10}$$

where

$$\begin{aligned} a_0^0 = & \int_{-\alpha}^{0} \left(q_0(t) \left(\int_{-\alpha}^{0} V_{0,-}^0(t,\tau)d\tau + U_{0,-}^0(t) \right) + \int_t^0 q_0(\tau)\mathcal{V}_{0,-}^0(\tau,t)d\tau \right) dt \\ & + \int_0^T \left(q_0(t) \left(\int_{-\alpha}^{0} V_{0,+}^0(t,\tau)d\tau + U_{0,+}^0(t) \right) + \int_t^T q_0(\tau)\mathcal{U}_{0,+}^0(\tau,t)d\tau \right) dt, \\ b_0^0 = & \int_{-\alpha}^{0} V_{0,+}^0(T,\tau)d\tau + U_{0,+}^0(T) + \int_0^T \mathcal{U}_{0,+}^0(T,\tau)d\tau, \\ a_{0,-}^0(t) = & \int_t^0 \left(\int_{-\alpha}^{0} q_0(\xi)V_{0,-}^0(\xi,\tau)d\xi \right) d\tau + \int_{-\alpha}^{0} q_0(\xi)U_{0,-}^0(\xi)d\xi \\ & + \int_t^0 q_0(\xi) \left(\int_t^{\xi} \mathcal{V}_{0,-}^0(\xi,\tau)d\tau \right) d\xi + \int_t^0 \left(\int_0^T q_0(\xi)V_{0,+}^0(\xi,\tau)d\xi \right) d\tau \\ & + \int_0^T q_0(\xi)U_{0,+}^0(\xi)d\xi + \int_0^T \int_{\tau}^{T} q_0(\xi)\mathcal{U}_{0,+}^0(\xi,\tau)d\xi d\tau, \\ b_{0,-}^0(t) = & \int_t^0 V_{0,+}^0(T,\xi)d\xi + U_{0,+}^0(T) + \int_0^T \mathcal{U}_{0,+}^0(T,\tau)d\tau, \end{aligned}$$

$$a^0_{0,+}(t) = \int_t^T q_0(\xi)\left(\int_t^\xi \mathcal{U}^0_{0,+}(\xi,\tau)d\tau\right)d\xi, \qquad b^0_{0,+}(t) = \int_t^T \mathcal{U}^0_{0,+}(T,\tau)d\tau,$$

$\left.\dfrac{\partial\, L_0}{\partial v_0}\right|(t)$ is the Frechet derivative of the functional L_0.

The parameter μ_0 in the equations (3.10) is determined from the moment equality (3.5) for $i = 0$ and has the form

$$\mu_0 = \frac{m\Big(\psi_0 - \Phi^0_{0,+}(T)\varphi_0\Big) - r_0}{R_0}\ , \tag{3.11}$$

where

$$r_0 = a^0_0 b^0_0 + \int_{-\alpha}^0 a^0_{0,-}(t)b^0_{0,-}(t)dt + \int_0^T a^0_{0,+}(t)b^0_{0,+}(t)dt,$$
$$R_0 = (b^0_0)^2 + \int_{-\alpha}^0 (b^0_{0,-}(t))^2dt + \int_0^T (b^0_{0,+}(t))^2dt.$$

In (3.11) $R_0 > 0$. Indeed, let $R_0 = 0$. Then

$$b^0_0 = 0, \qquad b^0_{0,-}(t) = 0,\ t \in [-\alpha, 0], \qquad b^0_{0,+}(t) = 0,\ t \in (0, T]. \tag{3.12}$$

From the last equation of the system (3.12) it follows

$$\mathcal{U}^0_{0,+}(T,t) = 0, \quad t \in (0, T].$$

Since $b^0_{0,-}(-\alpha) = b^0_0$, then subtracting from the first equation of the system (3.12) second one, we get

$$V^0_{0,+}(T,t) = 0, \quad t \in [-\alpha, 0].$$

Taking into account the two equalities obtained above, it follows from the first equation of the system (3.12) that $U^0_{0,+}(T) = 0$. The resulting equalities contradict the definition of the functions $\mathcal{U}^0_{0,+}(t,\tau)$, $V^0_{0,+}(t,\tau)$, $U^0_{0,+}(t)$, i.e. $R_0 > 0$.

Then from (3.10) – (3.11) we find

$$\hat{u}_0(-\alpha) = \frac{1}{m}\mathcal{A}^0_0 + \mathcal{B}^0_0, \qquad v_{0,\mp}(t) = \frac{1}{m}\mathcal{A}^0_{0,\mp}(t) + \mathcal{B}^0_{0,\mp}(t), \tag{3.13}$$

where

$$\mathcal{A}^0_0 = a^0_0 - \frac{b^0_0 r_0}{R_0}\ , \qquad \mathcal{B}^0_0 = \frac{b^0_0\Big(\psi_0 - \Phi_{0,+}(T)\varphi_0\Big)}{R_0}\ ,$$
$$\mathcal{A}^0_{0,\mp}(t) = a^0_{0,\mp}(t) - \frac{b^0_{0,\mp}(t) r_0}{R_0}\ , \qquad \mathcal{B}^0_{0,\mp}(t) = \frac{b^0_{0,\mp}(t)\Big(\psi_0 - \Phi_{0,+}(T)\varphi_0\Big)}{R_0}\ .$$

For the problem (3.9) we get

$$
\begin{aligned}
\frac{\partial L_{2k-1,2k}}{\partial \hat{u}_{2k-1}(-\alpha)} &= a_{2k-1}^{2k-1} + \sum_{j=2k-1}^{2k} \mu_j b_j^{2k-1} - m\hat{u}_{2k-1}(-\alpha) = 0,\\
\frac{\partial L_{2k-1,2k}}{\partial \hat{u}_{2k}(-\alpha)} &= a_{2k}^{2k} + \sum_{j=2k-1}^{2k} \mu_j b_j^{2k} - m\hat{u}_{2k}(-\alpha) = 0,\\
0 = \left.\frac{\partial L_{2k-1,2k}}{\partial v_{2k-1}}\right|(t) &= \begin{cases} a_{2k-1,-}^{2k-1}(t) + \sum_{j=2k-1}^{2k} \mu_j b_{j,-}^{2k-1}(t) - mv_{2k-1,-}(t), \ t<0,\\ a_{2k-1,+}^{2k-1}(t) + \sum_{j=2k-1}^{2k} \mu_j b_{j,+}^{2k-1}(t) - mv_{2k-1,+}(t), \ t>0,\end{cases}\\
0 = \left.\frac{\partial L_{2k-1,2k}}{\partial v_{2k}}\right|(t) &= \begin{cases} a_{2k,-}^{2k}(t) + \sum_{j=2k-1}^{2k} \mu_j b_{j,-}^{2k}(t) - mv_{2k,-}(t), \ t<0,\\ a_{2k,+}^{2k}(t) + \sum_{j=2k-1}^{2k} \mu_j b_{j,+}^{2k}(t) \ - \ m\ v_{2k,+}(t), \ t>0,\end{cases}
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
a_{2k-1}^{2k-1} = &\int_{-\alpha}^{0}\left(q_{2k-1}(t)\Big(\int_{-\alpha}^{0} V_{2k-1,-}^{2k-1}(t,\tau)d\tau + U_{2k-1,-}^{2k-1}(t)\Big)\right.\\
&\left.+\int_t^0 q_{2k-1}(\tau)\mathcal{V}_{2k-1,-}^{2k-1}(\tau,t)d\tau\right)dt\\
&+\int_0^T\left(q_{2k-1}(t)\Big(\int_{-\alpha}^{0} V_{2k-1,+}^{2k-1}(t,\tau)d\tau + U_{2k-1,+}^{2k-1}(t)\Big)\right.\\
&\left.+\int_t^T q_{2k-1}(\tau)\mathcal{U}_{2k-1,+}^{2k-1}(\tau,t)d\tau\right)dt\\
&+\int_{-\alpha}^{0}\left(q_{2k}(t)\Big(\int_{-\alpha}^{0} V_{2k-1,-}^{2k}(t,\tau)d\tau + U_{2k-1,-}^{2k}(t)\Big)\right.\\
&\left.+\int_t^0 q_{2k}(\tau)\mathcal{V}_{2k-1,-}^{2k}(\tau,t)d\tau\right)dt\\
&+\int_0^T\left(q_{2k}(t)\Big(\int_{-\alpha}^{0} V_{2k-1,+}^{2k}(t,\tau)d\tau + U_{2k-1,+}^{2k}(t)\Big)\right.\\
&\left.+\int_t^T q_{2k}(\tau)\mathcal{U}_{2k-1,+}^{2k}(\tau,t)d\tau\right)dt,\\
b_{2k-1}^{2k-1} = &\int_{-\alpha}^{0} V_{2k-1,+}^{2k-1}(T,\tau)d\tau + U_{2k-1,+}^{2k-1}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau,\\
b_{2k}^{2k-1} = &\int_{-\alpha}^{0} V_{2k-1,+}^{2k}(T,\tau)d\tau + U_{2k-1,+}^{2k}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k}(T,\tau)d\tau,
\end{aligned}
$$

$$a_{2k}^{2k}=\int_{-\alpha}^{0}\left(q_{2k}(t)\Big(\int_{-\alpha}^{0}V_{2k,-}^{2k}(t,\tau)d\tau+U_{2k,-}^{2k}(t)\Big)+\int_{t}^{0}q_{2k}(\tau)\mathcal{V}_{2k,-}^{2k}(\tau,t)d\tau\right)dt$$
$$+\int_{0}^{T}\left(q_{2k}(t)\Big(\int_{-\alpha}^{0}V_{2k,+}^{2k}(t,\tau)d\tau+U_{2k,+}^{2k}(t)\Big)+\int_{t}^{T}q_{2k}(\tau)\mathcal{U}_{2k,+}^{2k}(\tau,t)d\tau\right)dt$$
$$b_{2k-1}^{2k}=\ 0,\qquad b_{2k}^{2k}=\int_{-\alpha}^{0}V_{2k,+}^{2k}(T,\tau)d\tau+U_{2k,+}^{2k}(T)+\int_{0}^{T}\mathcal{U}_{2k,+}^{2k}(T,\tau)d\tau,$$

$$a_{2k-1,-}^{2k-1}(t)=\int_{t}^{0}\left(\int_{-\alpha}^{0}q_{2k-1}(\xi)V_{2k-1,-}^{2k-1}(\xi,\tau)d\xi\right)d\tau+\int_{-\alpha}^{0}q_{2k-1}(\xi)U_{2k-1,-}^{2k-1}(\xi)d\xi$$
$$+\int_{t}^{0}q_{2k-1}(\xi)\left(\int_{t}^{\xi}\mathcal{V}_{2k-1,-}^{2k-1}(\xi,\tau)d\tau\right)d\xi+\int_{t}^{0}\left(\int_{0}^{T}q_{2k-1}(\xi)V_{2k-1,+}^{2k-1}(\xi,\tau)d\xi\right)d\tau$$
$$+\int_{0}^{T}q_{2k-1}(\xi)U_{2k-1,+}^{2k-1}(\xi)d\xi+\int_{0}^{T}\int_{\tau}^{T}q_{2k-1}(\xi)\mathcal{U}_{2k-1,+}^{2k-1}(\xi,\tau)d\xi d\tau$$
$$\int_{t}^{0}\left(\int_{-\alpha}^{0}q_{2k}(\xi)V_{2k-1,-}^{2k}(\xi,\tau)d\xi\right)d\tau+\int_{-\alpha}^{0}q_{2k}(\xi)U_{2k-1,-}^{2k}(\xi)d\xi$$
$$+\int_{t}^{0}q_{2k}(\xi)\left(\int_{t}^{\xi}\mathcal{V}_{2k-1,-}^{2k}(\xi,\tau)d\tau\right)d\xi+\int_{t}^{0}\left(\int_{0}^{T}q_{2k}(\xi)V_{2k-1,+}^{2k}(\xi,\tau)d\xi\right)d\tau$$
$$+\int_{0}^{T}q_{2k}(\xi)U_{2k-1,+}^{2k}(\xi)d\xi+\int_{0}^{T}\int_{\tau}^{T}q_{2k}(\xi)\mathcal{U}_{2k-1,+}^{2k}(\xi,\tau)d\xi d\tau,$$

$$b_{2k-1,-}^{2k-1}(t)=\int_{t}^{0}V_{2k-1,+}^{2k-1}(T,\xi)d\xi+U_{2k-1,+}^{2k-1}(T)+\int_{0}^{T}\mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau,$$

$$b_{2k,-}^{2k-1}(t)=\int_{t}^{0}V_{2k-1,+}^{2k}(T,\xi)d\xi+U_{2k-1,+}^{2k}(T)+\int_{0}^{T}\mathcal{U}_{2k-1,+}^{2k}(T,\tau)d\tau,$$

$$a_{2k-1,+}^{2k-1}(t)=\int_{t}^{T}q_{2k-1}(\xi)\left(\int_{t}^{\xi}\mathcal{U}_{2k-1,+}^{2k-1}(\xi,\tau)d\tau\right)d\xi$$
$$+\int_{t}^{T}q_{2k}(\xi)\left(\int_{t}^{\xi}\mathcal{U}_{2k-1,+}^{2k}(\xi,\tau)d\tau\right)d\xi,$$

$$b_{2k-1,+}^{2k-1}(t)=\int_{t}^{T}\mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau,\qquad b_{2k,+}^{2k-1}(t)=\int_{t}^{T}\mathcal{U}_{2k-1,+}^{2k}(T,\tau)d\tau,$$

$$a_{2k,-}^{2k}(t)=\int_{t}^{0}\left(\int_{-\alpha}^{0}q_{2k}(\xi)V_{2k,-}^{2k}(\xi,\tau)d\xi\right)d\tau+\int_{-\alpha}^{0}q_{2k}(\xi)U_{2k,-}^{2k}(\xi)d\xi$$
$$+\int_{t}^{0}q_{2k}(\xi)\left(\int_{t}^{\xi}\mathcal{V}_{2k,-}^{2k}(\xi,\tau)d\tau\right)d\xi+\int_{t}^{0}\left(\int_{0}^{T}q_{2k}(\xi)V_{2k,+}^{2k}(\xi,\tau)d\xi\right)d\tau$$
$$+\int_{0}^{T}q_{2k}(\xi)U_{2k,+}^{2k}(\xi)d\xi+\int_{0}^{T}\int_{\tau}^{T}q_{2k}(\xi)\mathcal{U}_{2k,+}^{2k}(\xi,\tau)d\xi d\tau,$$

$$\begin{aligned}
b^{2k}_{2k-1,-}(t) = 0, \quad b^{2k}_{2k,-}(t) &= \int_t^0 V^{2k}_{2k,+}(T,\xi)d\xi + U^{2k}_{2k,+}(T) + \int_0^T \mathcal{U}^{2k}_{2k,+}(T,\tau)d\tau, \\
a^{2k}_{2k,+}(t) &= \int_t^T q_{2k}(\xi) \int_t^\xi \mathcal{U}^{2k}_{2k,+}(\xi,\tau)d\tau d\xi, \\
b^{2k}_{2k-1,+}(t) = 0, \quad b^{2k}_{2k,+}(t) &= \int_t^T \mathcal{U}^{2k}_{2k,+}(T,\tau)d\tau.
\end{aligned}$$

The parameters $\mu_{2k-1}, \mu_{2k},\ k = 1, 2, \dots$ for each k are determined from the moment equalities (3.5) ($i = \overline{2k-1, 2k}$) that generate the system

$$\begin{aligned}
R^{2k-1}_{2k-1}\mu_{2k-1} + R^{2k-1}_{2k}\mu_{2k} &= m\left(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\right) - r^{2k-1}_{2k-1}, \\
R^{2k}_{2k-1}\mu_{2k-1} + R^{2k}_{2k}\mu_{2k} & \\
&= m\left(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\right) - r^{2k}_{2k},
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
r^{2k-1}_{2k-1} &= a^{2k-1}_{2k-1}b^{2k-1}_{2k-1} + \int_{-\alpha}^0 a^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k-1,-}(t)dt + \int_0^T a^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k-1,+}(t)dt, \\
R^{2k-1}_{2k-1} &= \left(b^{2k-1}_{2k-1}\right)^2 + \int_{-\alpha}^0 \left(b^{2k-1}_{2k-1,-}(t)\right)^2 dt + \int_0^T \left(b^{2k-1}_{2k-1,+}(t)\right)^2 dt, \\
R^{2k-1}_{2k} &= b^{2k-1}_{2k-1}b^{2k-1}_{2k} + \int_{-\alpha}^0 b^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k,-}(t)dt + \int_0^T b^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k,+}(t)dt, \\
r^{2k}_{2k} &= a^{2k-1}_{2k-1}b^{2k-1}_{2k} + \int_{-\alpha}^0 a^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k,-}(t)dt + \int_0^T a^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k,+}(t)dt \\
&\quad + a^{2k}_{2k}b^{2k}_{2k} + \int_{-\alpha}^0 a^{2k}_{2k,-}(t)b^{2k}_{2k,-}(t)dt + \int_0^T a^{2k}_{2k,+}(t)b^{2k}_{2k,+}(t)dt, \\
R^{2k}_{2k-1} &= b^{2k-1}_{2k-1}b^{2k-1}_{2k} + \int_{-\alpha}^0 b^{2k-1}_{2k-1,-}(t)b^{2k-1}_{2k,-}(t)dt + \int_0^T b^{2k-1}_{2k-1,+}(t)b^{2k-1}_{2k,+}(t)dt, \\
R^{2k}_{2k} &= \left(b^{2k-1}_{2k}\right)^2 + \int_{-\alpha}^0 \left(b^{2k-1}_{2k,-}(t)\right)^2 dt + \int_0^T \left(b^{2k-1}_{2k,+}(t)\right)^2 dt \\
&\quad + \left(b^{2k}_{2k}\right)^2 + \int_{-\alpha}^0 \left(b^{2k}_{2k,-}(t)\right)^2 dt + \int_0^T \left(b^{2k}_{2k,+}(t)\right)^2 dt.
\end{aligned}$$

Consider the solvability of the system (3.15). Its determinant has the form

$$\Delta_{2k-1,2k} = R^{2k-1}_{2k-1}R^{2k}_{2k} - \left(R^{2k-1}_{2k}\right)^2, \tag{3.16}$$

because $R^{2k-1}_{2k} = R^{2k}_{2k-1}$.

Let us show that for all $k : \Delta_{2k-1,2k} > 0$. To do this, we use the Hilbert space $\mathcal{H} = R^1 \times L_2(-\alpha, 0) \times L_2(0,T)$ with scalar product

$$\left(\hat{a}, \hat{b}\right)_{\mathcal{H}} = ab + \int_{-\alpha}^{0} a_-(t) b_-(t) dt + \int_0^T a_+(t) b_+(t) dt, \quad \hat{a}, \hat{b} \in \mathcal{H}. \tag{3.17}$$

Then the determinant (3.16) in terms of the characteristics of the space $\mathcal{H}$ can be represented in the form

$$\Delta_{2k-1,2k} = \left\|\hat{b}^{2k-1}_{2k-1}\right\|^2_{\mathcal{H}} \left(\left\|\hat{b}^{2k-1}_{2k}\right\|^2_{\mathcal{H}} + \left\|\hat{b}^{2k}_{2k}\right\|^2_{\mathcal{H}} \right) - \left(\hat{b}^{2k-1}_{2k-1}, \hat{b}^{2k-1}_{2k}\right)^2_{\mathcal{H}}, \tag{3.18}$$

where $\left(\hat{b}^{2k-1}_{2k-1}\right)' = \left(b^{2k-1}_{2k-1},\ b^{2k-1}_{2k-1,-}(t),\ b^{2k-1}_{2k-1,+}(t)\right)$, $\left(\hat{b}^{2k-1}_{2k}\right)' = \left(b^{2k-1}_{2k},\ b^{2k-1}_{2k,-}(t),\ b^{2k-1}_{2k,+}(t)\right)$, $\left(\hat{b}^{2k}_{2k}\right)' = \left(b^{2k}_{2k},\ b^{2k}_{2k,-}(t),\ b^{2k}_{2k,+}(t)\right)$.

Let us estimate from below the value of the determinant $\Delta_{2k-1,2k}$, using the Cauchy-Bunyakovsky inequality for the scalar product,

$$\begin{aligned} \Delta_{2k-1,2k} &\geq \|\hat{b}^{2k-1}_{2k-1}\|^2_{\mathcal{H}} \Big(\|\hat{b}^{2k-1}_{2k}\|^2_{\mathcal{H}} + \|\hat{b}^{2k}_{2k}\|^2_{\mathcal{H}} \Big) - \|\hat{b}^{2k-1}_{2k-1}\|^2_{\mathcal{H}} \|\hat{b}^{2k-1}_{2k}\|^2_{\mathcal{H}} \\ &= \|\hat{b}^{2k-1}_{2k-1}\|^2_{\mathcal{H}} \|\hat{b}^{2k}_{2k}\|^2_{\mathcal{H}} = \|\hat{b}^{2k-1}_{2k-1}\|^4_{\mathcal{H}}, \end{aligned} \tag{3.19}$$

since $\hat{b}^{2k}_{2k} = \hat{b}^{2k-1}_{2k-1}$ due to the integral representation (2.13) – (2.14) of the original boundary value problem solution.

Let us show that $\Delta_{2k-1,2k} > 0$ for any k.

Indeed, suppose that $\hat{b}^{2k-1}_{2k-1} = 0$. Then the system of equations

$$\begin{aligned} b^{2k-1}_{2k-1} &= \int_{-\alpha}^{0} V^{2k-1}_{2k-1,+}(T,\tau) d\tau + U^{2k-1}_{2k-1,+}(T) + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau) d\tau = 0, \\ b^{2k-1}_{2k-1,-}(t) &= \int_t^0 V^{2k-1}_{2k-1,+}(T,\xi) d\xi + U^{2k-1}_{2k-1,+}(T) \\ &\quad + \int_0^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau) d\tau = 0, \quad t \in [-\alpha, 0), \\ b^{2k-1}_{2k-1,+}(t) &= \int_t^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau) d\tau = 0, \quad t \in [0,T]. \end{aligned}$$

The written system will take place provided that

$$\mathcal{U}^{2k-1}_{2k-1,+}(T,\tau) = V^{2k-1}_{2k-1,+}(T,\tau) = U^{2k-1}_{2k-1,+}(T) = 0$$

(see the analysis of the $R_0 = 0$ case above). This contradicts the definition of functions $\mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)$, $V^{2k-1}_{2k-1,+}(T,\tau)$, $U^{2k-1}_{2k-1,+}(T)$.

Thus,

$$\begin{aligned}
\mu_{2k-1} = \Delta^{-1}_{2k-1,2k}\Bigg[m\Bigg(\Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k} - \Big(\psi_{2k} \\
- \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k}\Bigg) + r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Bigg], \\
\mu_{2k} = \Delta^{-1}_{2k-1,2k}\Bigg[\Bigg(m\Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)\ R^{2k-1}_{2k-1} \\
- \Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k-1}\Bigg) + r^{2k-1}_{2k-1}R^{2k}_{2k-1} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Bigg].
\end{aligned} \tag{3.20}$$

Then from (3.14) and (3.20) we find

$$\begin{gathered}
\hat{u}_j(-\alpha) = \frac{1}{m}\mathcal{A}^j_j + \mathcal{B}^j_j, \\
v_{j,\mp}(t) = \frac{1}{m}\mathcal{A}^j_{j,\mp}(t) + \mathcal{B}^j_{j,\mp}(t), \quad j = \overline{2k-1,2k},\ k = 1, 2, \ldots,
\end{gathered} \tag{3.21}$$

where

$$\begin{aligned}
\mathcal{A}^{2k-1}_{2k-1} &= a^{2k-1}_{2k-1} + \frac{1}{\Delta_{2k-1,2k}}\Bigg[b^{2k-1}_{2k-1}\Big(r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Big) \\
&\quad + b^{2k-1}_{2k}\Big(r^{2k-1}_{2k-1}R^{2k-1}_{2k} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Big)\Bigg], \\
\mathcal{B}^{2k-1}_{2k-1} &= \frac{1}{\Delta_{2k-1,2k}}\Bigg[b^{2k-1}_{2k-1}\Bigg(\Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k} \\
&\quad - \Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k}\Bigg) \\
&\quad + b^{2k-1}_{2k}\Bigg(\Big(\psi_{2k} - \Phi^{2k}_{2k-1,+}(T)\varphi_{2k-1} - \Phi^{2k}_{2k,+}(T)\varphi_{2k}\Big)R^{2k-1}_{2k-1} \\
&\quad - \Big(\psi_{2k-1} - \Phi^{2k-1}_{2k-1,+}(T)\varphi_{2k-1}\Big)R^{2k}_{2k-1}\Bigg)\Bigg], \\
\mathcal{A}^{2k-1}_{2k-1,\mp}(t) &= a^{2k-1}_{2k-1,\mp}(t) + \frac{1}{\Delta_{2k-1,2k}}\Bigg[b^{2k-1}_{2k-1,\mp}(t)\Big(r^{2k}_{2k}R^{2k-1}_{2k} - r^{2k-1}_{2k-1}R^{2k}_{2k}\Big) \\
&\quad + b^{2k-1}_{2k,\mp}(t)\Big(r^{2k-1}_{2k-1}R^{2k-1}_{2k} - r^{2k}_{2k}R^{2k-1}_{2k-1}\Big)\Bigg],
\end{aligned}$$

$$
\begin{aligned}
\mathcal{B}_{2k-1,\mp}^{2k-1}(t) &= \frac{1}{\Delta_{2k-1,2k}}\Bigg[b_{2k-1,\mp}^{2k-1}(t)\Bigg(\Big(\psi_{2k-1} - \Phi_{2k-1,+}^{2k-1}(T)\varphi_{2k-1}\Big)R_{2k}^{2k} \\
&\quad - \Big(\psi_{2k} - \Phi_{2k-1,+}^{2k}(T)\varphi_{2k-1} - \Phi_{2k,+}^{2k}(T)\varphi_{2k}\Big)R_{2k}^{2k-1}\Bigg) \\
&\quad + b_{2k,\mp}^{2k-1}(t)\Bigg(\Big(\psi_{2k} - \Phi_{2k-1,+}^{2k}(T)\varphi_{2k-1} - \Phi_{2k,+}^{2k}(T)\varphi_{2k}\Big)R_{2k-1}^{2k-1} \\
&\quad - \Big(\psi_{2k-1} - \Phi_{2k-1,+}^{2k-1}(T)\varphi_{2k-1}\Big)R_{2k-1}^{2k}\Bigg)\Bigg]; \\
\mathcal{A}_{2k}^{2k} &= a_{2k}^{2k} + \frac{1}{\Delta_{2k-1,2k}}\Bigg[b_{2k-1}^{2k}\Big(r_{2k}^{2k}R_{2k}^{2k-1} - r_{2k-1}^{2k-1}R_{2k}^{2k}\Big) \\
&\quad + b_{2k}^{2k}\Big(r_{2k-1}^{2k-1}R_{2k}^{2k-1} - r_{2k}^{2k}R_{2k-1}^{2k-1}\Big)\Bigg], \\
\mathcal{B}_{2k}^{2k} &= \frac{1}{\Delta_{2k-1,2k}}\Bigg[b_{2k-1}^{2k}\Bigg(\Big(\psi_{2k-1} - \Phi_{2k-1,+}^{2k-1}(T)\varphi_{2k-1}\Big)R_{2k}^{2k} \\
&\quad - \Big(\psi_{2k} - \Phi_{2k-1,+}^{2k}(T)\varphi_{2k-1} - \Phi_{2k,+}^{2k}(T)\varphi_{2k}\Big)R_{2k}^{2k-1}\Bigg) \\
&\quad + b_{2k}^{2k}\Bigg(\Big(\psi_{2k} - \Phi_{2k-1,+}^{2k}(T)\varphi_{2k-1} - \Phi_{2k,+}^{2k}(T)\varphi_{2k}\Big)R_{2k-1}^{2k-1} \\
&\quad - \Big(\psi_{2k-1} - \Phi_{2k-1,+}^{2k-1}(T)\varphi_{2k-1}\Big)R_{2k-1}^{2k}\Bigg)\Bigg], \\
\mathcal{A}_{2k,\mp}^{2k}(t) &= a_{2k,\mp}^{2k}(t) + \frac{1}{\Delta_{2k-1,2k}}\Bigg[b_{2k-1,\mp}^{2k}(t)\Big(r_{2k}^{2k}R_{2k}^{2k-1} - r_{2k-1}^{2k-1}R_{2k}^{2k}\Big) \\
&\quad + b_{2k,\mp}^{2k}(t)\Big(r_{2k-1}^{2k-1}R_{2k}^{2k-1} - r_{2k}^{2k}R_{2k-1}^{2k-1}\Big)\Bigg], \\
\mathcal{B}_{2k,\mp}^{2k}(t) &= \frac{1}{\Delta_{2k-1,2k}}\Bigg[b_{2k-1,\mp}^{2k}(t)\Bigg(\Big(\psi_{2k-1} - \Phi_{2k-1,+}^{2k-1}(T)\varphi_{2k-1}\Big)R_{2k}^{2k} \\
&\quad - \Big(\psi_{2k} - \Phi_{2k-1,+}^{2k}(T)\varphi_{2k-1} - \Phi_{2k,+}^{2k}(T)\varphi_{2k}\Big)R_{2k}^{2k-1}\Bigg) \\
&\quad + b_{2k,\mp}^{2k}(t)\Bigg(\Big(\psi_{2k} - \Phi_{2k-1,+}^{2k}(T)\varphi_{2k-1} - \Phi_{2k,+}^{2k}(T)\varphi_{2k}\Big)R_{2k-1}^{2k-1} \\
&\quad - \Big(\psi_{2k-1} - \Phi_{2k-1,+}^{2k-1}(T)\varphi_{2k-1}\Big)R_{2k-1}^{2k}\Bigg)\Bigg].
\end{aligned}
$$

Lagrange multiplier m, according to condition (3.4) and formulas (3.13), (3.21), determined from the equation

$$c\left(\frac{1}{m}\right)^2 + 2d\left(\frac{1}{m}\right) + e = 0, \tag{3.22}$$

where

$$c = \sum_{i=0}^{\infty}\left[\left(\mathcal{A}_i^i\right)^2 + \int_{-\alpha}^{0}\left(\mathcal{A}_{i,-}^i(t)\right)^2 dt + \int_0^T \left(\mathcal{A}_{i,+}^i(t)\right)^2 dt\right],$$
$$d = \sum_{i=0}^{\infty}\left[\mathcal{A}_i^i\mathcal{B}_i^i + \int_{-\alpha}^{0}\mathcal{A}_{i,-}^i(t)\mathcal{B}_{i,-}^i(t)dt + \int_0^T \mathcal{A}_{i,+}^i(t)\mathcal{B}_{i,+}^i(t)dt\right],$$
$$e = -\nu^2 + \sum_{i=0}^{\infty}\left[\left(\mathcal{B}_i^i\right)^2 + \int_{-\alpha}^{0}\left(\mathcal{B}_{i,-}^i(t)\right)^2 dt + \int_0^T \left(\mathcal{B}_{i,+}^i(t)\right)^2 dt\right].$$

Let the series defining the coefficients c, d, e of the equation (3.22) converge. Then

$$\left(\frac{1}{m}\right)_{1,2} = \frac{-d \pm \sqrt{D}}{c}, \tag{3.23}$$

where $D = d^2 - ce$ is the discriminant of the equation (3.22).

The number ν in the constraint (2.18) must be such that the inequality

$$\begin{aligned} D = &\left(\sum_{i=0}^{\infty}\left[\mathcal{A}_i^i\mathcal{B}_i^i + \int_{-\alpha}^{0}\mathcal{A}_{i,-}^i(t)\mathcal{B}_{i,-}^i(t)dt + \int_0^T \mathcal{A}_{i,+}^i(t)\mathcal{B}_{i,+}^i(t)dt\right]\right)^2 \\ &+ \left(\sum_{i=0}^{\infty}\left[\left(\mathcal{A}_i^i\right)^2 + \int_{-\alpha}^{0}\left(\mathcal{A}_{i,-}^i(t)\right)^2 dt + \int_0^T \left(\mathcal{A}_{i,+}^i(t)\right)^2 dt\right]\right) \\ &\times \left(\nu^2 - \sum_{i=0}^{\infty}\left[\left(\mathcal{B}_i^i\right)^2 + \int_{-\alpha}^{0}\left(\mathcal{B}_{i,-}^i(t)\right)^2 dt + \int_0^T \left(\mathcal{B}_{i,+}^i(t)\right)^2 dt\right]\right) \geq 0. \end{aligned} \tag{3.24}$$

Then the formulas (3.13), (3.21) are a formal solution to the problem (2.17) – (2.19).

4. Substantiation of the solution of an extremal problem

Let us find the conditions on the data of the original problem under which the above formal results hold.

Let us first estimate the right-hand side of the inequality (3.19) from below.

$$\|\hat{b}_{2k-1}^{2k-1}\|_{\mathcal{H}}^4 =\Bigg(\Big(\int_{-\alpha}^0 V_{2k-1,+}^{2k-1}(T,\tau)d\tau + U_{2k-1,+}^{2k-1}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 + \int_{-\alpha}^0 \Big(\int_t^0 V_{2k-1,+}^{2k-1}(T,\xi)d\xi + U_{2k-1,+}^{2k-1}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 dt + \int_0^T \Big(\int_t^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 dt\Bigg)^2. \tag{4.1}$$

Applying to the right side of equality (4.1) the inequality about the relationship between the arithmetic mean and the geometric mean, we get

$$\|\hat{b}_{2k-1}^{2k-1}\|_{\mathcal{H}}^4 \geq 9\Bigg[\Big(\int_{-\alpha}^0 V_{2k-1,+}^{2k-1}(T,\tau)d\tau + U_{2k-1,+}^{2k-1}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 \times \int_{-\alpha}^0 \Big(\int_t^0 V_{2k-1,+}^{2k-1}(T,\xi)d\xi + U_{2k-1,+}^{2k-1}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 dt \times \int_0^T \Big(\int_t^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 dt\Bigg]^{\frac{2}{3}}. \tag{4.2}$$

Let us estimate the factors of the right-hand side of the inequality (4.2) from below.

$$\Big(\int_{-\alpha}^0 V_{2k-1,+}^{2k-1}(T,\tau)d\tau + U_{2k-1,+}^{2k-1}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 = \Big(-\frac{\exp(-\lambda_k^2 T)}{\delta_k(\alpha)\lambda_k^2}\Big(1-\cos\lambda_k\alpha\Big) + \frac{\exp(-\lambda_k^2 T)}{\delta_k(\alpha)\lambda_k}\sin(\lambda_k\alpha) + \frac{1}{\lambda_k^2}\Big(1-\exp(-\lambda_k^2 T)\Big)\Big)^2 \geq \frac{1}{\lambda_k^4}\left(1-\frac{\exp(-\lambda_1^2 T)}{\lambda_1}\right)^2;$$

$$\int_{-\alpha}^0 \Big(\int_t^0 V_{2k-1,+}^{2k-1}(T,\xi)d\xi + U_{2k-1,+}^{2k-1}(T) + \int_0^T \mathcal{U}_{2k-1,+}^{2k-1}(T,\tau)d\tau\Big)^2 dt = \int_{-\alpha}^0 \Big(\frac{\exp(-\lambda_k^2 T)}{\delta_k(\alpha)\lambda_k^2}\Big(\cos\lambda_k\alpha - \cos\lambda_k(\alpha+t)\Big) + \frac{\exp(-\lambda_k^2 T)}{\delta_k(\alpha)\lambda_k}\sin(\lambda_k\alpha) + \frac{1}{\lambda_k^2}\Big(1-\exp(-\lambda_k^2 T)\Big)\Big)^2 dt \geq \frac{\alpha}{\lambda_k^4}\left(1-\frac{\exp(-\lambda_1^2 T)}{\lambda_1}\right)^2;$$

$$\int_0^T\Big(\int_t^T \mathcal{U}^{2k-1}_{2k-1,+}(T,\tau)d\tau\Big)^2 dt = \frac{1}{\lambda_k^4}\int_0^T\bigg(1-\exp\Big(-\lambda_k^2(T-t)\Big)\bigg)^2 dt$$
$$> \frac{1}{\lambda_k^4}\int_0^T\bigg(1-\exp\Big(-\lambda_1^2(T-t)\Big)\bigg)^2 dt.$$

The above estimates imply the inequality

$$\Delta_{2k-1,2k} > \frac{C}{\lambda_k^8}\ . \tag{4.3}$$

The optimal control formally found in the previous subsection must be an absolutely continuous function, i.e., the series must converge

$$U = \sum_{i=0}^{\infty}\Big(\hat{u}_i^2(-\alpha)+\int_{-\alpha}^T v_i^2(t)dt\Big) \le C\left[\frac{1}{\hat{m}^2}\sum_{i=0}^{\infty}\Big(\big(\mathcal{A}_i^i\big)^2+\int_{-\alpha}^0\big(\mathcal{A}_{i,-}^i(t)\big)^2dt\ + \right.$$
$$\left. +\int_0^T\big(\mathcal{A}_{i,+}^i(t)\big)^2dt\Big)+\sum_{i=0}^{\infty}\Big(\big(\mathcal{B}_i^i\big)^2+\int_{-\alpha}^0\big(\mathcal{B}_{i,-}^i(t)\big)^2dt+\int_0^T\big(\mathcal{B}_{i,+}^i(t)\big)^2dt\Big)\right], \tag{4.4}$$

where

$$\frac{1}{\hat{m}} < \frac{|d|+\sqrt{D}}{c}\ .$$

Due to the representation for the constant c,, the estimate for the quantity U will be a fair

$$U < C\left[\frac{d^2+D}{c_0}\ +\sum_{i=0}^{\infty}\Big(\big(\mathcal{B}_i^i\big)^2+\int_0^T\big(\mathcal{B}_{i,-}^i(t)\big)^2dt+\int_{-\alpha}^0\big(\mathcal{B}_{i,+}^i(t)\big)^2dt\Big)\right], \tag{4.5}$$

where

$$c_0 = \big(\mathcal{A}_0^0\big)^2+\int_{-\alpha}^0\big(\mathcal{A}_{0,-}^0(t)\big)^2dt+\int_0^T\big(\mathcal{A}_{0,+}^0(t)\big)^2dt \neq 0.$$

Further, two variants of sufficient conditions on the initial data of the problem under consideration are possible, under which the above series will converge, defining the formal solution of the problem.

Case 1. Let the input data be $\varphi(x),\ \psi(x)\in C_0^\infty(0,1)$, $q(x.t)\in C_0^{0,\infty}(D)$, i.e. by spatial coordinate they are infinitely differentiable and finite on the boundary.

Consider, for example, how it will look for the function $\varphi(x)$. The above series will converge if the series is convergent

$$\sum_{k=0}^{\infty}\lambda_k^l\Big(\varphi_{2k-1}^2+\varphi_{2k}^2\Big), \tag{4.6}$$

where l – some positive integer and let here $\varphi_{-1}=0$. Really,

$$\varphi_{2k-1} = \int_0^1\varphi(x)Y_{2k-1}(x)dx,\quad \varphi_{2k} = \int_0^1\varphi(x)Y_{2k}(x)dx.$$

Integrating by parts, for the coefficient φ_{2k-1} we obtain

$$\begin{aligned}
\varphi_{2k-1} = & \int_0^1 \varphi(x) Y_{2k-1}(x) dx = 4\int_0^1 \varphi(x)\cos(\lambda_k x)dx \\
= & \frac{4}{\lambda_k}\,\varphi(x)\sin(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k}\int_0^1 \frac{d\varphi(x)}{dx}\sin(\lambda_k x)dx \\
= & \frac{4}{\lambda_k}\,\varphi(x)\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\varphi(x)}{dx}\cos(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^2}\,\int_0^1 \frac{d^2\varphi(x)}{dx^2}\cos(\lambda_k x)dx \\
= & \frac{4}{\lambda_k}\,\varphi(x)\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\varphi(x)}{dx}\cos(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^3}\,\frac{d^2\varphi(x)}{dx^2}\sin(\lambda_k x)\Big|_0^1 \\
& + \frac{4}{\lambda_k^3}\int_0^1 \frac{d^3\varphi(x)}{dx^3}\sin(\lambda_k x)dx = \frac{4}{\lambda_k}\varphi(x)\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\varphi(x)}{dx}\cos(\lambda_k x)\Big|_0^1 \\
& - \frac{4}{\lambda_k^3}\frac{d^2\varphi(x)}{dx^2}\sin(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^4}\frac{d^3\varphi(x)}{dx^3}\cos(\lambda_k x)\Big|_0^1 \\
& + \frac{4}{\lambda_k^4}\int_0^1 \frac{d^4\varphi(x)}{dx^4}\cos(\lambda_k x)dx = \ldots = 4\sum_{i=1}^{m}\frac{(-1)^{\mu_i+1}}{\lambda_k^i}\frac{d^{i-1}\varphi(x)}{dx^{i-1}}\varrho_i(\lambda_k, x)\Big|_0^1 \\
& + (-1)^{\mu_m}\frac{4}{\lambda_k^m}\int_0^1 \frac{d^m\varphi(x)}{dx^m}\varrho_m(\lambda_k, x)dx,
\end{aligned} \tag{4.7}$$

where

$$\mu_m = \begin{cases} j, & m = 2j-1, \quad j = 1, 2, \ldots, \\ n, & m = 2n, \qquad\; n = 1, 2, \ldots\ , \end{cases}$$

$$\varrho_m(\lambda_k, x) = \begin{cases} \sin(\lambda_k x), & m - \text{odd}, \\ \cos(\lambda_k\ x), & m - \text{even}. \end{cases}$$

Since the function $\varphi(x)$ is finite, the equality (4.7) takes the form

$$\varphi_{2k-1} = (-1)^{\mu_m}\frac{4}{\lambda_k^m}\int_0^1 \frac{d^m\varphi(x)}{dx^m}\varrho_m(\lambda_k, x)dx. \tag{4.8}$$

Similarly, setting $\omega(x) = \varphi(x)(1-x)$, for the coefficient φ_{2k} we have

$$\begin{aligned}
\varphi_{2k} = & \int_0^1 \varphi(x) Y_{2k}(x) dx = 4\int_0^1 \omega(x)\sin(\lambda_k x)dx \\
= & -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k}\int_0^1 \frac{d\omega(x)}{dx}\cos(\lambda_k x)dx \\
= & -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\omega(x)}{dx}\,\sin(\lambda_k x)\Big|_0^1 - \frac{4}{\lambda_k^2}\int_0^1 \frac{d^2\omega(x)}{dx^2}\sin(\lambda_k x)dx \\
= & -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^2}\,\frac{d\omega(x)}{dx}\,\sin(\lambda_k x)\Big|_0^1 + \frac{4}{\lambda_k^3}\,\frac{d^2\omega(x)}{dx^2}\cos(\lambda_k x)\Big|_0^1 \\
& - \frac{4}{\lambda_k^3}\int_0^1 \frac{d^3\omega(x)}{dx^3}\,\cos(\lambda_k x)dx = -\frac{4}{\lambda_k}\,\omega(x)\cos(\lambda_k x)\Big|_0^1
\end{aligned}$$

$$+\frac{4}{\lambda_k^2}\frac{d\omega(x)}{dx}\sin(\lambda_k x)\Big|_0^1+\frac{4}{\lambda_k^3}\frac{d^2\omega(x)}{dx^2}\cos(\lambda_k x)\Big|_0^1-\frac{4}{\lambda_k^4}\frac{d^3\omega(x)}{dx^3}\sin(\lambda_k x)\Big|_0^1$$
$$+\frac{4}{\lambda_k^4}\int_0^1\frac{d^4\omega(x)}{dx^4}\sin(\lambda_k x)dx=\dots$$
$$=-\frac{4}{\lambda_k}\omega(x)\cos(\lambda_k x)\Big|_0^1+4\sum_{i=1}^{m-1}\frac{(-1)^{\mu_i+1}}{\lambda_k^{i+1}}\frac{d^i\omega(x)}{dx^i}\varrho_i(\lambda_k,x)\Big|_0^1+$$
$$+(-1)^m\frac{4}{\lambda_k^m}\int_0^1\frac{d^m\omega(x)}{dx^m}\varrho_{m+1}(\lambda_k,x)dx. \tag{4.9}$$

Taking into account that the function $\varphi(x)$, is finite, the equality (4.9) takes the form

$$\varphi_{2k}=(-\,1)^m\frac{4}{\lambda_k^m}\int_0^1\frac{d^m\omega(x)}{dx^m}\,\varrho_{m+1}(\lambda_k,x)\,dx. \tag{4.10}$$

Due to the fact that

$$\frac{d^n\omega(x)}{dx^n}=(1-x)\frac{d^n\varphi(x)}{dx^n}-n\,\frac{d^{n-1}\varphi(x)}{dx^{n-1}},$$

from the formulas (4.8), (4.10) we get the estimate

$$|\varphi_j|\le\frac{C}{\lambda_k^m}\sum_{\nu=0}^m\max_{x\in[0,1]}\left|\frac{d^\nu\varphi(x)}{dx^\nu}\right|,\quad j=\overline{2k-1,2k}. \tag{4.11}$$

Taking into account (4.11), for the series (4.6), we obtain the estimate

$$\sum_{k=0}^\infty\lambda_k^l\left(\varphi_{2k-1}^2+\varphi_{2k}^2\right)\le C\sum_{\nu=0}^m\max_{x\in[0,1]}\left|\frac{d^\nu\varphi(x)}{dx^\nu}\right|^2\sum_{k=0}^\infty\frac{1}{\lambda_k^{2m-l}}\,. \tag{4.12}$$

We choose the number m from the condition

$$2m-l\ge 2. \tag{4.13}$$

Then the series (4.6) will converge.

Case 2. The class of original functions can be extended for the convergence of the above series, defining the formal solution of the problem. Indeed, let a priori estimates of the type (4.5) result in series of the form (4.6). Then due to (4.7), (4.9), we set $\varphi(x)\in C^m(0,1)$ и

$$\frac{d^{2k-1}\varphi(0)}{dx^{2k-1}}=\frac{d^{2k-1}\varphi(1)}{dx^{2k-1}},\quad 2k-1>0,$$
$$\frac{d^{2k}\varphi(0)}{dx^{2k}}=0,\quad k=\overline{0,m}.$$

If the conditions (4.13) are met, the series (4.6) will converge.

For other inputs, the analog result will take place. The general case is not presented here due to the cumbersomeness of a priori estimates of the form (4.5).

References

1. A. I. Egorov, *Optimal control of linear system*, Vyscha Shkola, Kyiv, 1988.
2. O. V. Kapustyan, O. A. Kapustyan, A. V. Sukretna *Approximate bounded synthesis for one weakly nonlinear boundary-value problem* Nonlinear Oscillations **12(3)**(2009), 297–304.
3. O. V. Kapustyan, O. A. Kapustyan, A. V. Sukretna *Approximate stabilization for a nonlinear parabolic boundary-value problem* Ukrainian Mathematical Journal **63(5)**(2011), 759–767.
4. A. V. Sukretna *Bounded approximate synthesis of the optimal control for the wave equation* Ukrainian Mathematical Journal **59(8)**(2007), 1212–1223.
5. V. E. Kapustyan, I. A. Pyshnograev, *Optimal control and minimax estimation for parabolic-hyperbolic equations with nonlocal boundary conditions*, KPI im. Igor Sikorskogo, Kyiv, 2020.
6. K. B. Sabitov , *Boundary value problem for a parabolic-hyperbolic equation with a nonlocal integral condition*, Differential Equations, **46**(2010), 1472–1481.
7. N. I. Ionkin, *The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition*, Differentsial'nye Uravneniya, **13**(1977), 294–304.
8. A. Yu. Mokin , *On a family of initial-boundary value problems for the heat equation*, Differential Equations, **45**(2009), 126–141.
9. V. O. Kapustyan, I. O. Pyshnograev, *Conditions of existence and unity solution of a parabolic – hyperbolic equation with nonlocal boundaries conditions*, Naukovi Visti NTUU "KPI". Teoretychni ta prykladni problemy matematyky, **4**(2012), 72–76.

Received 15.05.2023

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 74–94, DOI 10.15421/142304

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

MODEL OF NON-ISOTHERMAL CONSOLIDATION IN THE PRESENCE OF GEOBARRIERS AND THE TOTAL APPROXIMATION PROPERTIES OF ITS FINITE ELEMENT SOLUTIONS

Petro M. Martyniuk*, Oksana V. Ulianchuk-Martyniuk †

Abstract. The boundary value problem for the system of quasi-linear parabolic equations in the presence of integral conjugation conditions is considered. The boundary value problem is a mathematical model of the process of non-isothermal filtration consolidation of the soil mass which contains a thin geobarrier. Geobarriers exposed to non-isothermal conditions are a component of waste storage facilities. The change of hydromechanical and thermal properties the geobarriers, as well as the phenomenon of thermal osmosis, require modification of both the equations in the mathematical model and the conjugation conditions. The finite element method is used to find approximate solutions of the corresponding system of quasi-linear parabolic equations. The existence and uniqueness of the approximate generalized solution is proved. The accuracy of finite element solutions in the sense of total approximation are also estimated. The differences in the values of pressure and temperature distributions for the classical case and the case considered in the article were analyzed on the test model example.

Key words: system of quasi-linear parabolic equations, finite element method, generalized solution, accuracy of finite element solutions, thermo-osmosis, geobarrier, conjugation condition.

2010 Mathematics Subject Classification: 35K05, 35K59, 35Q35, 65M60, 86-08.

Communicated by Prof. V. Ye. Belozyorov

1. Introduction

Waste dumps have become integral elements of the daily life of people on planet Earth [22]. This applies to both the industrial scale and the household life. [22] state: "Disposal of municipal solid waste (MSW) in engineered landfills is one of the most widely used waste management practices in the USA and worldwide." It is clear that the consequence of this most common practice is the problems of the impact of waste landfills on the environment. One of the engineering elements of waste storage facilities designed to reduce the level of such negative impact is geobarriers, both of natural geomaterials (mainly clay and artificial geotextiles. The use of geosynthetic clay liners (GCLs) is noted in [5] as a common practice in geologic engineering for waste storage facilities to protect groundwaters from contamination.

*Department of Computer Science and Applied Mathematics, National University of Water and Environmental Engineering, 11, Soborna st., Rivne, 33028, Ukraine, p.m.martyniuk@nuwm.edu.ua

†Department of Computer Science and Applied Mathematics, National University of Water and Environmental Engineering, 11, Soborna st., Rivne, 33028, Ukraine, o.v.ulianchukmartyniuk@nuwm.edu.ua

Complex processes that take place in the waste storage facilities can affect the physical, mechanical and chemical properties of geobarrier materials. The internal temperature in waste dumps can reach $55-60^oC$ due to the biochemical reactions present there [27]. Geobarriers are also mandatory components of radioactive waste and spent fuel storage [24] and are exposed to temperatures over 60^oC. Geobarriers are also used in bioreactors during the operation of which a significant amount of thermal energy is released [9]. Therefore, studying and modeling the behavior of geobarriers in non-isothermal conditions is a problem of practical importance.

One of the main issues of the impact of waste storage facilities on the environment is the spread of harmful substances due to their filtration through the soil base of the facilities into underground water. Therefore, the factors that affect the parameters of filtration through geobarriers should be taken into account both in field experiments and in predictive mathematical models. The importance of researching heat transfer processes in geobarriers in relation to the impact on the environment was shown in field experiments [4]. The interactions of thermal, hydrological and mechanical properties of porous media are manifested in the change in the filtration coefficient of the porous material of the geobarrier; presence of thermo-osmosis [10, 36]; dependence of the thermal conductivity coefficient of the porous medium on the porosity. For example, the problem of thermal conductivity of porous media from the point of view of the important task of burying radioactive waste which still emits thermal energy for many years is considered in [37]. Specifically, eight mathematical formula models for determining the thermal conductivity coefficient were analyzed in detail, depending on the characteristics of the porous medium (porosity, saturation, geometry of the structure). We will consider each particular factor separately on the examples of the analysis of scientific literature.

The results of experimental studies of the effect of temperature on the filtration coefficient of illite clay are presented in [27]. The research is summarized in the form of analytical dependence

$$k\left(T\right)=\frac{\rho_w\,g\,\exp\left(-30.894-0.0109\,T\right)}{0.2601+1.517\,\exp\left(-0.034688\,T\right)}.$$

Here $k\left(T\right)$ is the filtration coefficient (cm/s); ρ_w is the density of water; g is the gravity acceleration; T is temperature (^oC). For instance, when the temperature increases from $25°C$ to 60^oC, the filtration coefficient increases monotonously and non-linearly from $3.2\cdot10^{-9}cm/s$ to $4.5\cdot10^{-9}cm/s$.

A detailed review of experimental studies of thermo-osmotic properties of membrane systems is provided in [2]. It is the clay soils used in geobarriers that are known to have the properties of semipermeable membranes [7, 28]. A new theoretical explanation of thermo-osmotic filtration of solutions in clays is proposed in [11]. However, the authors do not deviate from the law for the rate of filtration known from the scientific literature which takes into account thermal osmosis:

$$q = -\frac{k}{\eta}\left(\nabla p + \rho g \nabla z\right) - \frac{k\nabla H}{\eta T}\nabla T,$$

where q is the pore fluid-specific discharge (m/s), k is the Darcy permeability (m^2), η is the dynamic viscosity $(Pa \cdot s)$, p is the pressure (Pa), ρ is the fluid density (kg/m^3), g is the gravity acceleration (m/s^2), ∇z is $(0,0,1)$ if the z axis is vertical upward, T is the temperature (K), and ∇H is the macroscopic volume-averaged excess specific enthalpy due to fluid-solid interactions (J/m^3). According to the equation, the thermo-osmotic permeability is $k_T = k\nabla H/T$ $(Pa \cdot m^2/K)$ [11]. Most often, $\nabla H > 0$ in clays, and fluid flow occurs in the direction of decreasing temperature, but negative values have also been reported.

Attention not just on the effect of thermal osmosis in the presence of a temperature gradient, but especially on the dependence of the thermo-osmotic coefficient on the porosity of the medium is focused in [12]. Qualitatively, the thermo-osmotic coefficient decreases monotonously with increasing porosity. However, in the vicinity of porosity values of 0.4, the authors observed an anomalous slight increase when porosity increases to 0.5. This effect is reasonably explained by the authors. It is important that quantitative indicators are also given (e.g. see Figure 8 of the work). For instance, the thermo-osmotic coefficient is approximately halved with an increase in porosity from 0.35 to 0.55.

An extensive review of thermal conductivity models of sands is presented in [16], and the model performance is evaluated for different types of sand, from dry to saturated. A total of 14 models were evaluated to predict the thermal conductivity of sands using a large data set collection consisting of 1025 measurements on 62 samples from 20 studies. According to the results of research, the authors selected two models of thermal conductivity of sands which show the best agreement with the data of field experiments. This article is important from the point of view that such studies are relevant not only for clays, but also for sands. After all, sands are used in nuclear power plant waste repositories, and for them, too, the issues of thermal conductivity and its non-linear dependence on the parameters of the porous medium (porosity and moisture saturation) are important.

The coefficient of thermal conductivity of the soil, as shown in [26], depends non-linearly on many factors. Among these factors, soil moisture, density, soil organic matter (SOM), as well as clay content, were singled out. The following trends were noted and quantified: the coefficient of thermal conductivity of the soil monotonously increases with humidity; the coefficient of thermal conductivity decreases with increasing clay content; the coefficient of thermal conductivity increases with an increase in SOM content. In particular, an analytical dependence is proposed

$$\lambda = -2.35 + 3.58\,S - 2.04\,S^2 + 1.82\,BD + 2.88\,SOM - 1.48\,clay\,S,$$

where λ is soil thermal conductivity, $[\lambda] = \frac{W}{m \cdot K}$; SOM is soil organic matter,

$[SOM] = 1$; BD is bulk density, $[BD] = \frac{g}{cm^3}$; $clay$ is relative content of clay particles, $[clay] = \frac{kg}{kg}$; S is degree of pore water saturation, $[S] = 1$. Similarly, experimental studies of thermal conductivity of sands, sandy loams and clays depending on humidity, the presence of salts ($NaCl$ and $CaCl_2$) and SOM were performed in [1]. The data of field experiments showed that these influencing factors cannot be neglected. For instance, for clays, the coefficient of thermal conductivity varies from 0.36 to $0.65 \frac{W}{m \cdot K}$ when the water content varies from 1.4 to 21.2 %.

Based on the analysis of scientific sources and data [35], eight models were selected for determining the thermal conductivity of a fully saturated porous medium, depending on the coefficients of thermal conductivity of solid particles, pore fluid and porosity. Qualitative conclusions from research are: 1) the thermal conductivity coefficient of the medium increases with water saturation; 2) thermal conductivity decreases with increasing porosity and vice versa. In the context of our problem, it is important that in the process of consolidation of a fully saturated porous medium, its porosity changes. Therefore, the coefficient of thermal conductivity will also change.

Mathematical modeling of various processes in porous media is also being developed [13, 14], taking into account the influence of non-isothermal conditions and thermal osmosis. The effect of thermal osmosis on the occurrence of excess pressures in the pores of a porous medium and the displacements of the skeleton of the porous medium resulting from the change in pressures and temperature were investigated in [3]. This was done by modifying Darcy's law, constructing the appropriate mathematical model and performing numerical experiments. The authors confirm that taking thermal osmosis into account can lead to negative pressures in the pore fluid. The effect of thermo-osmotic effects on pressure in the pore fluid of saturated clays was also modeled in [38]. Unfortunately, the authors disregarded that similar effects had been studied earlier [33, 34]. It is shown in all these works that under certain conditions thermal effects can change the field of pressures and concentrations of chemical substances. However, these reports do not take into account the presence of geobarriers. Additionally, the work [3] did not take into account the dependence of the filtration coefficient on temperature, and the study was conducted for homogeneous soils without the presence of geobarriers.

The importance of the phenomenon of thermal osmosis in the study of deformations of clay soils was proved in [21] on the basis of numerical simulations. However, the presence of thin geobarriers in a porous medium was not considered.

Analyzing the results of the above field experiments, the non-linear dependences of parameters and influencing factors should be noted (e.g. porosity, temperature, filtration coefficient, thermal conductivity coefficient). In the presence of thin inclusions, the conjugation conditions of non-ideal contact for inclusions should take into account the change in the physical and mechanical parameters of the inclusion under the effect of the studied factors. It was shown in [6] that the conjugation conditions of non-ideal contact can apply to the contact problems of heterogeneous media even without the presence of fine inclusions. Quite detailed studies of the effect of modified conjugation conditions in the presence of geobarriers on forecast distributions of moisture and excess pressures under the action of chemical and biological (bioclogging) factors were performed in [7, 18–20, 28–32]. However, problems with the effect of thermal factors have not yet been considered.

2. Formulation of the problem in the physical domain

Consider a soil massif with a total thickness of l which consists of two subregions Ω_1 and Ω_2. Moreover $\Omega_1 \cap \Omega_2 = \emptyset$. We consider the area $\Omega = \Omega_1 \cup \Omega_2$ to be inhomogeneous. By inhomogeneity, we mean the presence of a contact boundary $\omega = \overline{\Omega}_1 \cap \overline{\Omega}_2$ which, from a physical viewpoint, is a thin inclusion of the third material of the thickness d (Fig. 1). From a mathematical viewpoint, the thickness of the inclusion itself is neglected and the value d appears only in the so-called conjugation conditions of non-ideal contact for unknown functions.

Let us investigate the process of consolidation of this fully saturated porous medium in the region $\Omega = \Omega_1 \cup \Omega_2$ under the effect of temperature. The heterogeneous soil layer is considered the base of the waste repository, and the fine inclusion is the geobarrier. Soil consolidation is a consequence of applying an external load in the form of solid waste in storage. Chemical reactions in the storage result in the release of thermal energy. Therefore, the pressure $h(x,t)$ and temperature $T(x,t)$ functions are unknown. Although the processes in the geobarrier itself are not investigated, both the thickness of the inclusion and its characteristics appear in the conjugation conditions. The classical conjugation condition of non-ideal contact for pressures serves as an example [8, 25]

$$u^{\pm}\big|_{x=\xi} = -\frac{k_\omega}{d}\left(h^+ - h^-\right),$$

where $k_\omega = const$ is the filtration coefficient of the porous inclusion material, u is the filtration rate, h^+ and h^- are the values of the pressure at the inclusion at $x = \xi + 0$ and $x = \xi - 0$, respectively. However, the filtration coefficient depends on the parameters of the state of the environment (in the context of the considered problem, porosity and temperature). Then such dependences should be taken into account in conjugation conditions and their modification. The method of modification of the conjugation conditions and the conjugation conditions themselves for the cases of considering the effect of chemical and biological factors are given in [7, 29, 31, 32].

3. Mathematical model of the problem in the domain with a thin inclusion

The main elements of the mathematical model are known from the classical theory of filtration consolidation and heat transfer in porous media. However,

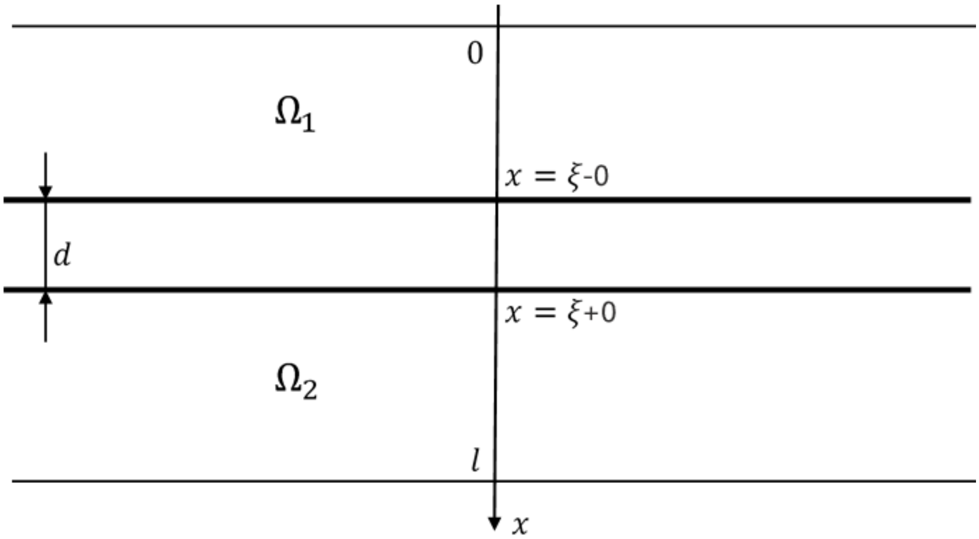


Fig. 2.1. A layer of soil of thickness l with a thin inclusion ω of thickness d ($d \ll l$).

some dependences will require further explanation and clarification. The interrelated process of changes in pressure and temperature of a completely saturated inhomogeneous soil mass in the one-dimensional case is described by the following boundary value problem:

$$\frac{\partial h}{\partial t} = \frac{1+e}{\gamma a}\frac{\partial}{\partial x}\left(k\left(h,T\right)\frac{\partial h}{\partial x} + \mu\left(h\right)\frac{\partial T}{\partial x}\right), \quad x \in \Omega_1 \cup \Omega_2, t > 0; \tag{3.1}$$

$$h(x,t)|_{x=0} = \overline{h}_0(t), t \geq 0; \tag{3.2}$$

$$u(x,t)|_{x=l} = \left(-k\left(h,T\right)\frac{\partial h}{\partial x} - \mu\left(h\right)\frac{\partial T}{\partial x}\right)\Bigg|_{x=l} = 0, t \geq 0; \tag{3.3}$$

$$h(x,0) = h_0(x), x \in \overline{\Omega}_1 \cup \overline{\Omega}_2; \tag{3.4}$$

$$c_s\left(h\right)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(h)\frac{\partial T}{\partial x}\right) - \rho_w c_w u\left(h,T\right)\frac{\partial T}{\partial x}, \quad x \in \Omega_1 \cup \Omega_2, t > 0; \tag{3.5}$$

$$T(x,t)|_{x=0} = \overline{T}_0(t), t \geq 0; \tag{3.6}$$

$$q_T(x,t)|_{x=l} = -\lambda(h)\frac{\partial T}{\partial x}\bigg|_{x=l} = 0, t \geq 0; \tag{3.7}$$

$$T(x,0) = T_0(x), \in \overline{\Omega}_1 \cup \overline{\Omega}_2; \tag{3.8}$$

$$u^{\pm}\big|_{x=\xi} = -\frac{[h]}{\int_0^d \frac{dx}{k_\omega(h,T)}} - \frac{[T]}{\int_0^d \frac{dx}{\mu_\omega(h)}}; \tag{3.9}$$

$$q_T^{\pm}\big|_{x=\xi} = -\frac{[T]}{\int_0^d \frac{dx}{\lambda_\omega(h)}}. \tag{3.10}$$

Here $\Omega_1 = (0;\xi)$, $\Omega_2 = (\xi;l)$, $0 < \xi < l$; $\overline{h}_0\,(t)$, $h_0\,(x)$, $\overline{T}_0\,(t)$, $T_0\,(x)$ are known functions; a is the soil compressibility coefficient; h is pressure; k, k_ω are the filtration coefficients of the main soil and inclusion soil, respectively; λ, λ_ω are the thermal conductivity coefficients of the main soil and inclusion soil, respectively; μ, μ_ω are the thermo-osmotic coefficients of of the main soil and inclusion soil, respectively; u is the filtration rate; e is the soil void ratio, with $e = \frac{n}{1-n}$, where n is the soil porosity; q_T is the thermal energy flow; $u^{\pm}$, $q_T^{\pm}$ are the values of filtration rates and flows at $x = \xi - 0$ and $x = \xi + 0$, respectively; $[h] = h^+ - h^-$, $[T] = T^+ - T^-$ are the pressure and temperature jumps on the thin inclusion; $c_s = \rho_w c_w n + \rho_{solid} c_{solid}(1-n)$ is the volume heat capacity coefficient of the soil; ρ_w, ρ_{solid} are the densities of pore fluid and solid soil particles; c_w, c_{solid} are specific heat capacities of pore fluid and solid soil particles.

Eq. (3.1) is the filtration consolidation equation for variable temperature and the presence of thermo-osmotic effects [33,34]. Conjugation conditions (3.9), (3.10) differ from classical ones [8,25] and take into account the dependence of the geobarrier filtration coefficient on porosity and temperature, the presence of thermal osmosis, as well as the dependence of the thermal conductivity coefficient on porosity. Conditions (3.9), (3.10) are derived similarly to those in [7,29,31,32].

We note that according to known field experiments the filtration, thermo-osmotic, and thermal conductivity coefficients depend on porosity n. However, in the consolidation problem, as explained in [20], $n = n\,(h)$. This is taken into account in the problem (3.1)–(3.10). Eqs. (3.1), (3.5) neglect internal sources (sinks) of pore fluid and thermal energy. Note that in (3.1)–(3.10) each of the functions $\alpha(x,t)$ is defined as

$$\alpha(x,t) = \{\alpha_i(x,t),\ x \in \Omega_i\ ,\quad i = 1,2.$$

Purely for convenience in this article $i = 2$. Generally, for $i > 2$ all calculations will be similar.

4. A system of quasi-linear equations of the parabolic type with homogeneous boundary conditions of the first kind and its generalized solution

To simplify the theoretical statements, which in principle do not reduce the generality of the problem, in problem (3.1)–(3.10) we will consider the boundary conditions of the first kind (3.2), (3.6) to be homogeneous for the time being. That is, let the conditions be fulfilled:

$$\overline{h}_0(t) \equiv 0, \overline{T}_0\,(t) \equiv 0, \quad t \geq 0. \tag{4.1}$$

How to take into account the inhomogeneity of boundary conditions of the first kind will be discussed in a separate section of the article.

Similarly to [8], we introduce the following notations: $Q_{\mathcal{T}} = \Omega \times (0;\mathcal{T}]$, $Q^1_{\mathcal{T}} = \Omega_1 \times (0;\mathcal{T}]$, $Q^2_{\mathcal{T}} = \Omega_2 \times (0;\mathcal{T}]$.

Suppose that the functions $h_0(x)$, $T_0(x)$ are continuous on each of the closures $\overline{\Omega}_1$, $\overline{\Omega}_2$. Also regarding the coefficients k, k_ω, μ, μ_ω, λ, λ_ω, c_s suppose that
1)

$$\begin{gathered} 0 < k_{min} \le k(s_1, s_2) \le k_{max} < \infty, \\ 0 < k_{\omega,min} \le k_\omega(s_1, s_2) \le k_{\omega,max} < \infty, \\ 0 < p_{min} \le p(s) \le p_{max} < \infty, \end{gathered}$$

for all $s,\ s_1,\ s_2 \in \mathbb{R}$; k_{min}, k_{max},$k_{\omega,min}$, $k_{\omega,max}$, p_{min}, p_{max} are positive constants; $p \in \{\mu,\ \lambda,\ \mu_\omega,\ \lambda_\omega, c_s\ \}$;
2)

$$\begin{gathered} \left|p\left(s_1\right) - p\left(s_2\right)\right| \le p_L \left|s_1 - s_2\right|,\ 0 < p_L < \infty; \\ \left|k\left(s_1, s_2\right) - k\left(s_1', s_2'\right)\right| \le k_L \left(\left|s_1 - s_1'\right| + \left|s_2 - s_2'\right|\right),\quad 0 < k_L < \infty; \\ \left|k_\omega\left(s_1, s_2\right) - k_\omega\left(s_1', s_2'\right)\right| \le k_{\omega,L} \left(\left|s_1 - s_1'\right| + \left|s_2 - s_2'\right|\right),\quad 0 < k_{\omega,L} < \infty; \end{gathered}$$

for all $s_1,\ s_2,\ s_1', s_2' \in \mathbb{R}$; $p \in \{\mu,\ \lambda,\ \mu_\omega,\ \lambda_\omega\}$.
3)

$$\begin{gathered} \left|u\left(s_1, s_2\right)\right| \le u_1,\ \forall s_1, s_2 \in \mathbb{R}; \\ \left|u\left(s_1, s_2\right) - u\left(s_1', s_2'\right)\right| \le u_L \left(\left|s_1 - s_1'\right| + \left|s_2 - s_2'\right|\right),\ 0 < u_1,\ u_L < \infty, \end{gathered}$$

for all $s_1,\ s_2,\ s_1', s_2' \in \mathbb{R}$.

Definition 4.1. The classical solution of the initial-boundary value problem (3.1)–(3.10), which admits a discontinuity of the first kind at the point $x = \xi$, is called a pair of functions $h(x,t) \in \Psi_h$, $T(x,t) \in \Psi_T$, which satisfy $\forall (x,t) \in \overline{Q}_{\mathcal{T}}$ equations (3.1), (3.5) and initial conditions (3.4), (3.8) respectively.

In the above definition Ψ_h, Ψ_T are the sets of functions $\psi_h(x,t)$, $\psi_T(x,t)$, which together with $\frac{\partial(\cdot)}{\partial x}$, are continuous on each of the closures $\overline{Q}^1_{\mathcal{T}}$, $\overline{Q}^2_{\mathcal{T}}$, have bounded continuous partial derivatives $\frac{\partial(\cdot)}{\partial t}$, $\frac{\partial^2(\cdot)}{\partial x^2}$ on $Q^1_{\mathcal{T}}$, $Q^2_{\mathcal{T}}$, and satisfy conditions (3.2), (3.3), (3.9) and (3.6), (3.7), (3.10) respectively.

For further explanations, similarly to the work [20], we note the following. Given condition 1), we get

$$k_{\omega,min} \frac{[h]}{d} \le \frac{[h]}{\int_0^d \frac{dx}{k_\omega(h,T)}} \le k_{\omega,max} \frac{[h]}{d},$$

$$\lambda_{\omega,min} \frac{[T]}{d} \le \frac{[T]}{\int_0^d \frac{dx}{\lambda_\omega(h)}} \le \lambda_{\omega,max} \frac{[T]}{d},$$

$$\mu_{\omega,min}\frac{[T]}{d} \leq \frac{[T]}{\int_0^d \frac{dx}{\mu_\omega(h)}} \leq \mu_{\omega,max}\frac{[T]}{d}.$$

The above estimates make it possible to apply well-known theoretical calculations [8, 25] for the classical conjugation condition (see [25], page 291, formula (7.4))

$$\left(\varkappa\left(x,u\right)\frac{\partial u}{\partial x}\right)\Bigg|_{x=\xi} = r\left[u\right],$$

in which r is some known constant; u is an unknown function. The classical conjugation condition and theoretical explanations for it require that

$$0 < r_0 \leq r < \infty.$$

Similarly to [8], let H_0 be the space of functions $s\left(x\right)$ that on each of the domains Ω_i belong to the Sobolev space $W_2^1\left(\Omega_i\right)$, $i = 1, 2$, and satisfy the condition

$$s(x)|_{x=0} = 0.$$

Let $h(x,t) \in \Psi_h$, $T(x,t) \in \Psi_T$ be the classical solution of the initial-boundary value problem (3.1)–(3.10). Take $s\left(x\right) \in H_0$. We multiply equation (3.1) and initial condition (3.4) by $s\left(x\right)$, and similarly, equation (3.5) and initial condition (3.8). Integrating them over the segment $[0; l]$ and taking into account the conjugation conditions (3.9), (3.10), we obtain

$$\int_0^l \frac{\gamma a}{1+e}\frac{\partial h}{\partial t}s\left(x\right)dx + \int_0^l k(h,T)\frac{\partial h}{\partial x}\frac{ds}{dx}\,dx + \int_0^l \mu(h)\frac{\partial T}{\partial x}\frac{ds}{dx}\,dx + \frac{[h]\,[s]}{\int_0^d \frac{dx}{k_\omega(h,T)}} + \frac{[T]\,[s]}{\int_0^d \frac{dx}{\mu_\omega(h)}} = 0, \tag{4.2}$$

$$\int_0^l h\left(x,0\right)s\left(x\right)\,dx = \int_0^l h_0\left(x\right)s\left(x\right)\,dx, \tag{4.3}$$

$$\int_0^l c_s\frac{\partial T}{\partial t}s\left(x\right)dx + \int_0^l \lambda(h)\frac{\partial T}{\partial x}\frac{ds}{dx}\,dx + \int_0^l \rho_w c_w u\frac{\partial T}{\partial x}s\left(x\right)\,dx + \frac{[T]\,[s]}{\int_0^d \frac{dx}{\lambda_\omega(h)}} = 0, \tag{4.4}$$

$$\int_0^l h\left(x,0\right)s\left(x\right)\,dx = \int_0^l h_0\left(x\right)s\left(x\right)\,dx. \tag{4.5}$$

Therefore, if $h(x,t) \in \Psi_h$, $T(x,t) \in \Psi_T$ is a classical solution of the initial-boundary value problem (3.1)–(3.10), then $h(x,t)$, $T(x,t)$ is the solution of the problem (4.2)–(4.5) in the weak formulation.

Let H be the space of functions $v\left(x,t\right)$ that are square integrable together with their first derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$ on each of the intervals $(0;\xi)$, $(\xi;l)$, $\forall t \in (0\,;\mathcal{T}]$, $\mathcal{T} > 0$, and they satisfy homogeneous boundary conditions of the first kind

$$v(x,t)|_{x=0} = 0, \quad t \geq 0.$$

Definition 4.2. The functions $h(x,t) \in H$, $T(x,t) \in H$, which for any $s\,(x) \in H_0$ satisfy the integral relation (4.2)–(4.5) are called the generalized solution of the initial-boundary problem (3.1)–(3.10) (if condition (4.1) is fulfilled).

5. Approximate generalized solution: its existence and uniqueness

An approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) will be sought in the form

$$\widehat{h}\,(x,t) = \sum_{i=1}^{N} h_i\,(t)\,\varphi_i\,(x), \quad \widehat{T}\,(x,t) = \sum_{i=1}^{N} T_i\,(t)\,\varphi_i\,(x), \tag{5.1}$$

where $\{\varphi_i(x)\}_{i=1}^{N}$ is the basis of the finite-dimensional subspace $M_0 \subset H_0$; $h_i\,(t)$, $T_i\,(t)$, $i = \overline{1,N}$ are unknown coefficients that depend only on time.

A set of functions that can be represented in the form (5.1) generate a finite-dimensional subspace $M \subset H$.

Definition 5.1. An approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) is a pair of functions $\widehat{h}(x,t) \in M$, $\widehat{T}(x,t) \in M$ which for an arbitrary function $S(x) \in M_0$ satisfy the integral relations

$$\int_0^l \frac{\gamma a}{1+e}\frac{\partial \widehat{h}}{\partial t} S\,(x)\;dx + \int_0^l k(\widehat{h},\widehat{T})\frac{\partial \widehat{h}}{\partial x}\frac{dS}{dx}\,dx + \int_0^l \mu(\widehat{h})\frac{\partial \widehat{T}}{\partial x}\frac{dS}{dx}\,dx + \frac{\left[\widehat{h}\right][S]}{\int_0^d \frac{dx}{k_\omega(\widehat{h},\widehat{T})}} + \frac{\left[\widehat{T}\right][S]}{\int_0^d \frac{dx}{\mu_\omega(\widehat{h})}} = 0, \tag{5.2}$$

$$\int_0^l \widehat{h}\,(x,0)\,S\,(x)\;dx = \int_0^l h_0\,(x)\,S\,(x)\;dx, \tag{5.3}$$

$$\int_0^l c_s\frac{\partial \widehat{T}}{\partial t} S\,(x)\;dx + \int_0^l \lambda(\widehat{h})\frac{\partial \widehat{T}}{\partial x}\frac{dS}{dx}\,dx + \int_0^l \rho_w c_w u\frac{\partial \widehat{T}}{\partial x} S\,(x)\;dx + \frac{\left[\widehat{T}\right][S]}{\int_0^d \frac{dx}{\lambda_\omega(\widehat{h})}} = 0, \tag{5.4}$$

$$\int_0^l \widehat{T}\,(x,0)\,S\,(x)\;dx = \int_0^l T_0\,(x)\,S\,(x)\;dx. \tag{5.5}$$

Next, from the weak formulation (5.2)–(5.5) of the problem (3.1)–(3.10), taking into account (5.1) (setting the function $S(x)$ equal to each basis function $\varphi_i(x)$, $i = \overline{1,N}$), we obtain the Cauchy problem for the system of non-linear differential equations

$$\mathbf{M_1}\left(\mathbf{H}\right)\frac{d\mathbf{H}}{dt}+\mathbf{L_1}\left(\mathbf{H},\mathbf{T}\right)\mathbf{H}\left(t\right)+\mathbf{L_{12}}\left(\mathbf{H}\right)\mathbf{T}\left(t\right)=\mathbf{0}, \tag{5.6}$$

$$\widetilde{\mathbf{M}}_{\mathbf{1}}\mathbf{H}^{(\mathbf{0})}=\widetilde{\mathbf{F}}_{\mathbf{1}}, \tag{5.7}$$

$$\mathbf{M_2}\left(\mathbf{H}\right)\frac{d\mathbf{T}}{dt}+\mathbf{L_2}\left(\mathbf{H},\mathbf{T}\right)\mathbf{T}\left(t\right)=\mathbf{0}, \tag{5.8}$$

$$\widetilde{\mathbf{M}}_{\mathbf{2}}\mathbf{T}^{(\mathbf{0})}=\widetilde{\mathbf{F}}_{\mathbf{2}}, \tag{5.9}$$

where

$$\widetilde{\mathbf{F}}_k=\left(\widetilde{f}_i^{(k)}\right)_{i=1}^N,\widetilde{\mathbf{M}}_k=\left(\widetilde{m}_{ij}^{(k)}\right)_{i,j=1}^N,\mathbf{M}_k=\left(m_{ij}^{(k)}\right)_{i,j=1}^N,\mathbf{L}_k=\left(l_{ij}^{(k)}\right)_{i,j=1}^N,k=1,2;$$
$$\mathbf{L}_{12}=\left(l_{ij}^{(12)}\right)_{i,j=1}^N\widetilde{m}_{ij}^{(k)}=\int_0^l\varphi_i\varphi_j\,dx,\ \ \widetilde{f}_i^{(1)}=\int_0^0 h_0\varphi_i\,dx,\widetilde{f}_i^{(2)}=\int_0^0 T_0\varphi_i\,dx,$$
$$\mathbf{H}=\left(h_i\left(t\right)\right)_{i=1}^N,\mathbf{T}=\left(T_i\left(t\right)\right)_{i=1}^N,\mathbf{H}^{(\mathbf{0})}=\left(h_i\left(0\right)\right)_{i=1}^N,\mathbf{T}^{(\mathbf{0})}=\left(T_i\left(0\right)\right)_{i=1}^N,$$
$$m_{ij}^{(1)}=\int_0^l\frac{\gamma a}{1+e}\varphi_i\varphi_j\,dx,l_{ij}^{(1)}=\int_0^l k\left(\widehat{h},\widehat{T}\right)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx+\frac{\left[\varphi_i\right]\left[\varphi_j\right]}{\int_0^d\frac{dx}{k_\omega(\widehat{h},\widehat{T})}},$$
$$l_{ij}^{(12)}=\int_0^l\mu\left(\widehat{h}\right)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx+\frac{\left[\varphi_i\right]\left[\varphi_j\right]}{\int_0^d\frac{dx}{\mu_\omega(\widehat{h})}},m_{ij}^{(2)}=\int_0^l c_s\varphi_i\varphi_j\,dx,$$
$$l_{ij}^{(2)}=\int_0^l\lambda(\widehat{h})\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx+\int_0^l\rho_w c_w u\frac{d\varphi_j}{dx}\varphi_i\,dx+\frac{\left[\varphi_i\right]\left[\varphi_j\right]}{\int_0^d\frac{dx}{\lambda_\omega(\widehat{h})}}.$$

The system of equations (5.6), (5.8) can be written in the form

$$\mathbf{M}\frac{d\mathbf{V}}{dt}+\mathbf{L}\left(\mathbf{V}\right)\mathbf{V}\left(t\right)=\mathbf{0}, \tag{5.10}$$

where

$$\mathbf{M}=\begin{pmatrix}\mathbf{M_1}&\mathbf{0}\\\mathbf{0}&\mathbf{M_2}\end{pmatrix},\quad\mathbf{V}=\begin{pmatrix}\mathbf{H}\\\mathbf{T}\end{pmatrix},\quad\mathbf{M}=\begin{pmatrix}\mathbf{L_1}&\mathbf{L_{12}}\\\mathbf{0}&\mathbf{L_2}\end{pmatrix}.$$

If the above conditions 1)–3) are met, the square matrix $\mathbf{M}$ is symmetric and positive definite, $\forall(x,t)\in\overline{Q}_{\mathcal{T}}$. Therefore, there exists a unique inverse matrix $\mathbf{M^{-1}}$. Then we write (5.10) in the form

$$\frac{d\mathbf{V}}{dt}=\mathbf{\Phi}\left(\mathbf{V}\right), \tag{5.11}$$

where $\mathbf{\Phi}\left(\mathbf{V}\right)=-\mathbf{M^{-1}}\mathbf{L}\left(\mathbf{V}\right)\mathbf{V}\left(t\right)$. Despite the fact that the matrix $\mathbf{L}\left(\mathbf{V}\right)$ will not be symmetric and positive definite, the functions $\mathbf{\Phi}\left(\mathbf{H}\right)$, $\frac{\partial\mathbf{\Phi}}{\partial\mathbf{V}}$ will be continuous and bounded. Then, similarly to [8] (Chapter 8, point 6), the solution $\mathbf{V}$ of the

Cauchy problem for the system of equations (5.11) exists and is unique. That is, there exists a single approximate generalized solution of the problem (3.1)–(3.10) with homogeneous boundary conditions of the first kind.

Let us introduce the following norms [8, page 380]:

$$\|u\|_{L_2}^2 = \int_0^l u^2(x,t)\,dx, \|u\|_{H_0^1}^2 = \left\|\frac{\partial u}{\partial x}\right\|_{L_2}^2, \|u\|_{L_2\times L_2}^2 = \|u\|_{L_2(Q_T)}^2 = \int_0^{\mathcal{T}}\int_0^l u^2\,dxdt,$$

$$\|u\|_{H_0^1\times L_2}^2 = \int_0^{\mathcal{T}} \|u\|_{H_0^1}^2\,dt = \int_0^{\mathcal{T}}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dxdt, \|u\|_{L_2\times L_\infty} = \sup_{t\in(0,\mathcal{T}]} \|u(\cdot,t)\|_{L_2},$$

$$\|\nabla_x u\|_{L_\infty\times L_\infty} = \sup_{(x,t)\in Q_{\mathcal{T}}}\left|\frac{\partial u(x,t)}{\partial x}\right|, \|u\|_{W_2^1\times L_2}^2 = \int_0^{\mathcal{T}}\int_0^l \left(u^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) dxdt,$$

$$\|[u]\|_{L_2}^2 = \int_0^{\mathcal{T}} [u]^2\,dt = \int_0^{\mathcal{T}} (u(\xi+0,t) - u(\xi-0,t))^2\,dt.$$

Theorem 5.1. *Let $h(x,t)$, $T(x,t)$ be the generalized solution of the initial-boundary value problem* (3.1)–(3.10)*, and $\widehat{h}\,(x,t)$, $\widehat{T}(x,t)$ be the approximate generalized solution of this problem. Then, if conditions 1)–3) are fulfilled, there exist such positive constant values c, δ_1, δ_2, that for arbitrary functions $\tilde{h}\,(x,t) \in M$, $\tilde{T}\,(x,t) \in M$ the inequality holds*

$$\begin{aligned}\left\|h-\widehat{h}\right\|_{L_2\times L_\infty} + \left\|T-\widehat{T}\right\|_{L_2\times L_\infty} + \delta_1\left(\left\|h-\widehat{h}\right\|_{H_0^1\times L_2} + \left\|T-\widehat{T}\right\|_{H_0^1\times L_2}\right) \\ +\delta_2\left(\left\|\left[h-\widehat{h}\right]\right\|_{L_2} + \left\|\left[T-\widehat{T}\right]\right\|_{L_2}\right) \le c\left(\left\|h-\tilde{h}\right\|_{L_2\times L_\infty}\right. \\ +\left\|h-\tilde{h}\right\|_{H_0^1\times L_2} + \left\|\left[h-\tilde{h}\right]\right\|_{L_2} + \left\|\frac{\partial\left(h-\tilde{h}\right)}{\partial t}\right\|_{L_2\times L_2} + \left\|T-\tilde{T}\right\|_{L_2\times L_\infty} \\ \left. +\left\|T-\tilde{T}\right\|_{H_0^1\times L_2} + \left\|\left[T-\tilde{T}\right]\right\|_{L_2} + \left\|\frac{\partial\left(T-\tilde{T}\right)}{\partial t}\right\|_{L_2\times L_2}\right)\end{aligned} \tag{5.12}$$

Proof. The theorem is proved similarly to [8, p. 380, Theorem 1; p. 438, Theorem 19]. However, there is one difference. At the initial stage of the proof, the equality obtained from (4.2), (5.2) for the functions $\left(h-\widehat{h}\right)$, $\left(\tilde{h}-\widehat{h}\right)$ and the equality obtained from (4.4), (5.4) for the functions $\left(T-\widehat{T}\right)$, $\left(\tilde{T}-\widehat{T}\right)$ must be added. As a result, estimate (5.12) is cumulative with respect to both functions $h(x,t)$, $T(x,t)$. The estimate (5.12) can be generalized to an arbitrary finite number of functions as a generalized solution of the initial-boundary value problem, the structure of which coincides with (3.1)–(3.10). □

Inequality (5.12) is used in estimating the accuracy of the finite element method.

6. Finite element method

Cover the region $\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with a finite element mesh with the total number of nodes N. Moreover, the point $x = \xi$ should have double numbering, of the node on the left $x = \xi - 0$ and the node on the right $x = \xi + 0$. Let in (5.1) $\varphi_i(x)$ be the basis functions of the finite element method which admit a discontinuity of the first kind at a point $x = \xi$ and are polynomials of degree m. Then the space of functions of the form (5.1) with the indicated basis functions is denoted by H_m^N.

Theorem 6.1. *Let the classical solution $h(x,t)$, $T(x,t)$ of the boundary value problem* (3.1)–(3.10) *have partial derivatives $\frac{\partial^{m+1}(\cdot)}{\partial x^{m+1}}$, $\frac{\partial^{m+2}(\cdot)}{\partial x^{m+1}\partial t}$, that are bounded on $Q_{\mathcal{T}}^i$, $i = 1, 2$. Then for the approximate generalized solution $\widehat{h}(x,t) \in H_m^N$, $\widehat{T}(x,t) \in H_m^N$, the following estimate is valid:*

$$\left\| h - \widehat{h} \right\|_{W_2^1 \times L_2} + \left\| T - \widehat{T} \right\|_{W_2^1 \times L_2} \leq c\, h_{max}^m,$$

where m is the degree of finite element polynomials, $c = const > 0$, $h_{max} = \max_{i=\overline{0,N-1}} (x_{i+1} - x_i)$, $[x_{i+1}; x_i]$ are finite elements.

Proof. The validity of the theorem follows from the estimate (5.12) of the previous theorem, taking into account the interpolation estimates [8, p. 387, Theorem 2]. Note that if the basis functions of different degrees m_1 and m_2 are used for the sought $\widehat{h}(x,t)$, $\widehat{T}(x,t)$, then $m = min\,(m_1; m_2)$. □

Problem (5.6)–(5.9) is a Cauchy problem for a system of non-linear differential equations of the first order. Finding its solution also requires the use of appropriate discretization schemes. [8, 25] substantiate the use of the Crank-Nicolson method

$$\mathbf{M_1}\left(\mathbf{H}^{(j+\frac{1}{2})}\right)\frac{\mathbf{H}^{(j+1)} - \mathbf{H}^{(j)}}{\tau} + \mathbf{L_1}\left(\mathbf{H}^{(j+\frac{1}{2})}, \mathbf{T}^{(j+\frac{1}{2})}\right)\mathbf{H}^{(j+\frac{1}{2})} + \mathbf{L_{12}}\left(\mathbf{H}^{(j+\frac{1}{2})}\right)\mathbf{T}^{(j+\frac{1}{2})} = \mathbf{0},$$

$$\mathbf{M_2}\left(\mathbf{H}^{(j+\frac{1}{2})}\right)\frac{\mathbf{T}^{(j+1)} - \mathbf{T}^{(j)}}{\tau} + \mathbf{L_2}\left(\mathbf{H}^{(j+\frac{1}{2})}, \mathbf{T}^{(j+\frac{1}{2})}\right)\mathbf{T}^{(j+\frac{1}{2})} = \mathbf{0},$$

$j = 0, 1, 2, ..., m_\tau - 1$. Here time segment $[0, \mathcal{T}]$ is split into m_τ equal parts with step $\tau = \frac{\mathcal{T}}{m_\tau}$; $\mathbf{H}^{(j)}$, $\mathbf{T}^{(j)}$ is the approximate solution of the Cauchy problem for $t = j\tau$, $\mathbf{H}^{(j+\frac{1}{2})} = \frac{1}{2}\left(\mathbf{H}^{(j+1)} + \mathbf{H}^{(j)}\right)$, $\mathbf{T}^{(j+\frac{1}{2})} = \frac{1}{2}\left(\mathbf{T}^{(j+1)} + \mathbf{T}^{(j)}\right)$. We also introduce the following notations: $h^{(j)}$, $T^{(j)}$ is the classical solution of the initial-boundary value problem (3.1)–(3.10) for $t = j\tau$; $\widehat{h}^{(j)}$, $\widehat{T}^{(j)}$ is an approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) for $t = j\tau$; $\phi^{\left(j+\frac{1}{2}\right)} = \frac{1}{2}(\phi^{(j+1)} + \phi^{(j)})$; $z_h^{(j)} = h^{(j)} - \widehat{h}^{(j)}$, $z_T^{(j)} = T^{(j)} - \widehat{T}^{(j)}$.

Similarly to Theorem 5 [8, Chap. 8],

Theorem 6.2. *Let $h(x,t)$, $T(x,t)$ be the classical solution of the initial boundary value problem* (3.1)–(3.10)*. Let the first derivatives $\frac{\partial(\cdot)}{\partial t}$, $\frac{\partial(\cdot)}{\partial x}$ of the classical solution be twice continuously differentiable with respect to time on $\overline{Q}^i_{\mathcal{T}}$, $i = 1, 2$. Also assume that the derivatives $\frac{\partial^3(\cdot)}{\partial t^3}$, $\frac{\partial^3(\cdot)}{\partial t^2 \partial x}$ are uniformly bounded in modulus by a constant c_1 $\forall(x,t) \in \overline{Q}_{\mathcal{T}}$. If conditions 1)–3) are fulfilled, then there exist positive constants c, δ_1, r_0, τ_0, which depend on the constants from conditions 1)–3), as well as $\mathcal{T}$, l, such that $\forall \tau \le \tau_0$ for the classical solution $h(x,t)$, $T(x,t)$ and for the approximate generalized solution $\widehat{h}(x,t) \in M$, $\widehat{T}(x,t) \in M$ obtained using the Crank-Nicolson method, of the problems* (3.1)–(3.10) *and* (5.6)–(5.9)*, respectively, the following inequality is valid:*

$$
\begin{aligned}
&\left\| z_h^{(m_\tau)} \right\|^2_{L_2} + \left\| z_T^{(m_\tau)} \right\|^2_{L_2} + \delta_1 \tau \left(\sum_{j=0}^{m_\tau - 1} \left\| z_h^{\left(j+\frac{1}{2}\right)} \right\|^2_{H_0^1} + \sum_{j=0}^{m_\tau - 1} \left\| z_T^{\left(j+\frac{1}{2}\right)} \right\|^2_{H_0^1} \right) \\
&\qquad + r_0 \tau \left(\sum_{j=0}^{m_\tau - 1} \left[z_h^{\left(j+\frac{1}{2}\right)} \right]^2 + \sum_{j=0}^{m_\tau - 1} \left[z_T^{\left(j+\frac{1}{2}\right)} \right]^2 \right) \\
&\le c \Biggl(\tau \sum_{j=0}^{m_\tau - 1} \left\| \left(h - \tilde{h}\right)^{\left(j+\frac{1}{2}\right)} \right\|^2_{H_0^1} + \tau \sum_{j=0}^{m_\tau - 1} \left\| \left(T - \tilde{T}\right)^{\left(j+\frac{1}{2}\right)} \right\|^2_{H_0^1} \\
&\qquad + \tau \sum_{j=1}^{m_\tau - 1} \left\| \frac{\left(h - \tilde{h}\right)^{\left(j+\frac{1}{2}\right)} - \left(h - \tilde{h}\right)^{\left(j-\frac{1}{2}\right)}}{\tau} \right\|^2_{L_2} \\
&\qquad + \tau \sum_{j=1}^{m_\tau - 1} \left\| \frac{\left(T - \tilde{T}\right)^{\left(j+\frac{1}{2}\right)} - \left(T - \tilde{T}\right)^{\left(j-\frac{1}{2}\right)}}{\tau} \right\|^2_{L_2} \\
&\quad + \tau \sum_{j=0}^{m_\tau - 1} \left[\left(h - \tilde{h}\right)^{\left(j+\frac{1}{2}\right)} \right]^2 + \left\| \left(h - \tilde{h}\right)^{(0)} \right\|^2_{L_2} + \left\| \left(h - \tilde{h}\right)^{\left(m_\tau - \frac{1}{2}\right)} \right\|^2_{L_2} \\
&\qquad + \left\| \left(h - \tilde{h}\right)^{\left(\frac{1}{2}\right)} \right\|^2_{L_2} \\
&\quad + \tau \sum_{j=0}^{m_\tau - 1} \left[\left(T - \tilde{T}\right)^{\left(j+\frac{1}{2}\right)} \right]^2 + \left\| \left(T - \tilde{T}\right)^{(0)} \right\|^2_{L_2} + \left\| \left(T - \tilde{T}\right)^{\left(m_\tau - \frac{1}{2}\right)} \right\|^2_{L_2}
\end{aligned}
$$

$$+\left\|\left(T-\tilde{T}\right)^{\left(\frac{1}{2}\right)}\right\|_{L_2}^2+O\left(\tau^4\right)\Bigg),\quad \forall\,\tilde{h}\in M,\forall\,\tilde{T}\in M. \tag{6.1}$$

Similarly to [8, Theorem 6, Chap. 8], taking into account estimate (6.1), it holds

Theorem 6.3. *Let the classical solution $h(x,t)$, $T(x,t)$ of the problem* (3.1)–(3.10) *satisfy the conditions of Theorem 6.2. Then for the errors z of the approximate generalized solution $\widehat{h}(x,t)\in H_m^N$, $\widehat{T}(x,t)\in H_m^N$ of the problem* (5.6)–(5.9) *obtained using the Crank-Nicolson method, the following estimate is valid:*

$$\left\|z_h^{(m_\tau)}\right\|_{L_2}^2+\left\|z_T^{(m_\tau)}\right\|_{L_2}^2+\delta_1\tau\left(\sum_{j=0}^{m_\tau-1}\left\|z_h^{\left(j+\frac{1}{2}\right)}\right\|_{H_0^1}^2+\sum_{j=0}^{m_\tau-1}\left\|z_T^{\left(j+\frac{1}{2}\right)}\right\|_{H_0^1}^2\right)$$
$$\leq c\cdot\left(h_{max}^{2m}+\tau^4\right).$$

7. Using finite element method in the case of inhomogeneous boundary conditions of the first kind

We now abandon the assumptions that the boundary conditions of the first kind (3.2), (3.6) are homogeneous. Then the approximate generalized solution of the initial-boundary value problem (3.1)–(3.10) is sought in a slightly modified form compared to (5.1)

$$\begin{aligned}\widehat{h}\left(x,t\right)&=\sum_{i=1}^{N}h_i\left(t\right)\varphi_i\left(x\right)+W_h\left(x,t\right),\\ \widehat{T}\left(x,t\right)&=\sum_{i=1}^{N}T_i\left(t\right)\varphi_i\left(x\right)+W_T\left(x,t\right),\end{aligned}\tag{7.1}$$

where $W_h\left(x,t\right)$, $W_T\left(x,t\right)$ are some known functions such that

$$W_h\left(x,t\right)|_{x=0}=\overline{h}_0(t),\quad W_T\left(x,t\right)|_{x=0}=\overline{T}_0(t),t\geq 0. \tag{7.2}$$

As a result, all theoretical statements, including the formulation and proof of Theorems 5.1-6.3, will not change in substance. However, they will become more bulky. Given conditions (7.2), the presence of the functions $W_h\left(x,t\right)$, $W_T\left(x,t\right)$ will not affect the accuracy estimates in Theorems 6.1 and 6.3.

In the practical application of the finite element method functions $W_h\left(x,t\right)$, $W_T\left(x,t\right)$ are approximated by expressions

$$W_h\left(x,t\right)\approx h_0\left(t\right)\varphi_0\left(x\right),\quad W_h\left(x,t\right)\approx T_0\left(t\right)\varphi_0\left(x\right), \tag{7.3}$$

where $\varphi_0(x)$ is the piecewise polynomial basis function of the finite element method defined at the node $x = 0$. Next, substituting (7.3) into (7.1), we get

$$\widehat{h}(x,t) = \sum_{i=0}^{N} h_i(t)\,\varphi_i(x), \quad \widehat{T}(x,t) = \sum_{i=0}^{N} T_i(t)\,\varphi_i(x).$$

Given that according to the properties of the basis functions of the finite element method

$$\varphi_0(x)|_{x=0} = 1$$

from (7.2) and (7.3) we obtain

$$h_0(t) = \overline{h}_0(t), \quad T_0(t) = \overline{T}_0(t), t \geq 0.$$

8. Results of numerical experiments

Soil parameters for the numerical experiments were taken from the Hydrus-1D freeware [15]. Specifically, sandy-clay loam was considered as the main soil, with $k_0 = 0.108\ \frac{m}{day}$, $n_0 = 0.45$. Clay with the following parameters was used as the fine inclusion soil: $k_{\omega,0} = 0.0048\ \frac{m}{day}$, $n_{\omega,0} = 0.36$.

Chung and Horton model was used the dependence of thermal conductivity coefficient of saturated soil on porosity according to [23]. In this model $\lambda = p_1 + p_2 n + p_3\sqrt{n}$, where $p_1 = b_1,\ p_2 = b_2,\ p_3 = b_2\sqrt{0.75n + 2b_1/b_2}$. According to Hydrus-1D models [15], for clays $b_1 = 17020.86\frac{J}{day\cdot m\cdot{}^\circ C}$, $b_2 = 83676.27\frac{J}{day\cdot m\cdot{}^\circ C}$, and for loams $b_1 = 20995.15\frac{J}{day\cdot m\cdot{}^\circ C}$, $b_2 = 33955.17\frac{J}{day\cdot m\cdot{}^\circ C}$. Also, we used for both clays and loams, similar to Hydrus-1D,

$$\rho_w c_w = 419995.50\frac{J}{m^3\cdot{}^\circ C},\ \rho_{solid} c_{solid} = 1919996.90\frac{J}{m^3\cdot{}^\circ C}.$$

Regarding the dependence of the filtration coefficient on temperature, the experimental dependences of [27] were used. The value of the filtration coefficient at $T = 20^\circ C$ was taken as a standard. Then, using the Kozeny-Carman formula [20],

$$k(n,T) = k_0\frac{\overline{k}(T)}{\overline{k}(20^\circ C)}\frac{1+e_0}{1+e}\left(\frac{e}{e_0}\right)^3.$$

Here k_0, e_0 are the initial values of the filtration coefficient and the void ratio; k, e are their variable values over time, with $e = \frac{n}{1-n}$, and according to [27]

$$\overline{k}(T) = \frac{\rho_w\, g\, \exp(-30.894 - 0.0109\,T)}{0.2601 + 1.517\,\exp(-0.034688\,T)}.$$

The conclusions of [12] were used for the dependence of thermo-osmotic coefficient on porosity. Particularly, in the model example

$$\mu_\omega(n_\omega) = \begin{cases} 2\mu_{\omega,0} & ,\ n_\omega < 0.75,\ n_{\omega,0}; \\ \mu_{\omega,0} & ,\ 0.75\,n_{\omega,0} \leq n_\omega \leq 1.25\,n_{\omega,0}; \\ 0.5\mu_{\omega,0} & ,\ n_\omega > 1.25\,n_{\omega,0}. \end{cases}$$

The initial value of the thermo-osmotic coefficient was taken as 10% of the value of the filtration coefficient.

The model problem considered a soil layer of $l = 10$ m thickness. The depth of the inclusion $\xi = 2$ m, and its thickness $d = 0.2$ m. The variable x step was 0.02 m. Time step $\tau = 3$ day. Initial pressure distribution $h_0(x) = 20$ m. Initial temperature distribution $T_0(x) = 14°C$. Functions in boundary conditions of the first kind on the soil surface $\overline{h}_0(t) = 0$ m, $\overline{T}_0(t) = 55$ $°C$. Boundary conditions of the second kind were set at the lower boundary. The results of numerical experiments are shown in Table 1 and Table 2. Case I corresponds to numerical experiments for the problem of filtration consolidation under conditions of variable porosity, but without taking into account the effect of temperature. Case II additionally considers the phenomena of thermal osmosis for the geobarrier and the effect of temperature on the hydraulic conductivity parameters of the entire porous media. Note that temperature values and jumps for both cases are practically the same. Such temperature values will differ if the classical conjugation condition is used, without accounting for the effect of variable porosity on the thermal conductivity coefficient.

Taking into account the effects of temperature changes the values of pressure below and above the inclusion, as well as their jumps. Such changes vary within 10% of the values for the isothermal case. By the time of 240 *days*, the pressure jump for Case II is greater than when neglecting thermal effect. However, later, as the soil warms up (because the temperature at the upper limit reaches 55 $°C$), such jumps become smaller. That is, the effect of temperature on the $[h]$ values will not be uniform (definite and predictable increase or decrease of pressure jumps compared to classical and isothermal cases). The following points should also be noted: 1. The values of pressure from the bottom and top of the inclusion are always smaller for the case of non-isothermal conditions. 2. If we take into consideration the classical conjugation condition with constant inclusion parameters, the pressures and their jumps will differ both for Case I and for Case II. 3. Temperature variation at the upper limit will lead to pressure fluctuations, the amplitude of which will depend on the ratio between the filtration and thermo-osmotic coefficients.

The results of the experiments in the assumption that the excess pressures in the area of the soil mass has already dissipated, i.e. the initial pressure distribution $h_0(x) = 0$ m, are also informative. The presence of non-isothermal conditions and taking into account thermo-osmotic properties of the geobarrier material leads to the appearance of a stable pressure jump in the vicinity of a thin inclusion. The pressures above the inclusion become negative, and under the inclusion, positive. If the value of the thermo-osmotic coefficient is 10% of the value of the filtration coefficient, then the maximum pressure jump is 11 cm, increasing to 17 cm if it is 20%. If the values of the filtration and thermo-osmotic coefficients are equal, the maximum pressure jump reaches fully 1.14 m. Such a situation is dangerous in case of the complex geometry of the base of a waste storage facility and the presence of slopes. After all, the stability of slopes decreases in the presence of high humidity and stable pressure fields [17].

Generally, the results of the model examples show that the distribution of pressures in soil structures and natural masses of porous media with fine inclusions depends on the temperature factor. The quantitative indicators of such effects may vary, depending on the ratio of the values of thermo-osmotic and filtration coefficients.

Table 1.Results of numerical experiments - Case I

Time moment	h^-	h^+	$[h]$
$t = 30$ *days*	6,40	12,99	6,59
$t = 60$ *days*	6,20	11,17	4,97
$t = 120$ *days*	6,06	9,86	3,80
$t = 180$ *days*	5,81	9,16	3,35
$t = 240$ *days*	5,50	8,60	3,10
$t = 360$ *days*	4,93	7,71	2,78
$t = 540$ *days*	4,24	6,65	2,41
$t = 720$ *days*	3,70	5,80	2,10
$t = 900$ *days*	3,27	5,08	1,81
$t = 1080$ *days*	2,91	4,48	1,57

Table 2.Results of numerical experiments - Case II

Time moment	h^-	h^+	$[h]$
$t = 30$ *days*	5,90	13,02	7,12
$t = 60$ *days*	5,73	10,98	5,25
$t = 120$ *days*	5,31	9,44	4,13
$t = 180$ *days*	4,91	8,50	3,59
$t = 240$ *days*	4,65	7,86	3,21
$t = 360$ *days*	4,20	6,97	2,77
$t = 540$ *days*	3,60	5,94	2,34
$t = 720$ *days*	3,12	5,10	1,98
$t = 900$ *days*	2,72	4,39	1,67
$t = 1080$ *days*	2,40	3,80	1,40

References

1. N.H. Abu-Hamdeh, R.C. Reeder, *Soil Thermal Conductivity Effects of Density, Moisture, Salt Concentration, and Organic Matter*, Soil Science Society of America Journal, **64**(4) (2000), 1285–1290.
2. V.Maria Barragan, Signe Kjelstrup, *Thermo-osmosis in Membrane Systems: A Review*, Journal of Non-Equilibrium Thermodynamics, **42**(3) (2017), 217–236.
3. Bin Zhu, Zhigang Ye, Lujun Wang, Wenjie Xu, Deqiong Kong, Thomas Nagel, Olaf Kolditz, Yunmin Chen, *Theoretical Investigation into Thermo-Osmosis and Thermofiltration Effects on Hydromechanical Behavior of Saturated Soils*, Journal of Engineering Mechanics, **147**(4) (2021), 04021005.

4. A. Bouazza, M. Asgar Ali, R. Kerry Rowe, Will P. Gates, A. El-Zein, *Heat mitigation in geosynthetic composite liners exposed to elevated temperatures*, Geotextiles and Geomembranes, **45**(5) (2017), 406–417.
5. Bowei Yu, Abbas El-Zein, *Challenges and solutions in using geosynthetic clay liners exposed to thermal desiccation risks*, Japanese Geotechnical Society Special Publication, **9**(2) (2021), 61–66.
6. O.Y. Chernukha, Y.I. Bilushchak, *Mathematical Modeling of the Mean Concentration Field in Random Stratified Structures with Regard for the Jumps of Sought Function on the Interfaces*, J Math Sci,**240** (2019), 70–85.
7. Yu. Chui, P. Martynyuk, M. Kuzlo, O. Ulianchuk-Martyniuk, *The conditions of conjugation in the tasks of moisture transfer on a thin clay inclusion taking into account salt salutions and temperature*, J. of Theoretical and Applied Mechanics(Bulgaria), **49**(1) (2019), 28–38.
8. V.S. Deineka, I.V. Sergienko, V.V. Skopetsky, *Models and methods for solving problems with conjugation conditions*, Naukova dumka, Kiev, 1998.
9. G. Dolle, O. Duran, N. Feyeux, E. Frenod, M. Giacomini, C. Prud'homme, *Mathematical modeling and numerical simulation of a bioreactor landfill using Feel++*, ESAIM: Proceedings and Surveys, **55**(1) (2017), 83–110.
10. J. Goncalves, G. de Marsily, J. Tremosa, *Importance of thermo-osmosis for fluid flow and transport in clay formations hosting a nuclear waste repository*, Earth and Planetary Science Letters, **339-340** (2012), 1–10.
11. J. Goncalves, C.J. Yu, J.-M. Matray, J. Tremosa, *Analytical expressions for thermo-osmotic permeability of clays*, Geophysical Research Letters,**45**(2) (2018), 691–698.
12. B. Hafskjold, D. Bedeaux, S. Kjelstrup, O. Wilhelmsen, *Soret separation and thermo-osmosis in porous media*, Eur. Phys. J. E, **45** (2022), Article number: 41.
13. V.A. Herus, N.V. Ivanchuk, P.M. Martyniuk, *A System Approach to Mathematical and Computer Modeling of Geomigration Processes Using Freefem++ and Parallelization of Computations*, Cybernetics and Systems Analysis, **54**(2) (2019), 284–294.
14. N. Ivanchuk, P. Martynyuk, T. Tsvetkova, O. Michuta, *Mathematical modeling and computer simulation of the filtration processes in earth dams*, Eastern-European Journal of Enterprise Technologies, **2** (6) (2017), 63–69.
15. D. Jacques,J. Simunek, D. Mallants, M.Th. van Genuchten, *The HPx software for multicomponent reactive transport during variably-saturated flow: Recent developments and applications*, J. Hydrol. Hydromech., **66**(2) (2018), 211–226.
16. Jiaming Wang, Hailong He, Miles Dyck, Jialong Lv, *A Review and Evaluation of Predictive Models for Thermal Conductivity of Sands at Full Water Content Range*, Energies, **13** (2020), 1083.
17. T.V. Kutya, P.N. Martynyuk, *Mathematical Simulation of Humidification of Earth on a Slope and Calculation of Its Safety Factor*, J. of Eng. Phys. and Thermophysics, **91** (2018), 1189–1198.
18. P.M. Martyniuk, O.R. Michuta, O.V. Ulianchuk-Martyniuk, M.T. Kuzlo, *Numerical investigation of pressure head jump values on a thin inclusion in one-dimensional non-linear soil moisture transport problem*, Int. J. of Appl. Mathem., **31**(4) (2018), 649–660.
19. O. Michuta, N. Ivanchuk, P. Martyniuk, O. Ostapchuk, *A finite element study of elastic filtration in soils with thin inclusions*, Eastern-European Journal of Enterprise Technologies, **5**(5-107) (2020), 41–48.

20. O. R. Michuta, P. M. Martyniuk, *Nonlinear Evolutionary Problem of Filtration Consolidation With the Non-Classical Conjugation Condition*, J. of Optimization, Differential Equations and their Applications, **30**(1) (2022), 71–87.
21. Mohammadreza Mir Tamizdoust, Omid Ghasemi-Fare, *Assessment of thermal, hydraulic, and mechanical constitutive relations on the temperature-induced stress and pore fluid pressure in saturated clays*, Computers and Geotechnics, **145** (2022), 104686.
22. K.R. Reddy, G. Kumar, R.K. Giri, *Modeling Coupled Processes in Municipal Solid Waste Landfills: An Overview with Key Engineering Challenges*, Int. J. of Geosynth. and Ground Eng., **3** (2017), Article number: 6.
23. M. Sadeghi, B. Ghanbarian, R. Horton, *Derivation of an explicit form of the percolation-based effective medium approximation for thermal conductivity of partially saturated soils*, Water Resources Research, **54**(2) (2018), 1389–1399.
24. M. Sanchez, B. Pomaro, A. Gens, *Coupled THM analysis of a full-scale test for high-level nuclear waste and spent fuel disposal under actual repository conditions during 18 years of operation*, Geotechnique, (2022), 1–21. https://doi.org/10.1680/jgeot.21.00106
25. I.V. Sergienko, V.V. Skopetsky, V.S. Deineka, *Mathematical modeling and research of processes in inhomogeneous environments*, Naukova dumka, Kiev, 1991.
26. Per Schjonning, *Thermal conductivity of undisturbed soil - Measurements and predictions*, Geoderma, **402** (2021), 115188.
27. Shenwei Wang, Wei Zhu, Xuede Qian, Haoqing Xu, Xihui Fan, *Temperature effects on non-Darcy flow of compacted clay*, Applied Clay Science, **135** (2017), 521–525.
28. O.V. Ulianchuk-Martyniuk, *Numerical simulation of the effect of semi-permeable properties of clay on the value of concentration jumps of contaminants in a thin geochemical barrier*, Eurasian Journal of Mathematical and Computer Applications, **8**(1) (2020), 91–104.
29. O. Ulianchuk-Martyniuk, O. Michuta, *Conjugation conditions in the problem of filtering chemical solutions in the case of structural changes to the material and chemical suffusion in the geobarrier*, JP Journal of Heat and Mass Transfer, **19** (1) (2020), 141–154.
30. O.V. Ulianchuk-Martyniuk, O.R. Michuta, *Numerical Simulation of the Effect of Chemical Osmosis on the Value of the Jumps of Pollution in the Geochemical Barrier*,Int. J. of Appl. Mathematics, **33**(6) (2020), 1067–1082.
31. O. Ulianchuk-Martyniuk, O. Michuta, N. Ivanchuk, *Biocolmatation and the finite element modeling of its influence on changes in the head drop in a geobarrier*, Eastern-European Journal of Enterprise Technologies, **4** (10-106) (2020), 18–26.
32. O.V. Ulianchuk-Martyniuk, O.R. Michuta, N.V. Ivanchuk, *Finite element analysis of the diffusion model of the bioclogging of the geobarrier*, Eurasian Journal of Mathematical and Computer Applications, **9**(4) (2021), 100–114.
33. A.P. Vlasyuk, P.M. Martynyuk, *Numerical solution of three-dimensional problems of filtration consolidation with regard for the influence of technogenic factors by the method of radial basis functions*, J. of Math. Sciences, **171**(5) (2010), 632–648.
34. A.P. Vlasyuk, P.M. Martynyuk, O.R. Fursovych, *Numerical solution of a one-dimensional problem of filtration consolidation of saline soils in a nonisothermal regime*, J. of Math. Sciences, **160**(4) (2009), 525–535.
35. Wei Zhu, XiaoSi Su, Qiang Liu, *Analysis of the relationships between the thermophysical properties of rocks in the Dandong Area of China*, European Journal of Remote Sensing, **54**(2) (2021), 122–131.

36. Weiqiang Chen, Majid Sedighi, Andrey P. Jivkov, *Thermo-osmosis in silica nanochannels*, Japanese Geotechnical Society Special Publication, **9**(5) (2021), 210–214.
37. Yanjun Shen, Xu Wang, Yongzhi Wang, Keping Zhou, Jinyuan Zhang, Huan Zhang, Jielin Li, *Thermal conductivity models of sandstone: applicability evaluation and a newly proposed model*, Heat Mass Transfer, **57** (2021), 985–998.
38. R. Zagorscak, M. Sedighi, H. R. Thomas, *Effects of Thermo-Osmosis on Hydraulic Behavior of Saturated Clays*, Int. J. of Geomechanics, **17**(3) (2017), Article num.: 04016068.

Received 21.04.2023

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 95–110, DOI 10.15421/142305

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

CAN A FINITE DEGENERATE 'STRING' HEAR ITSELF? NUMERICAL SOLUTIONS TO A SIMPLIFIED IBVP

Vladimir L. Borsch,* Peter I. Kogut†

Communicated by Prof. A. Plotnikov

Abstract. Some discrete models for a simplified (compared to that published earlier in JODEA, **28** (1) (2020), 1 – 42) initial boundary value problem for a 1D linear degenerate wave equation, posed in a space-time rectangle and solved earlier exactly (JODEA, **30** (1) (2022), 89 – 121), have been considered. It has been demonstrated that the correct evaluation of the degenerate grid flux can be possible.

Key words: degenerate wave equation, vibrating string, separation of variables, transmission condition, travelling wave, degenerate flux.

2010 Mathematics Subject Classification: 35C10, 35L05, 35L10, 35L20, 35L80.

1. Introduction and the problem formulation

The current study complementes [5], dealing with the following 1-parameter simplified initial boundary value problem (IBVP) for the degenerate wave equation, posed in the space-time rectangle $[0,T]\times[-1,+1]\subset\mathbb{R}_t^+\times\mathbb{R}_x$ wrt $u(t,x;\alpha)$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial u}{\partial x}\right)=0\,, & (t,|x|)\in(0,T]\times(0,1)\,,\\ \left.\begin{aligned}\dfrac{\partial u(0,x;\alpha)}{\partial t}&=\overset{**}{u}(x;\alpha)\\ u(0,x;\alpha)&=\overset{*}{u}(x;\alpha)\end{aligned}\right\}, & x\in[-1,+1]\,,\\ \left.\begin{aligned}u(t,-1;\alpha)&=h_1(t;\alpha)\\ u(t,+1;\alpha)&=h_2(t;\alpha)\end{aligned}\right\}, & t\in[0,T]\,,\end{cases}\tag{1.1}$$

where known control functions $h_{1,2}(t;\alpha)\in\mathscr{C}^1[0,T]\bigcap\mathscr{C}^2(0,T]$ obey the compatibility conditions: $h_1(0;\alpha)=\overset{*}{u}(-1;\alpha)$, $h_1'(0;\alpha)=\overset{**}{u}(-1;\alpha)$, $h_2(0;\alpha)=\overset{*}{u}(+1;\alpha)$, $h_2'(0;\alpha)=\overset{**}{u}(+1;\alpha)$, and the 1-parameter family of coefficient functions is defined as follows

$$a(x;\alpha)=|x|^\alpha,\qquad x\in[-1,+1]\,,\tag{1.2}$$

*Dept. of Math Analysis and Optimization, Faculty of Mech & Math, Oles Honchar Dnipro National University, 72, Gagarin av., Dnipro, 49010, Ukraine, bvl@dsu.dp.ua

†Dept. of Math Analysis and Optimization, Faculty of Mech & Math, Oles Honchar Dnipro National University, 72, Gagarin av., Dnipro, 49010, Ukraine, p.kogut@i.ua

the parameter of degeneracy $\alpha \in (0,2)$, and all the variables are nondimensional. The point $x = 0$, where the coefficient (1.2) vanishes, is referred below to as the degeneracy point, whereas $[0,T] \times [-1,+1] \supset [0,T] \times \{0\}$ is referred to as the degeneracy segment, or the dividing segment of the space-time rectangle. Dealing with (1.1), (1.2), we distinguish between the cases of: 1) weak degeneracy, $\alpha \in (0,1)$, 2) strong degeneracy, $\alpha \in (1,2)$, and 3) non-degeneracy, $\alpha = 0$ (the limiting case).

The following matching conditions must be imposed on the required solution to the IBVP at the degeneracy segment

$$\begin{cases} u(t,x;\alpha)\big|_{x=0-0} = u(t,x;\alpha)\big|_{x=0+0}\,, \\ f(t,x;\alpha)\big|_{x=0-0} = f(t,x;\alpha)\big|_{x=0+0}\,, \end{cases} \qquad t \in [0,T]\,. \tag{1.3}$$

where there is used a notion of the flux

$$f(t,x;\alpha) = a(x;\alpha)\,\frac{\partial u(t,x;\alpha)}{\partial x}\,. \tag{1.4}$$

The exact series solution to the IBVP (1.1) was obtained in [5], therefore we follow the notations, terminology and even an analogy of the problem and its solution to an imaginary ‘string’, wherever it is convenient.

In the current study our concern is numerical solving the IBVP (1.1). Formally, a proper grid approximation of the IBVP is not a problem. Nevertheless, any attempt to implement directly a numerical procedure to the IBVP involves a bulk of nested problems having relation to evaluating the flux (1.4) at the degeneracy, segment, where the flux degenerates. It was proved [2,4,5], using series solutions to the IBVP and to the degenerate wave equation alone, that the flux at the degeneracy segment does not vanish and is continuous. From this it immediately appears a problem to retain the above properties for the grid flux. Note, that in our previous study [1] we discussed some auxiliary problems arising in computational procedures applied to the IBVP. For example, it was attempted to introduce a regularization of the IBVP, unfortunately the convergence of the numerical solutions to the regularized problem was found not to exist.

The goal of the current study is:

1) to complement our previous study [5] in terms of suitable Bessel functions being linearly independent (refer to Sect.2);

2) to demonstrate that correct evaluating the degenerate flux on the grid is possible (refer to Sects. 3, 4, 5).

2. Some notes on separation of variables

Separation of variables applied to the original IBVP (1.1) is known [2–5] to involve us into solving the following boundary-value problem

$$\begin{cases} D'(x;\alpha) + \lambda(\alpha)\,X(x;\alpha) = 0\,, \qquad 0 < |x| < 1\,, \\ a)\ X(\mp 1;\alpha) = 0\,, \qquad b)\ X(x;\alpha)\big|_{x=0-0} = X(x;\alpha)\big|_{x=0+0}\,, \\ \qquad\qquad\qquad\qquad\quad c)\ D(x;\alpha)\big|_{x=0-0} = D(x;\alpha)\big|_{x=0+0}\,, \end{cases} \tag{2.1}$$

where $D(x;\alpha)=a(x;\alpha)\,X'(x;\alpha)$ is the flux of the solution $X(x;\alpha)$, referred to as the eigenfunction, whereas $\lambda(\alpha)$ is referred to as the eigenvalue.

To simplify further discussion, it is convenient to introduce the following α-dependent quantities

$$\nu(\alpha) = 1-\alpha\,, \qquad \theta(\alpha) = 2-\alpha\,, \qquad \varrho(\alpha) = \frac{\nu}{\theta} = \frac{1-\alpha}{2-\alpha}\,, \tag{2.2}$$

then in the case of weak degeneracy: 1) the eigenvalues $\lambda_{k,\mu}(\alpha)$ and the eigenfunctions $X_{k,\mu}(x;\alpha)$ of the problem (2.1) of the two kinds (marked with $k\in\{1,2\}$) are defined as follows

$$\begin{cases} \lambda_{1,\mu}(\alpha) = \left(\dfrac{\theta}{2}\, s_{1,\mu}\right)^2, & X_{1,\mu}(x;\alpha) = \phantom{\operatorname{sgn} x\,} Z_{1,\mu}(x;\alpha), \\[2ex] \lambda_{2,\mu}(\alpha) = \left(\dfrac{\theta}{2}\, s_{2,\mu}\right)^2, & X_{2,\mu}(x;\alpha) = \operatorname{sgn} x\; Z_{2,\mu}(x;\alpha), \end{cases} \tag{2.3}$$

where $\varrho\notin\mathbb{Z}$, $\{s_{k,\mu}\}_{\mu=1}^{\infty}$ are the unbounded monotonically increasing sequences of the zeros of the linearly independent Bessel functions $\mathtt{J}_{\mp\varrho}(s)$ of the first kind and orders $\mp\varrho$ [7], and

$$\begin{cases} Z_{1,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{J}_{-\varrho}\Big(s_{1,\mu}\,|x|^{\frac{\theta}{2}}\Big), \\[2ex] Z_{2,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{J}_{+\varrho}\Big(s_{2,\mu}\,|x|^{\frac{\theta}{2}}\Big). \end{cases} \tag{2.4}$$

The Bessel functions $\mathtt{J}_{\mp\varrho}(s)$ satisfy the ordinary differential equation

$$\mathtt{Z}''_{\mp\varrho}(s) + \frac{1}{s}\,\mathtt{Z}'_{\mp\varrho}(s) - \left(\frac{\varrho^2}{s^2} - 1\right)\mathtt{Z}_{\mp\varrho}(s) = 0 \tag{2.5}$$

and have the following power series representations

$$\mathtt{J}_{\mp\varrho}(s) = \left(\frac{s}{2}\right)^{\mp\varrho} \sum_{\gamma=0}^{\infty} \frac{(-1)^{\gamma}}{\gamma!\,\Gamma(1\mp\varrho+\gamma)} \left(\frac{s}{2}\right)^{2\gamma}. \tag{2.6}$$

Now we shortly recall the underlying idea [2] to reduce the governing equation of the BVP (2.1) to the Bessel equation (2.5). To this end, we introduce the following ansatz

$$Z(x;\alpha) = x^{o}\, V_{\alpha}(r)\,, \qquad r = x^{\omega}, \tag{2.7}$$

where o, ω are undetermined real exponents, and for the sake of brevity we assume that $x>0$. Substituting the ansatz (2.7) and its flux

$$D(x;\alpha) = aZ'(x;\alpha) = o\, x^{o+\alpha-1} V_{\alpha}(r) + \omega\, x^{o+\omega+\alpha-1} V'_{\alpha}(r) \tag{2.8}$$

into the equation of the IBVP gives the following relation involving the undetermined exponents and the degeneracy parameter α

$$D' + \lambda Z = \omega^2 x^{o+2\omega+\alpha-2} V''_\alpha + \omega\,[2o+\omega+\alpha-1]\, x^{o+\omega+\alpha-2} V'_\alpha$$
$$+\, o\,[o+\alpha-1]\, x^{o+\alpha-2} V_\alpha + \lambda\, x^o\, V_\alpha = 0\,.$$

Dividing by x^o simplifies the above relation to the following one

$$\omega^2 x^{2\omega-\theta} V''_\alpha + \omega\,[2o+\omega+\alpha-1]\, x^{\omega-\theta} V'_\alpha + o\,[o+\alpha-1]\, x^{-\theta} V_\alpha + \lambda\, V_\alpha = 0\,, \quad (2.9)$$

where quantities (2.2) are used. To agree (2.9) with the Bessel equation (2.6), we assume that $2\omega - \theta = 0$, then (2.9) simplifies as follows

$$\left(\frac{\theta}{2}\right)^2 V''_\alpha + \frac{\theta}{2}\left(2o + \frac{\theta}{2} - \nu\right) x^{-\frac{\theta}{2}} V'_\alpha + o\,(o-\nu)\, x^{-\frac{\theta}{2}2} V_\alpha + \lambda\, V_\alpha = 0\,,$$

and we have to assume that $2o - \nu$ to complete the agreement with the Bessel equation (2.6) in the form

$$\left(\frac{\theta}{2}\right)^2 V''_\alpha + \left(\frac{\theta}{2}\right)^2 x^{-\frac{\theta}{2}} V'_\alpha - \left(\frac{\nu}{2}\right)^2 x^{-\frac{\theta}{2}2} V_\alpha + \lambda\, V_\alpha = 0\,. \quad (2.10)$$

From (2.10) we immediately find the eigenvalues and the eigenfunctions (2.3), where the functions $Z_{1,\mu}(x;\alpha)$, $Z_{2,\mu}(x;\alpha)$ are defined in (2.4) and are linearly independent, provided that $\varrho \notin \mathbb{Z}$. In the case $\varrho \in \mathbb{Z}$ we should take following pairs of the eigenvalues and the eigenfunctions

$$\lambda_{3,\mu}(\alpha) = \left(\frac{\theta}{2}\, s_{3,\mu}\right)^2, \qquad \lambda_{4,\mu}(\alpha) = \left(\frac{\theta}{2}\, s_{4,\mu}\right)^2, \quad (2.11)$$

$$\begin{cases} Z_{3,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{Y}_\varrho\Big(s_{3,\mu}\, |x|^{\frac{\theta}{2}}\Big), \\ Z_{4,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{J}_\varrho\Big(s_{4,\mu}\, |x|^{\frac{\theta}{2}}\Big), \end{cases} \quad (2.12)$$

where $\mathtt{Y}_\varrho(s)$ are the Bessel functions of the second kind and orders $\mp\varrho$ [7] (referred to as the Neumann functions), $\{s_{k,\mu}\}_{\mu=1}^{\infty}$, $k = 3, 4$, are the unbounded monotonically increasing sequences of the zeros of the linearly independent functions $\mathtt{Y}_\varrho(s)$, $\mathtt{J}_\varrho(s)$, and it is evident that $s_{1,\mu} = s_{3,\mu}$, $\sigma_{1,\mu} = \sigma_{3,\mu}$, $X_{2,\mu} = X_{4,\mu}$.

The above reducing (2.7) to the Bessel equation (2.5) is not valid for the intermediate case $\alpha = 1$, therefore we repeat reducing especially for the case. Again taking the ansatz of the form (2.7)

$$Z(x;1) = x^\sigma\, V_\alpha\,(r)\,, \qquad r = x^\omega, \quad (2.13)$$

calculating its flux

$$D(x;1) = aZ'(x;1) = \sigma\, x^{\sigma-1} V_0\,(r) + \sigma\omega\, x^{\sigma+\omega} V_0'\,(r)\,, \tag{2.14}$$

and substituting into the equation of the IBVP gives the following relation involving the undetermined exponents

$$\begin{aligned} D' + \lambda X = \omega^2 x^{\sigma+2\omega-1} V_0''\,(r) + \omega\,[2\sigma+\omega]\, x^{\sigma+\omega-1} V_0'\,(r) \\ + \sigma^2 x^{\sigma-1} V_0\,(r) + \lambda x^{\sigma} V_0\,(r) = 0\,. \end{aligned}$$

Dividing by x^o yields to the simplified relation

$$\omega^2 x^{2\omega-1} V_0''(r) + \omega\,[2\sigma+\omega]\, x^{\omega-1} V_0'(r) + \sigma^2 x^{-1} V_0(r) + \lambda V_0(r) = 0\,, \tag{2.15}$$

where we have to assume $2\omega - 1 = 0$, $\sigma = 0$, to obtain the required equation

$$\frac{1}{4}\, V_0''(r) + \frac{1}{4}\,\frac{1}{\sqrt{x}}\, V_0'(r) + \lambda V_0(r) = 0\,, \tag{2.16}$$

consistent with the Bessel equation (2.5) of the order zero, provided $s = 2\sqrt{\lambda x}$. In this case we have the following pairs of the eigenvalues and the eigenfunctions.

$$\lambda_{5,\mu}(\alpha) = s_{5,\mu}^2\,, \qquad \lambda_{6,\mu}(\alpha) = s_{6,\mu}^2\,, \tag{2.17}$$

$$\left\{ \begin{aligned} Z_{5,\mu}(x;\alpha) &= |x|^{\frac{\nu}{2}}\, \mathtt{Y}_0\Big(2\, s_{5,\mu}\, |x|^{\frac{1}{2}}\Big), \\ X_{6,\mu}(x;\alpha) &= |x|^{\frac{\nu}{2}}\, \mathtt{J}_0\Big(2\, s_{6,\mu}\, |x|^{\frac{1}{2}}\Big), \end{aligned} \right. \tag{2.18}$$

where notation used is exactly the same as that in (2.11), (2.12).

The Neumann function has a series representation different from that for the Bessel function of the first kind, for example in the case of the zero order it reads

$$\mathtt{Y}_0(s) = \frac{2}{\pi}\Big(C + \ln\frac{s}{2}\Big)\, \mathtt{J}_0(s) - \frac{2}{\pi}\sum_{\gamma=0}^{\infty} \frac{(-1)^{\gamma}\,\Phi(\gamma)}{(\gamma!)^2}\Big(\frac{s}{2}\Big)^{2\gamma}, \quad \Phi(\gamma) = \sum_{\rho=1}^{\gamma}\frac{1}{\rho}\,, \tag{2.19}$$

where $C = 0.5772\ldots$ is the Euler constant and $\Phi(0) = 0$, nevertheless all the properties of the solution expressed in terms of (2.12) are exactly the same as those expressed in terms of (2.4).

3. Discrete formulation of the problem

In order to develop discrete models of the IBVP(1.1), we first introduce an orthogonal grid with space-time nodes (x_k, t^n), $k = 1, \ldots, K$, $n = 0, \ldots, N$, in rectangle $[-1,+1] \times [0,T]$. Nodes t^n are distributed uniformly on segment $[0,T]$, whereas spatial nodes x_k cluster in some way on segment $[-1,+1]$ towards the midpoint $x=0$. Second, we introduce the following grid operators

$$\begin{aligned}
\Delta_k^{\mp} x_k &:= \mp\left(x_{k\mp 1} - x_k\right), \\
2\Delta_k^0 x_k &:= \Delta_k^- x_k + \Delta_k^+ x_k = \left(\Delta_k^- + \Delta_k^+\right) x_k, \\
\Delta_{\mp}^n t^n &:= \mp\left(t^{n\mp 1} - t^n\right) \equiv \Delta t.
\end{aligned} \tag{3.1}$$

Third, we integrate the degenerate wave equation over the cell, centered at an arbitrary interior node $\left(x_k, t^n\right)$ ($x_{k-h} \leqslant x \leqslant x_{k+h}$, $t^{n-h} \leqslant t^{n+h}$, $h = \frac{1}{2}$) of the grid

$$0 = \iint_{\omega_k^n} \left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a\,\frac{\partial u}{\partial x}\right)\right] \mathrm{d}x\,\mathrm{d}t = \int_{x_{k-h}}^{x_{k+h}} \int_{t^{n-h}}^{t^{n+h}} \left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a\,\frac{\partial u}{\partial x}\right)\right] \mathrm{d}x\,\mathrm{d}t$$

$$= \int_{x_{k-h}}^{x_{k+h}} \left(\frac{\partial u}{\partial t}\right)\Bigg|_{t^{n-h}}^{t^{n+h}} \mathrm{d}x - \int_{t^{n-h}}^{t^{n+h}} \left(a\,\frac{\partial u}{\partial x}\right)\Bigg|_{x_{k-h}}^{x_{k+h}} \mathrm{d}t$$

and evaluate the integrals by applying the midpoint rule of Calculus as follows

$$\Delta_k^0 x_k \left(\frac{\partial u}{\partial t}\right)\Bigg|_{(k,n-h)}^{(k,n+h)} = \Delta t \left(a\,\frac{\partial u}{\partial x}\right)\Bigg|_{(k-h,n)}^{(k+h,n)}, \tag{3.2}$$

where notations $(k, n\mp h) = \left(x_k, t^{n\mp h}\right)$, $(k\mp h, n) = \left(x_{k\mp h}, t^n\right)$ are used for the sake of brevity. Fourth, using spatial averaging, introduce grid functions

$$u_k^n = \frac{1}{\Delta_k^0 x_k} \int_{x_{k-h}}^{x_{k+h}} u(t^n, x; \alpha)\,\mathrm{d}x, \tag{3.3}$$

$$f_{k\mp h}^n = \frac{\mp 1}{\Delta_k^{\mp} x_k} \int_{x_k}^{x_{k\mp h}} a(x;\alpha)\,\frac{\partial u(t^n, x; \alpha)}{\partial x}\,\mathrm{d}x, \tag{3.4}$$

then integration in (3.2) yields to

$$\Delta_k^0 x_k \left(\frac{\Delta_+^n u_k^n}{\Delta t} - \frac{\Delta_-^n u_k^n}{\Delta t}\right) = \Delta t \left(f_{k+h}^n - f_{k-h}^n\right). \tag{3.5}$$

From (3.5) it follows the required explicit computational formula for finding the values of grid function u_k^n at the upper time level $n+1$, being a well known three-layer finite-difference scheme [6]

$$u_k^{n+1} = 2\,u_k^n - u_k^{n-1} + \sigma_k \left(f_{k+h}^n - f_{k-h}^n\right), \qquad \sigma_k = \frac{(\Delta t)^2}{\Delta_k^0 x_k}\,. \tag{3.6}$$

4. Calculation of the fluxes

The inter-cell fluxes, playing the key role in (3.6), can be evaluated in various ways, but the most obvious one reads

$$f^n_{k\mp h} = a_{k\mp h}\,\frac{\Delta^{\mp}_k\, u^n_k}{\Delta^{\mp}_k\, x_k}\,, \tag{4.1}$$

where the inter-cell coefficients $a_{k\mp h}$ are not determined uniquely. For example, they can be directly taken as the inter-cell values of the coefficient function $a(x;\alpha)$, as follows

$$a_{k\mp h} = a(x_{k\mp h};\alpha) = |x_{k\mp h}|^{\alpha}. \tag{4.2}$$

Simplicity of the direct approach is in contrast to nature of the phenomenon being under consideration. Indeed, in original IBVP (1.1), the flux on the degeneracy segment was proved [1, 5] not to vanish, whereas the grid flux computed due to (4.1), (4.2) on the degeneracy segment, vanishes whatever values grid function u^n_k takes.

To overcome this fault of the direct approach due to (4.2), we refer to ‘the best scheme’ [6]. Following [6], we resolve the definition of the flux(1.4) wrt to ‘the string’ inclination

$$\frac{\partial u(t,x;\alpha)}{\partial x} = \frac{f(t,x;\alpha)}{a(x;\alpha)}$$

and integrate the above relation over segment $[x_k, x_{k+h}]$ at instant t^n

$$\int_{x_k}^{x_{k+h}} \frac{\partial u(t^n,x;\alpha)}{\partial x}\,\mathrm{d}x = \int_{x_k}^{x_{k+h}} \frac{f(t^n,x;\alpha)}{a(x;\alpha)}\,\mathrm{d}x\,.$$

Applying the fundamental and the midpoint theorems of Calculus to the above relation and dividing both sides of the resulting equality by the length of the segment yields to

$$\frac{u(t^n,x_{k+1};\alpha) - u(t^n,x_k;\alpha)}{\Delta^{+}_k\, x_k} = f(t^n,x_{k+h};\alpha)\frac{1}{\Delta^{+}_k\, x_k}\int_{x_k}^{x_{k+h}} \frac{\mathrm{d}x}{a(x;\alpha)}\,.$$

Comparing the obtained equation with (4.1) prompts the way used in ‘the best scheme’ to calculate the inter-cell coefficient

$$a_{k\mp h} = \mathring{a}_{k\mp h} = \left[\frac{\mp 1}{\Delta^{\mp}_k x_k}\int_{x_k}^{x_{k\mp h}} \frac{\mathrm{d}x}{a(x;\alpha)}\right]^{-1}. \tag{4.3}$$

In further discussion we will distinguish between the approaches to calculate the inter-cell fluxes outside the degeneracy segment (the regular fluxes, or the fluxes at the regular inter-cells) and exactly on the the degeneracy segment (the degenerate flux).

For example, the first approach, based on (4.2), is applicable only for the regular fluxes, whereas the second one, based on (4.3), is valid for fluxes of both kinds. Unfortunately, the second approach can not be applied in the case of strong degeneracy, therefore we consider some other approaches to evaluate the inter-cell fluxes.

Again, refer to the definition of the flux (1.4), written in its original form at instant t^n, integrate it over the same segment

$$\int_{x_k}^{x_{k+h}} f(t^n, x; \alpha)\, \mathrm{d}x = \int_{x_k}^{x_{k+h}} a(x; \alpha)\, \frac{\partial u(t^n, x; \alpha)}{\partial x}\, \mathrm{d}x\,,$$

apply the midpoint theorem of Calculus, divide both sides of the resulting equality by the length of the segment, and account for (3.4), to obtain

$$f^n_{k+h} = \left(\frac{\partial u(t^n, x; \alpha)}{\partial x}\right)\Bigg|_{x=x_{k+h}} \frac{1}{\Delta^+_k x_k} \int_{x_k}^{x_{k+h}} a(x; \alpha)\, \mathrm{d}x\,.$$

Evaluating the inter-cell inclination of 'the string' similarly to (4.1), we easily obtain one more approach for the inter-cell coefficients

$$a_{k\mp h} = a^*_{k\mp h} = \frac{\mp 1}{\Delta^{\mp}_k x_k} \int_{x_k}^{x_{k\mp h}} a(x; \alpha)\, \mathrm{d}x\,. \tag{4.4}$$

Other approaches to evaluate the inter-cell fluxes, we are going to discuss, refer to the degenerate flux and do not involve any direct way to calculate the inter-cell coefficient $a_{k\mp h}$. The first group of such approaches utilizes the continuity of the flux across the degenerate segment, for example, the simplest averaging of the regular fluxes calculated at the inter-cells adjacent to the degenerate inter-cell $k+h$

$$f_{k+h} = \frac{1}{2}\left(f_{k-h} + f_{k+3h}\right). \tag{4.5}$$

A more sophisticated approach, utilizing the flux continuity, reads as follows

$$f_{k+h} = \frac{1}{2}\left(f^-_{k+h} + f^+_{k+h}\right), \tag{4.6}$$

where $f^{\mp}_{k+h}$ are 'one-sided' values of the required degenerate flux, obtained using extrapolation

$$\begin{cases} f^-_{k+h} = f_{k-1h} + (\Delta x)_k \quad D^{-m}_k f_{k-1h}\,, \\ f^+_{k+h} = f_{k+3h} - (\Delta x)_{k+1}\, D^{+m}_k f_{k+3h}\,, \end{cases} \tag{4.7}$$

where $(\Delta x)_k = x_{k+h} - x_{k-h}$, $(\Delta x)_{k+1} = x_{k+1+h} - x_{k+h}$, $D^{\mp m}_k$, $m \geqslant 2$, are one-sided m-nodal grid operators of the first order differentiation (involving the regular grid fluxes calculated at m nodes). For example, the left (or backward) operator reads

$$D^{-m}_k f_{k-1h} = b_{k-1h}\, f_{k-1h} + b_{k-3h}\, f_{k-3h} + \ldots + b_{k-(2m-1)h}\, f_{k-(2m-1)h}\,, \tag{4.8}$$

where the coefficients b_{k-1h}, etc, are undetermined. The proper well-conditioned linear algebraic system $m \times m$ wrt the coefficients can be set up and solved easily.

The second group of approaches utilizes the possibility of building the solution to the original IBVP (1.1) in space-time rectangle $[-1,+1]\times[0,T]$ using matching the solutions to the auxiliary IBVPs posed in subrectangles $[-1,0]\times[0,T]$, $[0,+1]\times[0,T]$. Proper matching may involve other conditions in addition to (1.3). For example, it was shown [2] that the flux across the degenerate segment can be continuously differentiable. This property can be easily implemented to evaluate the degenerate flux. Indeed, using the above one-sided grid operators $D_k^{\mp m}$, we can represent the property as the equality of two one-sided derivatives of the first order at both sides of the degenerate inter-cell

$$D_k^{-m} f_{k+h} = D_k^{+m} f_{k+h}\,, \tag{4.9}$$

involving the required degenerate flux f_{k+1h}. The above equality is nothing but the linear algebraic equation wrt the required flux f_{k+h}.

5. The test case of the problem

To estimate and compare the approaches of Sect. 4 for the flux evaluation, we refer to test case A of [5], as a benchmark. Recall that in that test case: 1) the initially ($t=0$) disturbed 'string' is at rest

$$\overset{**}{u}(x;\alpha) \equiv \overset{**}{u}_0 = 0\,, \quad \overset{*}{u}(x;\alpha) = \begin{cases} 0\,, & |x-x_0| > \delta\,, \\ \overset{*}{u}_0\,, & |x-x_0| \leqslant \delta\,, \end{cases} \qquad x\in[-1,+1]\,; \tag{5.1}$$

and 2) both ends of the 'string' are fixed

$$u(-1,t;\alpha) = u(+1,t;\alpha) = 0\,, \qquad t\in[0,T]\,, \tag{5.2}$$

i. e., both controls are not applied: $h_1(t;\alpha) = h_2(t;\alpha) \equiv 0$. The initial step function was smoothed using a mollifier.

To resolve the structure of the grid solutions to the IBVP near the degenerate segment we first introduce the uniform grid on $[-1,+1]\subset\mathbb{R}_\xi$ with spacing $\Delta\xi$ between the nodes, the coordinates ξ_k of the nodes are calculated as follows

$$(K-1)\,\Delta\xi = 2\,, \qquad \xi_k = -1 + (k-1)\,\Delta\xi\,, \qquad k = 1,\dots,K\,,$$

N being the number of the nodes. In the case of even K, two central nodes are biased wrt the degeneracy point $\xi=0$ by half of $\Delta\xi$. Second, a nonlinear transformation $\xi\to x$ is applied to calculate the coordinates x_k of the grid nodes on segment $[-1,+1]\subset\mathbb{R}_x$, for the nodes to cluster near the degeneracy point $x=0$. To obtain the results partially presented below in Figs. 5.1 − 5.6, we assigned the number 2000 to K, and two values 0.25 and 0.75 to the parameter of degeneracy. Exact solutions to the test case A [5] are drawn as dashed lines. We will not give any comments to the behavior of the solution plots, since any curve should be studied individually to evaluate the possibilities of the approaches used to model vibrations of 'the damaged string'. The results, we believe, will be useful to develop proper discrete models for the case of strong degeneracy as well.

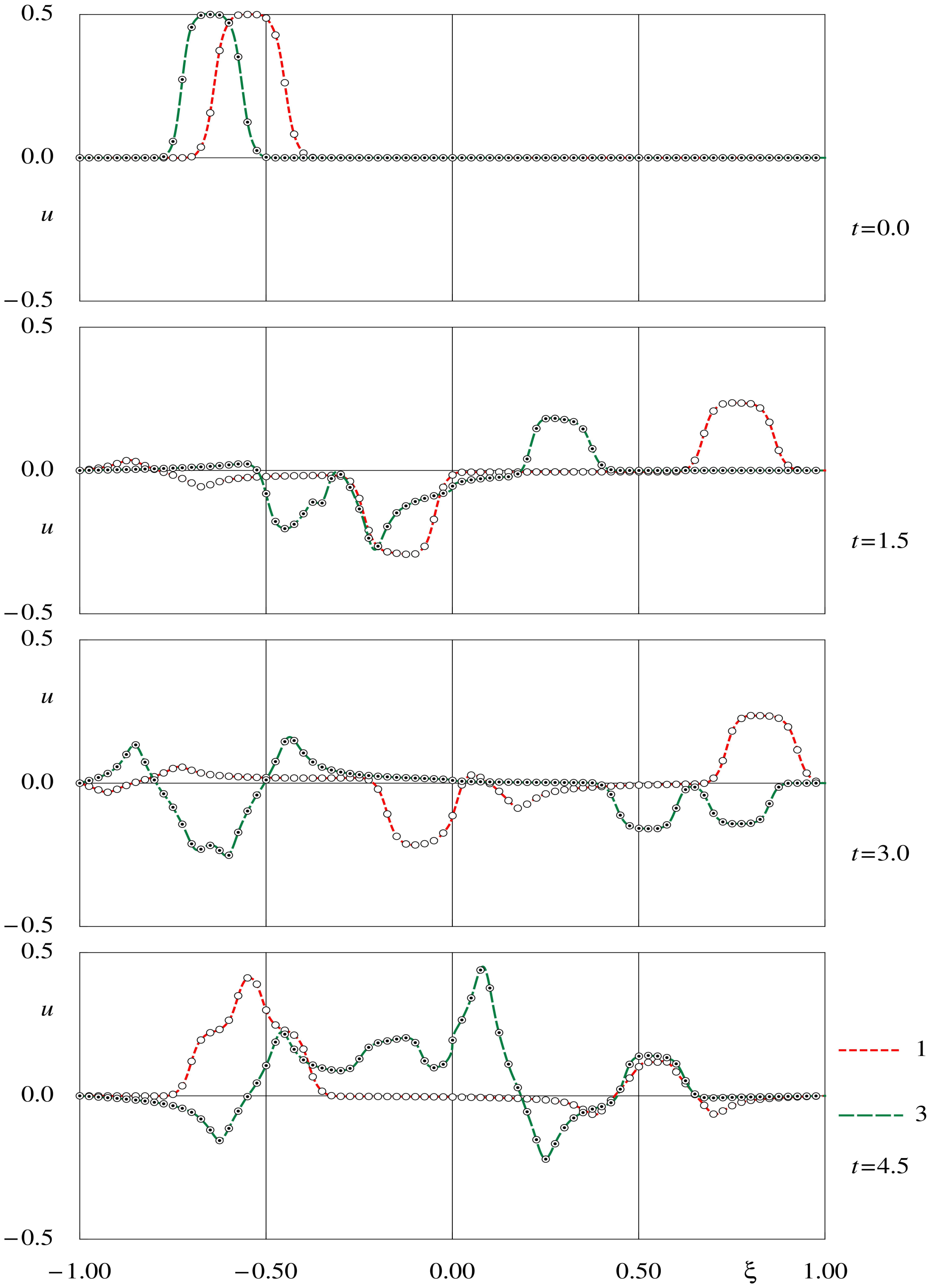


Fig. 5.1. Test case A: the regular fluxes are calculated due to (4.1), (4.3) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

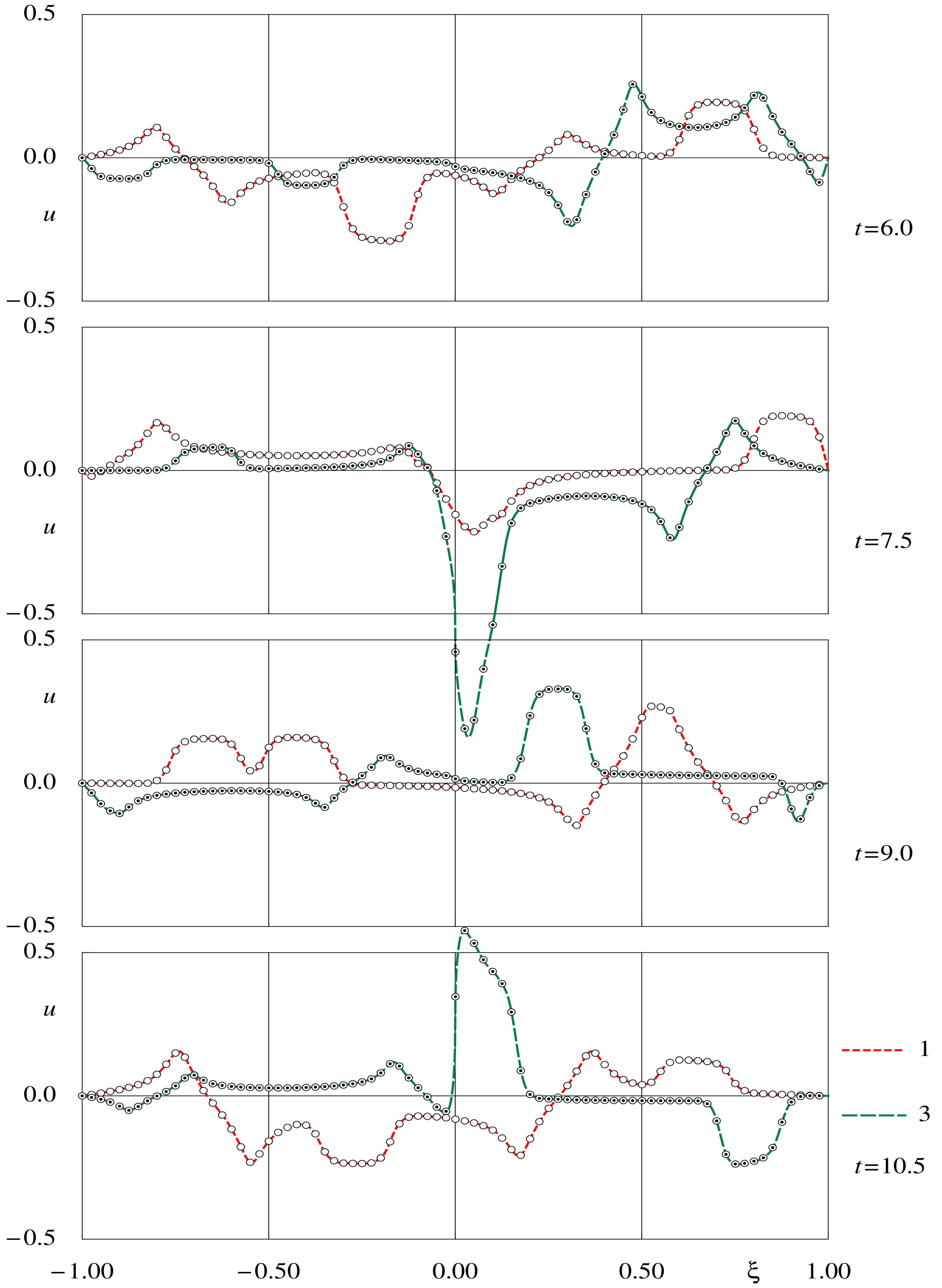


Fig. 5.2. Test case A: the regular fluxes are calculated due to (4.1), (4.3) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

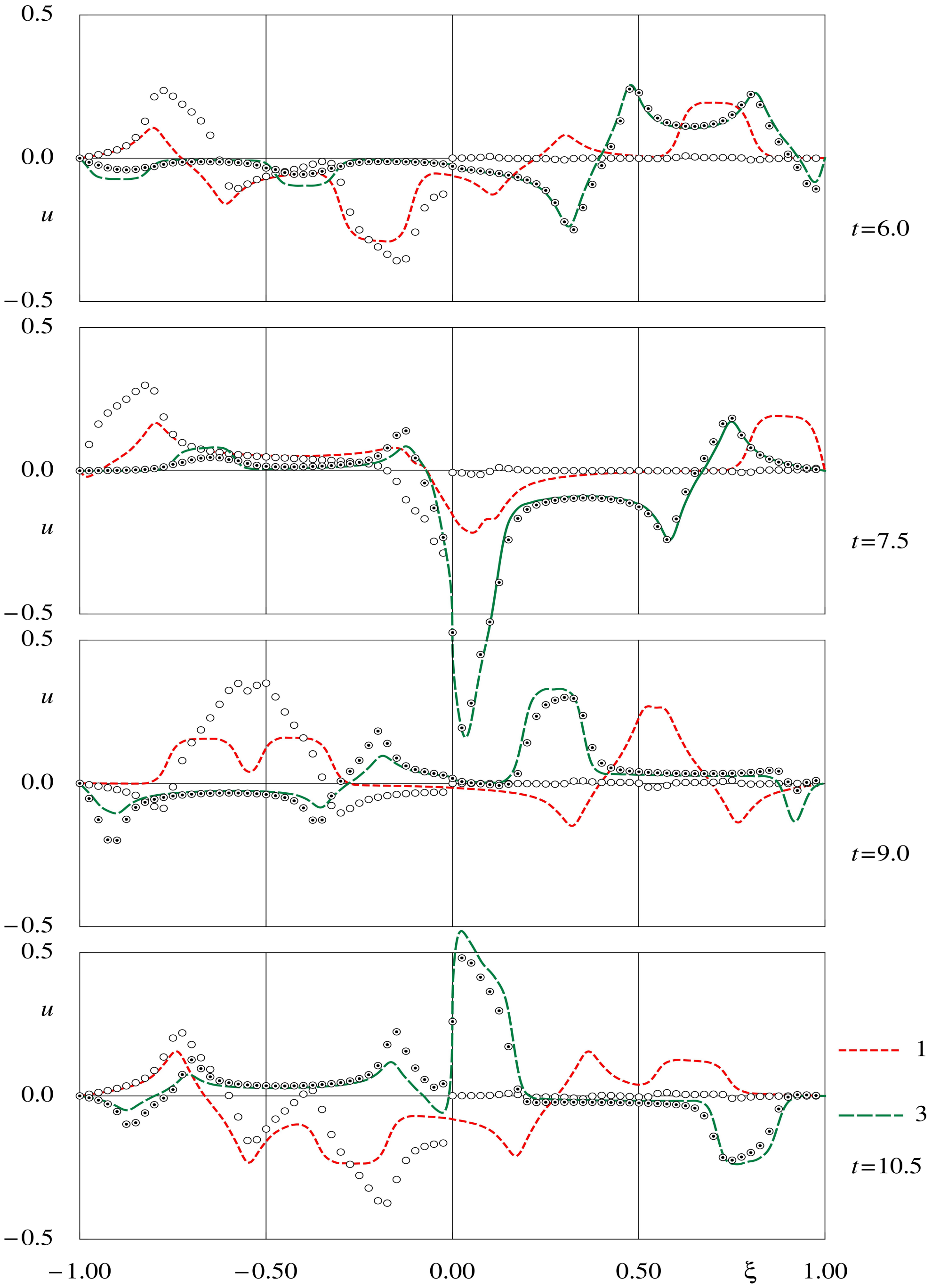


Fig. 5.3. Test case A: the regular fluxes are calculated due to (4.1), (4.3), whereas the degenerate fluxes are calculated using averaging (4.5) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

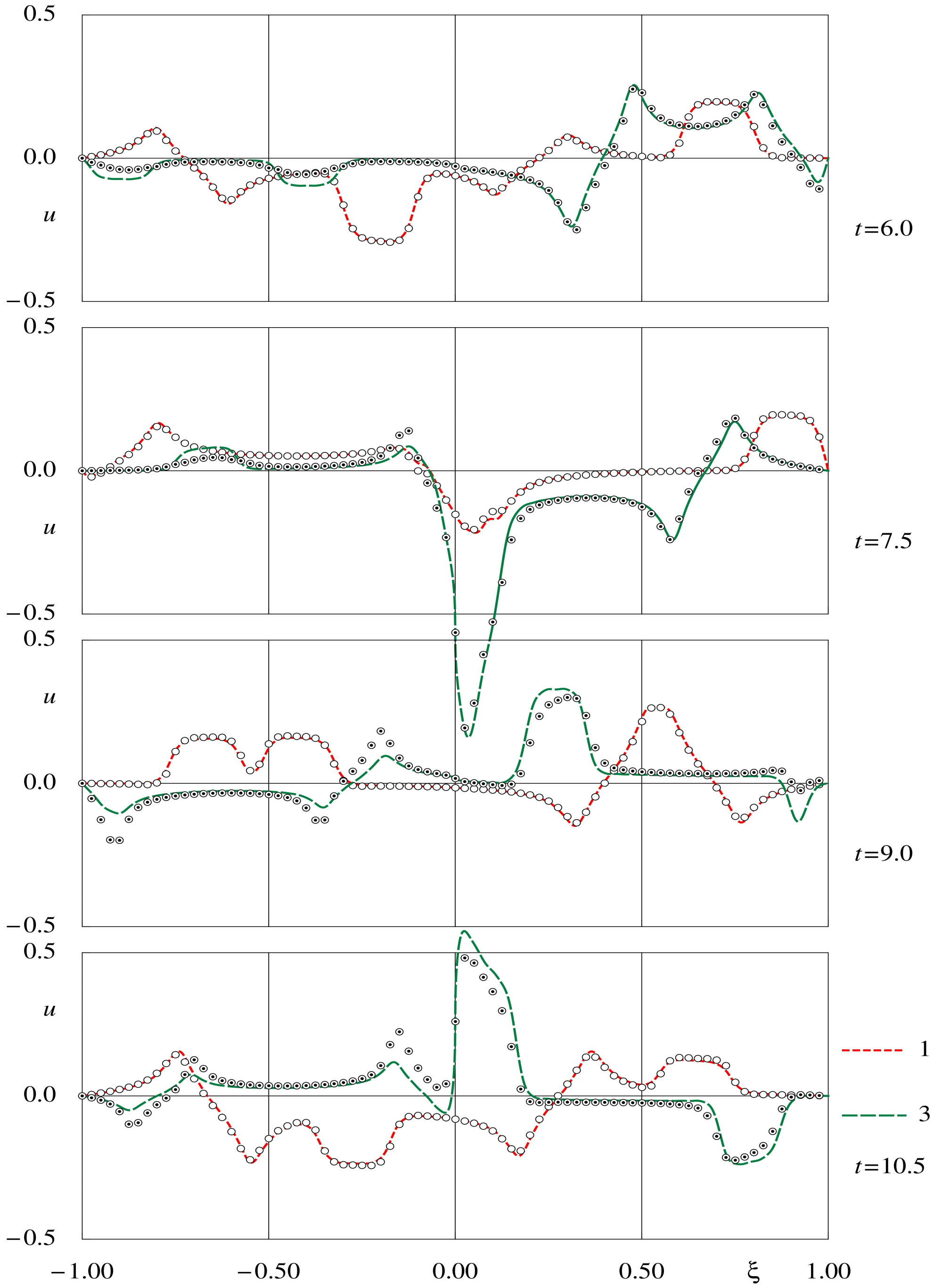


Fig. 5.4. Test case A: the regular fluxes are calculated due to (4.1), (4.3), whereas the degenerate fluxes are calculated using 2-nodal grid operators $D_k^{\mp m}$ and equality (4.9) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

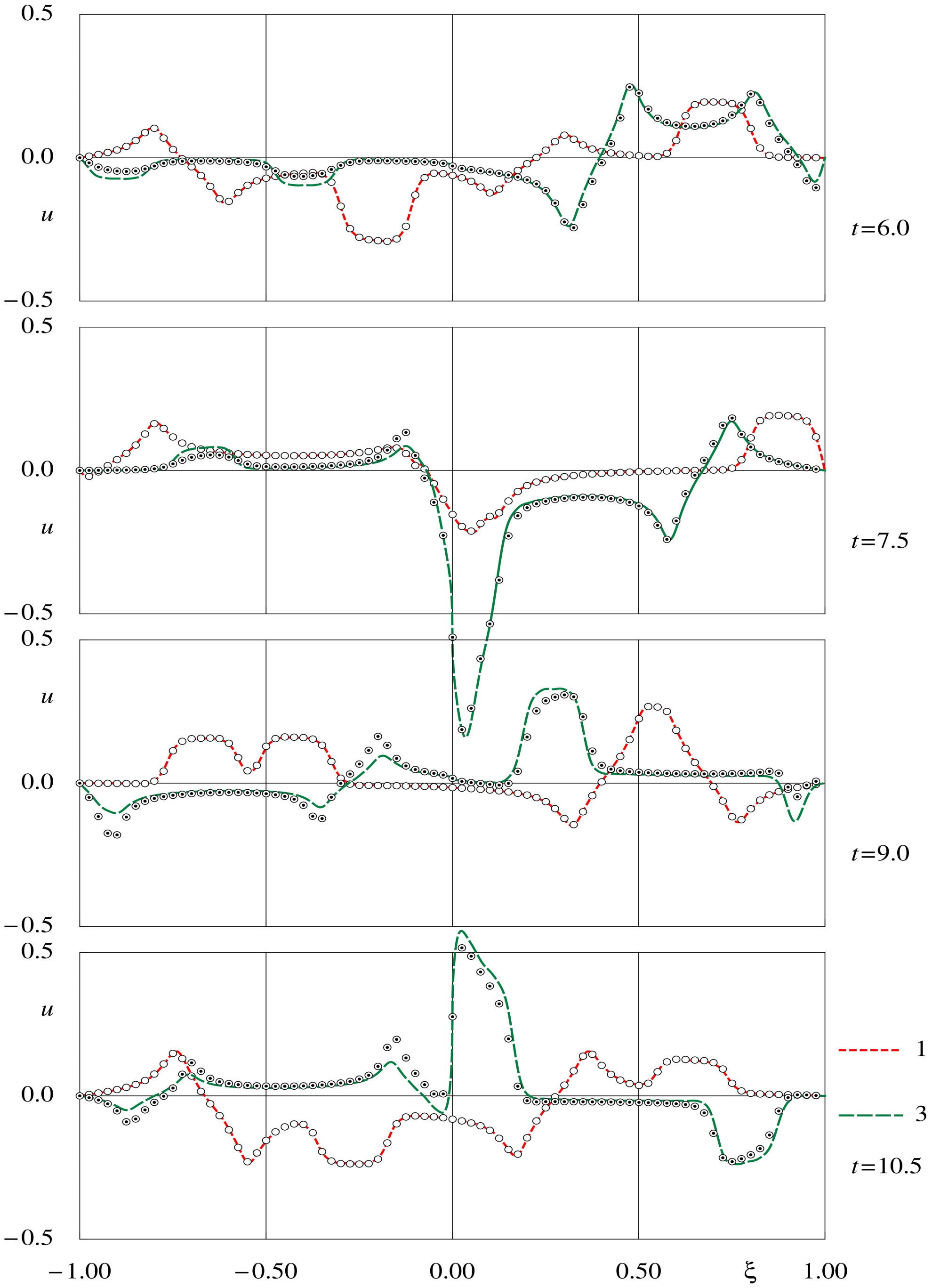


Fig. 5.5. Test case A: the regular fluxes are calculated due to (4.1), (4.3), whereas the degenerate fluxes are calculated using 3-nodal grid operators $D_k^{\mp m}$ and equality (4.9) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

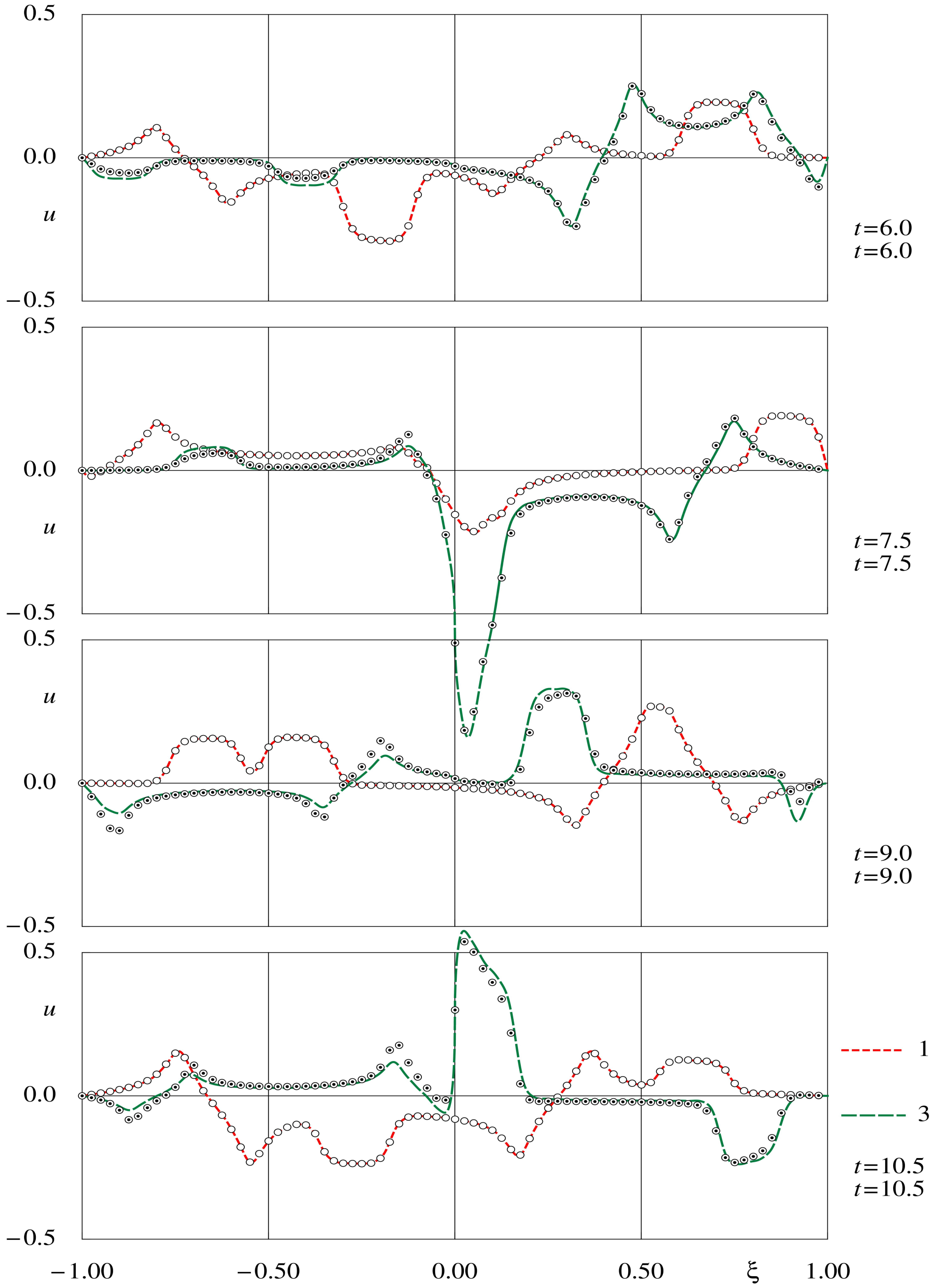


Fig. 5.6. Test case A: the regular and the degenerate fluxes are calculated due to (4.1), (4.4) (α = 0.25 — curves 1 of short dashes, α = 0.75 — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

6. Conclusions

We have demonstrated for test case A [5], treated as a benchmark, that the problem of correct evaluating the inter-cell fluxes at the degeneracy segment can be solved using various approaches. The first group of approaches is based on a proper (or efficient) calculation of the coefficient function $a(x;\alpha)$ at the degenerate inter-cell. The second and third group utilize respectively the properties of the flux continuity and continuous differentiability across the degeneracy segment and do not involve any calculation of the coefficient function $a(x;\alpha)$ at the degenerate inter-cell.

The preliminary results of the current study will be further used to develop discrete models of the IBVP(1.1) in the case of strong degeneracy.

References

1. V. L. Borsch, *On initial boundary value problems for the degenerate 1D wave equation*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **27** (2) (2019), 27–44.
2. V. L. Borsch, P. I. Kogut, G. Leugering, *On an initial boundary-value problem for 1D hyperbolic equation with interior degeneracy: series solutions with the continuously differentiable fluxes*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **28** (1) (2020), 1 – 42.
3. V. L. Borsch, P. I. Kogut, *The exact bounded solution to an initial boundary value problem for 1D hyperbolic equation with interior degeneracy. I. Separation of Variables*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **28** (2) (2020), 2 – 20.
4. V. L. Borsch, P. I. Kogut, *Solutions to a simplified initial boundary value problem for 1D hyperbolic equation with interior degeneracy*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **29** (1) (2021), 1 – 31.
5. V. L. Borsch, P. I. Kogut, *Can a finite degenerate 'string' hear itself? The exact solution to a simplified IBVP*, Journal of Optimization, Differential Equations, and their Applications (JODEA), **30** (1) (2022), 89 – 121.
6. N. N. Kalitkin, *Numerical Methods (In Russian)*, Nauka, Moscow, 1986.
7. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1922.

Received 30.04.2023

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 111–124, DOI 10.15421/142306

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

STABILITY W.R.T. DISTURBANCES FOR THE GLOBAL ATTRACTOR OF MULTI-VALUED SEMIFLOW GENERATED BY NONLINEAR WAVE EQUATION

Oleksiy V. Kapustyan[*], Oleksandr B. Kurylko[†], Taras V. Yusypiv[‡], Andriy V. Pankov[§]

Abstract. The paper investigates the issue of stability with respect to external disturbances for the global attractor of the wave equation under conditions that do not ensure the uniqueness of the solution to the initial problem. Under general conditions for nonlinear terms, it is proved that the global attractor of the undisturbed problem is locally stable in the sense of ISS and has the AG property with respect to disturbances.

Key words: global attractor, multi-valued semiflow, local input-to-state stability, asymptotic gain, wave equation.

2010 Mathematics Subject Classification: 35B41, 35K57, 37L30.

Communicated by Prof. P. O. Kasyanov

1. Introduction

Properties of global attractors of nonlinear wave equations with dissipation under different assumptions on the interaction functions have been under investigation in many papers (see [1, 2] and references therein). With the appearance of the works [3, 4], it became possible to study invariant uniformly attracting sets of infinite-dimensional dynamical systems without uniqueness of the solution of the initial problem, considering instead of a classical semigroup its multivalued counterpart called an m-semiflow. In particular, for the wave equation with nonsmooth nonlinear term f the existence and properties of the global attractor of the corresponding m-semiflow were investigated in [5].

In the presence of external disturbances, the problem becomes non-autonomous and its dynamics can be described in terms of uniform attractors of semi-processes [6–9]. It turned out, that this theory also allows us to solve the problem of estimating the deviation of the solution of the disturbed equation from the global attractor of the undisturbed system. In the case of a trivial attractor consisting

[*]Department of Mathematics and Mechanics, Taras Shevchenko National University of Kyiv, Volodymyrska Street 64, 01601 Kyiv, Ukraine alexkap@univ.kiev.ua

[†]Department of Mathematics and Mechanics, Taras Shevchenko National University of Kyiv, Volodymyrska Street 64, 01601 Kyiv, Ukraine v.v.sobchuk@gmail.com

[‡]Department of Mathematics and Mechanics, Taras Shevchenko National University of Kyiv, Volodymyrska Street 64, 01601 Kyiv, Ukraine yusypivt7@gmail.com

[§]State Scientific Institution "Institute of Education Content Modernization", Metropolitan Vasyl Lipkovsky 36, 03035 Kyiv, Ukraine andriy.pankov@gmail.com

of a single asymptotically stable equilibrium point, for the simplest partial differential equation of the reaction-diffusion type, such results first appeared in [10]. The technique of this work was based on the classical ISS approach of Lyapunov functions [11–13] and could not be applied to systems with non-trivial attractors. The corresponding technique was developed in the works of [14, 15] and applied to the wave equation with a smooth interaction function f and disturbances of the type $h(x)d(t)$ in the work [16]. The extension of this theory to the case of non-uniqueness of solution of the initial problem was carried out in [17], where the local ISS property of the attractor was established for the reaction-diffusion system.

In the present paper, we consider a wave equation with a non-smooth nonlinearity $f(y)$ and a $g(y)d(t)$-type disturbance with a non-smooth function g. Local ISS and AG stability properties with respect to disturbances are established for the global attractor of the undisturbed problem ($d \equiv 0$).

2. Setting of the problem

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ we consider the following boundary-value problem

$$\begin{cases} \frac{\partial^2 y(t,x)}{\partial t^2} + \alpha \frac{\partial y(t,x)}{\partial t} - \triangle y(t,x) + f(y(t,x)) = g(y(t,x))d(t), \ t > 0, \\ y(t,x)|_{x\in\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\alpha > 0$, $f, g \in C(\mathbb{R})$ are given, $d \in L^\infty(\mathbb{R}_+)$ is a disturbance parameter.

We prove (see Lemma 3.1) that under rather general assumptions on f, g the problem (2.1) is globally resolvable (in weak sense) in the phase space $X = H_0^1(\Omega) \times L^2(\Omega)$. The uniqueness of solutions is not guaranteed.

Let us consider a multi-valued map $S_d : \mathbb{R}_+ \times X \mapsto 2^X$,

$$S_d(t, z_0) = \{z(t) \mid z = \begin{pmatrix} y \\ y_t \end{pmatrix} \text{ is a solution of (2.1), } z(0) = z_0\}. \tag{2.2}$$

For $d \equiv 0$ (undisturbed problem) the multi-valued map $S_0 : \mathbb{R}_+ \times X \mapsto 2^X$ is a multi-valued semigroup (m-semiflow), which possesses a global attractor $\Theta \subset X$, i.e., there exists a compact set $\Theta \subset X$ such that

$$\Theta = S_0(t, \Theta) \ \forall t \geq 0,$$

$$\forall r > 0 \quad \sup_{\|z_0\| \leq r} \operatorname{dist}(S_0(t, z_0), \Theta) \to 0, t \to \infty.$$

Here and after we use denotations:

$$\operatorname{dist}(A, B) = \sup_{\xi\in A} \inf_{\eta\in B} \|\xi - \eta\|_X, \ \|A\|_\Theta = \operatorname{dist}(A, \Theta).$$

Thus, in the undisturbed case, all trajectories (2.1) eventually end up in an arbitrarily small neighborhood of Θ. The paper investigates the issue of estimating the deviation of the trajectory of the disturbed problem (2.1) from the set Θ depending on the value of $\|d\|_\infty = \operatorname{ess\,sup}_{t\in(0,+\infty)} |d(t)|$.

This question in terms of Input-to-State Stability (ISS) theory can be solved by setting the estimate (ISS property): $\forall t \geq 0$

$$\|S_d(t, z_0)\|_\Theta \leq \beta(\|z_0\|_\Theta, t) + \gamma(\|d\|_\infty). \tag{2.3}$$

Here $\gamma : [0, +\infty) \mapsto [0, +\infty)$ is a continuous strictly increasing function with $\gamma(0) = 0$ ($\gamma \in \mathcal{K}$), $\beta : [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty)$ is a continuous function, $\forall t \geq 0$ $\beta(, t) \in \mathcal{K}$, $\forall s \geq 0$ $\beta(s,)$ decreases to 0 ($\beta \in \mathcal{KL}$).

The main results of this work are a local variant of (2.3) (local ISS) (see Theorem 4.1) and Asymptotic Gain (AG) property: $\forall z_0 \in X$

$$\overline{\lim_{t\to\infty}} \|S_d(t, z_0)\|_\Theta \leq \gamma(\|d\|_\infty). \tag{2.4}$$

3. Existence, a priori estimates, and regularity of solutions.

Assume that there exist positive constants m, c_1, c_2, c_3, c_4 such that $\forall s \in \mathbb{R}$

$$|f(s)| \leq c_1(1 + |s|^{\frac{n}{n-2}}), \tag{3.1}$$

$$F(s) \geq -as^2 - c_2,\ f(s)s - F(s) + as^2 \geq -c_3, \tag{3.2}$$

$$|g(s)| \leq c_4, \tag{3.3}$$

where $a < \frac{\lambda_1}{2}$, λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$, $F(s) := \int_0^s f(t)\,dt$.

Remark 3.1. In all further arguments in the case $n = 2$ we can assume that in (3.1) f has arbitrary power growth because of embedding $H_0^1(\Omega) \subset L^p(\Omega)$, $\forall\ p \geq 1$, and in the case $n = 1$ assumption (3.1) is not needed because of embedding $H_0^1(\Omega) \subset C(\overline{\Omega})$.

A solution of (2.1) we will understand in a weak sense, i.e., a pair of functions $z() = \begin{pmatrix} y() \\ y_t() \end{pmatrix} \in L^\infty(0, T; X)$ is called a solution of (2.1) on $(0, T)$ if $\forall \psi \in H_0^1(\Omega)$, $\forall \eta \in C_0^\infty(0, T)$ the following equality holds

$$-\int_0^T (y_t, \psi)\eta_t + \int_0^T \Big(\alpha(y_t, \psi) + (y, \psi)_{H_0^1} + (f(y), \psi) - (g(y), \psi)d(t)\Big)\,\eta = 0, \tag{3.4}$$

where by $\|\|$ and $(,)$ we denote the norm and scalar product in $L^2(\Omega)$.

If $z \in L^\infty_{loc}(\mathbb{R}_+; X)$ satisfies (3.4) $\forall T > 0$, then z is called a global solution (a solution for short) of (2.1).

Lemma 3.1. *Under assumptions* (3.1)-(3.3) $\forall z_0 \in X$, $\forall d \in L^2_{loc}(\mathbb{R}_+)$ *there exists at least one solution of* (2.1) *with* $z|_{t=0} = z_0$.

Proof. First, it should be noted that due to embedding $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$, $n \geq 3$, from conditions (3.1), (3.3) we deduce that for $y \in L^\infty(0,T;H_0^1(\Omega))$

$$f(y) \in L^2(0,T;L^2(\Omega)),\ g(y)d(t) \in L^2(0,T;L^2(\Omega)).$$

So, results of [1] allow us to claim that for every solution of (2.1) and $\forall T > 0$

$$z = \begin{pmatrix} y \\ y_t \end{pmatrix} \in C([0,T];X).$$

In particular, the initial condition $z|_{t=0} = z_0$ makes sense.

We prove an existence of solution of (2.1) by Galerkin method [1]. Let $z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X, T > 0$ be given. For every $m \geq 1$ we consider an approximation function

$$y_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i,$$

where $\{\omega_i\}_{i\geq 1}$ are eigenfunctions of $-\triangle$ in $H_0^1(\Omega)$, and $\{g_{im}()\}$ are solutions of ODE system

$$\begin{aligned} \frac{d^2}{dt^2}(y_m,\omega_j) + \alpha\frac{d}{dt}(y_m,\omega_j) + (y_m,\omega_j)_{H_0^1} \\ + (f(y_m),\omega_j) - (g(y_m),\omega_j)d(t) = 0,\ j = \overline{1,m} \end{aligned} \tag{3.5}$$

$$y_m|_{t=0} = y_m(0) \to y_0 \text{ in } H_0^1(\Omega),\ y'_m|_{t=0} = y'_m(0) \to y_1 \text{ in } L^2(\Omega).$$

Due to $Carath\acute{e}odory's$ theorem we have a solution of (3.5) on $[0,T_m]$. Let us derive a priori estimates which would imply that $T_m = T$. For this purpose, we introduce a function

$$Y_m(t) = \frac{1}{2}\|y'_m(t)\|^2 + \frac{1}{2}\|y_m(t)\|^2_{H_0^1} + (F(y_m(t)),1) + \delta(y'_m(t),y_m(t)),$$

where $\delta \in (0,\alpha)$ we will choose later.

Due to (3.5) we get:

$$\begin{aligned} \frac{dY_m}{dt} &= -(\alpha-\delta)\|y'_m(t)\|^2 - \delta\|y_m(t)\|^2_{H_0^1} - \alpha\delta(y'_m,y_m) \\ &\quad - \delta(f(y_m),y_m) + (y'_m,g(y_m))d(t) - \delta(y_m,g(y_m))d(t) \\ &= -\delta Y_m(t) + \left(-\alpha + \frac{3\delta}{2}\right)\|y'_m(t)\|^2 - \frac{\delta}{2}\|y_m(t)\|^2_{H_0^1} \\ &\quad + \delta\left((F(y_m),1) - (f(y_m),y_m)\right) - \alpha\delta(y'_m,y_m) + \delta^2(y'_m,y_m) \\ &\quad + (y'_m,g(y_m))d(t) - \delta(y_m,g(y_m))d(t) \\ &\leq -\delta Y_m(t) + \left(-\alpha + \frac{3\delta}{2}\right)\|y'_m(t)\|^2 - \frac{\delta}{2}\|y_m(t)\|^2_{H_0^1} \\ &\quad + \delta m\|y_m\|^2 - \delta c_3 - \delta(\alpha-\delta)(y'_m,y_m) \\ &\quad + (y'_m,g(y_m))d(t) - \delta(y_m,g(y_m))d(t). \end{aligned}$$

Taking into account the *Poincaré* inequality $\|y_m\|^2_{H^1_0} \geq \lambda_1 \|y_m\|^2$ and assumption

$$\lambda_1 - 2a > 0,$$

we derive that for sufficiently small $\delta \in (0, \alpha)$ there exists a constant $c_5 > 0$ such that

$$\frac{d}{dt} Y_m(t) \leq -\delta Y_m(t) + c_5(1 + \|d\|^2). \tag{3.6}$$

Using estimate (3.6) and assumption (3.2) we get

$$\begin{aligned}\frac{1}{2}\|y_m'\|^2 &+ \left(\frac{1}{2} - \frac{a}{\lambda_1}\right)\|y_m\|^2_{H^1_0} + \delta(y_m', y_m) - c_2|\Omega| \\ &\leq \left(\frac{1}{2}\|y_m'(0)\|^2 + \frac{1}{2}\|y_m(0)\|^2_{H^1_0} + (F(y_m(0)), 1)\right) e^{-\delta t} \\ &\quad + \delta(y_m'(0), y_m(0))e^{-\delta t} + c_5\left(\frac{1}{\alpha} + \int_0^t \|d(s)\|^2 e^{-\delta(t-s)}\, ds\right).\end{aligned}$$

Thus, there exists a constant $c_6 > 0$ such that for sufficiently small $\delta > 0$ and for every $m \geq 1$ the following estimate holds:

$$\begin{aligned}\|y_m'(t)\|^2 + \|y_m(t)\|^2_{H^1_0} \leq c_6\Big(&(\|y_m'(0)\|^2 + \|y_m(0)\|^2_{H^1_0} \\ &+ \|y_m(t)\|^{\frac{2n-2}{n-2}}_{H^1_0})e^{-\delta t} + 1 + \int_0^t \|d(s)\|^2 e^{-\delta(t-s)}\, ds\Big).\end{aligned} \tag{3.7}$$

This estimate allows us to claim that solutions y_m exist on $[0, T]$ and for some function $z = \begin{pmatrix} y \\ y_t \end{pmatrix} \in L^\infty(0, T; X)$ up to subsequence

$$\begin{aligned}&y_m \to y \text{ weak-* in } L^\infty(0, T; H^1_0(\Omega)), \\ &y_m' \to y_t \text{ weak-* in } L^\infty(0, T; L^2(\Omega)).\end{aligned} \tag{3.8}$$

So, due to the Compactness Lemma [18]

$$y_m \to y \text{ in } L^2(0, T; L^2(\Omega)) \text{ and almost everywhere (a.e.) on } (0, T) \times \Omega. \tag{3.9}$$

Then

$$f(y_m) \to f(y),\ g(y_m) \to g(y) \text{ weakly in } L^2(0, T; L^2(\Omega)). \tag{3.10}$$

Passing to the limit in (3.5), we get that the function $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ satisfies (3.4) with $z(0) = z_0$. Therefore, z is the required solution of (2.1), and estimate (3.7) takes place. Lemma is proved. □

Remark 3.2. Since for the solution $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$

$$f(y),\ g(y)d(t) \in L^2(0,T;L^2(\Omega)),$$

then from [1] it follows that functions

$$t \mapsto \|y_t(t)\|^2 + \|y(t)\|^2_{H_0^1},\ t \mapsto (F(y(t)),1)\,,\ t \mapsto (y_t(t),y(t))$$

are absolutely continuous. Therefore, for the function

$$Y(t) = \frac{1}{2}\|y_t(t)\|^2 + \frac{1}{2}\|y(t)\|^2_{H_0^1} + (F(y(t)),1) + \delta(y_t(t),y(t))$$

we can repeat all arguments (3.6), (3.7) and obtain that every solution of (2.1) satisfies (3.7).

Moreover, if $d \in L^\infty(\mathbb{R}_+)$, then from (3.7) we deduce that every solution of (2.1) $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ satisfies the following estimate: $\forall t \geq 0$

$$\|y_t(t)\|^2 + \|y(t)\|^2_{H_0^1} \leq c_6 \Big((\|y_t(0)\|^2 + \|y(0)\|^2_{H_0^1} + \|y(0)\|^{\frac{2n-2}{n-2}}_{H_0^1})e^{-\delta t} + 1 + \frac{1}{\delta}\|d\|^2_\infty \Big). \quad (3.11)$$

Remark 3.3. For $n = 1,2$ in estimates (3.7), (3.11) the term with degree $\frac{2n-2}{n-2}$ is absent.

Lemma 3.2. *Let* $\{z_n = \begin{pmatrix} y \\ y_{n_t} \end{pmatrix}\}$ *be solutions of* (2.1) *on* $(0,T)$ *with disturbances* $\{d_n\} \subset L^2(0,T)$, *initial conditions* $\{z_n^0\} \subset X$, *and* $t_n \to t_0$. *If*

$$z_n^0 \to z^0 \ \textit{weakly in}\ X,\ d_n \to d \ \textit{weakly in}\ L^2(0,T), \quad (3.12)$$

then there exists a solution of (2.1) $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ *on* $(0,T)$ *such that* $z(0) = z_0$ *and up to subsequence*

$$z_n(t_n) \to z(t_0) \ \textit{weakly in}\ X. \quad (3.13)$$

If convergence in (3.12) *is strong, then*

$$z_n(t_n) \to z(t_0) \ \textit{in}\ X.$$

Proof. Assume that (3.12) are fulfilled. Using estimate (3.7) and the Compactness Lemma we can repeat arguments (3.9) and claim that z_n converges to z in the sense of (3.8), (3.9). Moreover,

$$y_n(t_n) \to y(t_0) \text{ in } L^2(\Omega),\ y_{n_t}(t_n) \to y_t(t_0) \text{ in } H^{-1}(\Omega). \quad (3.14)$$

Due to (3.9) and *Lebesgue's* dominated convergence theorem we get

$$(g(y_n),\psi) \to (g(y),\psi) \text{ in } L^2(0,T). \quad (3.15)$$

So, we can pass to the limit in (3.4) and obtain that $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ is a solution of (2.1), $z(0) = z_0$.

Estimate (3.7), convergence (3.14) and compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ guarantee that (3.13) is fulfilled.

Let convergence in (3.12) be strong. Taking into account Remark 3.2, for the absolutely continuous function

$$V_n(t) = \frac{1}{2}\|y_{n_t}(t)\|^2 + \frac{1}{2}\|y_n(t)\|^2_{H_0^1} + (F(y_n(t)), 1)$$

we have the following equality: for almost all $t \in (0, T)$

$$\frac{d}{dt}V_n(t) = -\alpha\|y_{n_t}(t)\|^2 + (g(y_n(t)), y_{n_t}(t))\, d_n(t).$$

So, for all $t \in [0, T]$, in particular, for $t = t_n$, we deduce:

$$\begin{aligned} &\frac{1}{2}\left(\|y_{n_t}(t_n)\|^2 + \|y_n(t_n)\|^2_{H_0^1}\right) + \alpha \int_0^{t_n} \|y_{n_t}(s)\|^2\, ds \\ &\qquad = V_n(0) - (F(y_n(t_n)), 1) + \int_0^{t_n} (g(y_n(s)), y_{n_t}(s))\, d_n(s)\, ds. \qquad (3.16) \end{aligned}$$

Let us justify the limit transition in the right-hand part of (3.16). It is clear that $V_n(0) \to V(0)$. Due to (3.14)

$$F(y_n(t_n, x)) \to F(y(t_0, x)) \text{ for a.a. } x \in \Omega.$$

Additionally, due to the compact embedding $H_0^1(\Omega) \subset L^{\frac{2n-2}{n-2}}(\Omega)$ we have

$$y_n(t_n) \to y(t_0) \text{ in } L^{\frac{2n-2}{n-2}}(\Omega).$$

Since from (3.1) we get the estimate

$$|F(s)| \leq c_7\left(1 + |s|^{\frac{2n-2}{n-2}}\right),$$

so due to $Lebesgue's$ dominated convergence theorem

$$(F(y_n(t_n)), 1) \to (F(y(t_0)), 1)\,. \qquad (3.17)$$

From the same reasons

$$g(y_n) \to g(y) \text{ in } L^2(0, T; L^2(\Omega)).$$

Thus, from (3.8) and strong convergence $d_n \to d$ in $L^2(0, T)$ we derive:

$$\int_0^T (g(y_n(\tau)), y_{n_t}(\tau))\, d_n(\tau)\, d\tau \to \int_0^T (g(y(\tau)), y_t(\tau))\, d(\tau)\, d\tau. \qquad (3.18)$$

Estimate (3.7) implies

$$\int_{t_0}^{t_n} |\,(g(y_n(s)), y_{n_t}(s))\, d_n(s)|\, ds \leq c \int_{t_0}^{t_n} |d_n(s)|\, ds \to 0 \ as\ n \to \infty.$$

Therefore, from (3.18) we can justify the limit transition in the last term of equality (3.16). Then (3.16) yields

$$\begin{aligned} \frac{1}{2} \varliminf_{n\to\infty} \left(\|y_{n_t}(t_n)\|^2 + \|y_n(t_n)\|^2_{H_0^1} \right) + \alpha \int_0^{t_0} \|y_t(s)\|^2\, ds \\ \leq V(0) - (F(y(t_0)), 1) + \int_0^{t_0} (g(y(s)), y_t(s))\, d(s)\, ds \\ = \frac{1}{2} \left(\|y_t(t_0)\|^2 + \|y(t_0)\|^2_{H_0^1} \right) + \alpha \int_0^{t_0} \|y_t(s)\|^2\, ds. \end{aligned} \quad (3.19)$$

From (3.19) we deduce that $\varliminf_{n\to\infty} \|z_n(t_n)\|_X \leq \|z(t_0)\|_X$, which means that $z_n(t_n)$ converges to $z(t_0)$ strongly in X. Lemma is proved. □

4. Local ISS property for the attractor.

We consider the undisturbed problem

$$\begin{cases} \frac{\partial^2 y(t,x)}{\partial t^2} + \alpha \frac{\partial y(t,x)}{\partial t} - \triangle y(t,x) + f(y(t,x)) = 0,\ t > 0,\ x \in \Omega, \\ y(t,x)|_{x\in\partial\Omega} = 0. \end{cases} \quad (4.1)$$

Under assumptions (3.1), (3.2) it is known [5], that the m-semiflow

$$S_0(t, z_0) = \{z(t) \mid z = \begin{pmatrix} y \\ y_t \end{pmatrix} \text{ is a solution of } (4.1), z(0) = z_0\} \quad (4.2)$$

possesses global attractor Θ in the phase space $X = H_0^1(\Omega) \times L^2(\Omega)$.

Lemma 3.2 and estimate (3.11) guarantee the following properties of S_0:

$$\forall t_n \to t_0 \geq 0,\ \forall z_0^n \to z_0,\ \forall \xi_n \in S_0(t_n, z_0^n)$$

$$\text{up to subsequence } \xi_n \to \xi_0 \in S_0(t_0, z_0), \quad (4.3)$$

$$\forall r > 0 \text{ the set } \{S_0(t, z_0) \mid t \geq 0,\ \|z_0\|_X \leq r\} \text{ is bounded in } X. \quad (4.4)$$

Properties (4.3), (4.4) imply stability of Θ in the following sense [17]:

$$\exists \beta \in \mathcal{KL}\ \forall z_0 \in X,\ \forall t \geq 0\ \|S_0(t, z_0)\|_\Theta \leq \beta(\|z_0\|_\Theta, t). \quad (4.5)$$

Let us consider the family of maps $\{S_d\}_{d\in U}$ defined in (2.2). Here $U = L^\infty(\mathbb{R}_+)$ describes the set of disturbances in (2.1).

In addition to conditions (3.1)-(3.3), we will make an additional assumption:

$$f \in C^1(\mathbb{R}) \text{ and } \exists c_8 > 0 \ \forall s \in \mathbb{R} \ |f'(s)| \leq c_8(1 + |s|^r), \ r < \frac{n}{n-2}. \tag{4.6}$$

It is known [6], that assumption (4.6) ensures the uniqueness of solution in (4.1), i.e., the map S_0 defined by (4.2) is single-valued and generates a classical semi-group. It should be noted that the function g can be non-smooth, so we cannot expect uniqueness for the disturbed problem (2.1).

Theorem 4.1. *Assume that conditions* (3.1)-(3.3), (4.6) *are fulfilled. Then the family*

$$\{S_d\}_{d\in U}, \ U = L^\infty(\mathbb{R}_+)$$

possesses local ISS property for the global attractor Θ*, i.e.,*

$$\exists r > 0, \ \exists \beta \in \mathcal{KL}, \ \exists \gamma \in \mathcal{K} \text{ such that}$$

$$\forall \|z_0\|_\Theta \leq r, \ \forall \|d\|_\infty \leq r, \ \forall t \geq 0$$
$$\|S_d(t, z_0)\|_\Theta \leq \beta(\|z_0\|_\Theta, t) + \gamma(\|d\|_\infty). \tag{4.7}$$

Proof. According to [17], it is enough to verify the following properties:

$$\forall r > 0 \text{ the set } \{S_d(t, z_0) \mid t \geq 0, \ \|d\|_\infty \leq r, \ \|z_0\|_X \leq r\} \text{ is bounded in } X, \tag{4.8}$$

$$\forall r > 0 \ \exists c(r) > 0 \ \forall \|z_0^{(1)}\|_X \leq r, \ \|z_0^{(2)}\|_X \leq r, \ \forall t \geq 0$$
$$\|S_0(t, z_0^{(1)}) - S_0(t, z_0^{(2)})\|_X \leq e^{c(r)t}\|z_0^{(1)} - z_0^{(2)}\|_X, \tag{4.9}$$

$$\exists \ \kappa \in \mathcal{K}, \ \exists \ \eta : \mathbb{R}_+^2 \mapsto \mathbb{R}_+ \text{ such that } \forall r > 0$$
$$\overline{\lim_{t\to 0+}} \frac{\eta(r,t)}{t} < \infty \text{ and } \forall t \geq 0, \ \forall \|z_0\|_X \leq r, \ \forall \|d\|_\infty \leq r$$
$$dist(S_d(t, z_0), S_0(t, z_0)) \leq \eta(r, t)\kappa(\|d\|_\infty). \tag{4.10}$$

Property (4.8) is a consequence of estimate (3.11). Property (4.9) can be derived from the following arguments [16]: for $\|y_1\|_{H_0^1} \leq r$, $\|y_2\|_{H_0^1} \leq r$ from (4.6), $H\ddot{o}lder's$ inequality and embedding $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$ we get

$$\int_\Omega |f(y_1) - f(y_2)|^2 \, dx \leq$$

$$c\left(1 + \|y_1\|_{L^{\frac{2n}{n-2}}}^{\frac{n}{n-2}} + \|y_2\|_{L^{\frac{2n}{n-2}}}^{\frac{n}{n-2}}\right)\|y_1 - y_2\|_{L^{\frac{2n}{n-2}}}^2 \leq c(r)\|y_1 - y_2\|_{H_0^1}^2. \tag{4.11}$$

Let $z^{(1)} = \begin{pmatrix} y^{(1)} \\ y_t^{(1)} \end{pmatrix}$, $z^{(2)} = \begin{pmatrix} y^{(2)} \\ y_t^{(2)} \end{pmatrix}$ be solutions of (4.1), and $\|z^{(1)}(0)\|_X \leq r$, $\|z^{(2)}(0)\|_X \leq r$. Then from (4.11) for the function $\omega(t) = y^{(1)}(t) - y^{(2)}(t)$, we deduce:

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega_t\|^2 + \|\omega\|_{H_0^1}^2\right) + \alpha\|\omega_t\|^2 \leq c^{\frac{1}{2}}(r)\|\omega\|_{H_0^1}\|\omega_t\|,$$

$$\frac{d}{dt}\left(\|\omega_t\|^2 + \|\omega\|^2_{H_0^1}\right) \leq c^{\frac{1}{2}}(r)\left(\|\omega_t\|^2 + \|\omega\|^2_{H_0^1}\right).$$

After applying *Grönwall's* lemma we obtain (4.9).

For proving (4.10) we consider arbitrary solution $z^{(1)} = \begin{pmatrix} y^{(1)} \\ y_t^{(1)} \end{pmatrix}$ of (2.1) with disturbance d, $\|d\|_\infty \leq r$ and initial data z_0. Let $z^{(2)} = \begin{pmatrix} y^{(2)} \\ y_t^{(2)} \end{pmatrix}$ be a unique solution of (4.1) with initial data z_0, $\|z_0\|_X \leq r$. Then for the function $\omega(t) = y^{(1)}(t) - y^{(2)}(t)$ we have the following estimate: for a.a. $t \in (0, T)$

$$\frac{d}{dt}\left(\|\omega_t\|^2 + \|\omega\|^2_{H_0^1}\right) \leq c^{\frac{1}{2}}(r)\left(\|\omega_t\|^2 + \|\omega\|^2_{H_0^1}\right) + c_4|\Omega|^{\frac{1}{2}}\|d\|_\infty \sup_{t\in[0,T]}\left(\|\omega_t\| + \|\omega\|_{H_0^1}\right). \tag{4.12}$$

Integrating over $[0, t]$, we get: $\forall\ t \in (0, T)$

$$\|\omega_t(t)\|^2 + \|\omega(t)\|^2_{H_0^1} \leq c^{\frac{1}{2}}(r)\int_0^t \left(\|\omega_t(s)\|^2 + \|\omega(s)\|^2_{H_0^1}\right) ds + c_4 T|\Omega|^{\frac{1}{2}}\|d\|_\infty \sup_{t\in[0,T]}\left(\|\omega_t\| + \|\omega\|_{H_0^1}\right). \tag{4.13}$$

After applying *Grönwall's* lemma from (4.13) we derive the existence of $c > 0$, $\eta(r) > 0$ such that

$$\sup_{t\in[0,T]} \|z^{(1)}(t) - z^{(2)}(t)\|_X \leq c\|d\|_\infty T e^{\eta(r)T}.$$

So, we have (4.10). Theorem is proved. □

5. AG property for the attractor.

In this part of the work we show that under assumptions (3.1)-(3.3) for sufficiently wide class of disturbances $U_1 \subset L^\infty(\mathbb{R}_+)$ the global attractor Θ of the m-semiflow S_0 is globally stable in the AG sense, i.e., robust estimate (2.4) takes place.

Assume that the set of disturbances U_1 consists of all functions $d \in L^\infty(\mathbb{R}_+)$ with

$$\sup_{t\geq 0}\int_t^{t+1} |d(s+\tau) - d(s)|^2\, ds \leq \psi(|l|), \tag{5.1}$$

where ψ may depend on d and $\psi(p) \to 0,\ p \to 0+$.

Property (5.1) is true for absolutely continuous functions $d \in L^\infty(\mathbb{R}_+)$ with $d' \in L^\infty(\mathbb{R}_+)$.

It is clear that the set U_1 is translation-invariant, i.e.,

$$\forall d() \in U_1,\ \forall h \geq 0\ d(+h) \in U_1.$$

Moreover, it is known [6] that for every $d \in U_1$ the set

$$\Sigma(d) := cl_{L^2_{loc}} \{d(+h) \mid h \geq 0\}$$

is a translation-invariant compact subset of $L^2_{loc}(\mathbb{R}_+)$, $d \in \Sigma(d)$, $\Sigma(0) = \{0\}$ i $\forall \sigma \in \Sigma(d)$

$$\sup_{t\geq 0} \int_t^{t+1} |\sigma(s)|^2\, ds \leq \sup_{t\geq 0} \int_t^{t+1} |d(s)|^2\, ds \leq \|d\|^2_\infty. \tag{5.2}$$

Theorem 5.1. *Assume that conditions* (3.1)-(3.3), (5.1) *are fulfilled. Then the family* $\{S_d\}_{d\in U_1}$ *possesses AG property for the global attractor* Θ, *i.e.,*

$$\exists \gamma \in \mathcal{K}\ \forall d \in U_1,\ \forall z_0 \in X$$

$$\overline{\lim_{t\to\infty}} \|S_d(t, z_0)\|_\Theta \leq \gamma(\|d\|_\infty). \tag{5.3}$$

Proof. As $\forall t \geq 0,\ \forall d \in U_1,\ \forall \sigma \in \Sigma(d)$ due to (5.2)

$$\int_0^t |\sigma(s)|^2 e^{-\sigma(t-s)}\, ds \leq \frac{1}{\sigma}\|d\|^2_\infty, \tag{5.4}$$

so from (3.7) we derive: $\exists c > 0\ \forall r > 0\ \exists T(r)\ \forall t \geq T(r),\ \forall \|z_0\|_X \leq r$ and for arbitrary solution $z()$ of (2.1) with $z(0) = z_0$ and disturbance $\sigma \in \Sigma(d)$ the following estimate holds

$$\|z(t)\|_X \leq c(1 + \|d\|_\infty). \tag{5.5}$$

Taking into account dissipative property (5.5), compactness of $\Sigma(d)$, estimate (5.2), and abstract results from [16], we conclude that for proving robust estimate (5.3) it is sufficient to verify the following properties:

$$\sigma_n \to \sigma \text{ in } L^2_{loc}(\mathbb{R}_+),\ z_0^n \to z_0 \text{ in } X,\ \xi_n \in S_{\sigma_n}(t, z_0^n),\ \xi_n \to \xi \text{ in } X \Rightarrow$$

$$\Rightarrow \xi \in S_\sigma(t, z_0), \tag{5.6}$$

$$\{\sigma_n\} \subset \Sigma(d),\ d \in U_1 \text{ (or } \sigma_n \in \Sigma(d_n),\ \|d_n\|_\infty \to 0),\ z_0^n \to z_0 \text{ weakly in } X,\ t_n \nearrow \infty,$$

$$\xi_n \in S_{\sigma_n}(t_n, z_0^n) \Rightarrow \{\xi_n\} \text{ is precompact in } X. \tag{5.7}$$

Property (5.6) is a direct consequence of Lemma 3.2.

Let us prove (5.7). We put $\xi_n = z_n(t_n)$, where $z_n()$ is a solution of (2.1) with $d = \sigma_n,\ z_n(0) = z_0^n$.

From estimates (3.7),(5.2) and assumption (5.7) we derive that the sequence $\{\xi_n\}$ is bounded in X. So, up to subsequence

$$\xi_n \to \xi \text{ weakly in } X. \tag{5.8}$$

We can extract a subsequence such that $\forall M \geq 1$

$$z_n(t_n - M) \to \xi_M \text{ weakly in } X.$$

Moreover, $\forall t \geq 0$ for sufficiently large n we have from the cocycle property:

$$z_n(t_n - M + t) \in S_{\sigma_n(+t_n - M)}(t, 0, z_n(t_n - M)).$$

Let us put $\bar{\sigma_n}(t) := \sigma_n(t + t_n - M)$. Assumption (5.7) allows us to claim that for some $\bar{\sigma}$ we have that

$$\bar{\sigma_n} \to \bar{\sigma} \text{ in } L^2_{loc}(\mathbb{R}_+). \tag{5.9}$$

Therefore, from Lemma 3.2 for $\bar{z_n}(t) = z_n(t + t_n - M)$ we have that $\forall t \geq 0$

$$\bar{z_n}(t) \to \bar{z}(t) \text{ weakly in } X,$$

$$\bar{z}(t) \in S_{\bar{\sigma}}(t, 0, \xi_M).$$

In particular,

$$\bar{z_n}(M) = \xi_n \to \bar{z}(M) = \xi \text{ weakly in } X.$$

It is known [5] that every solution $z()$ of (2.1) with disturbance $d()$ satisfies the equality

$$\frac{d}{dt} I(z(t)) + \alpha I(z(t)) = H_d(t, z(t)), \tag{5.10}$$

where

$$\begin{aligned} I(z) &= \frac{1}{2}\|y_t\|^2 + \frac{1}{2}\|y\|^2_{H^1_0} + (F(y), 1) + \frac{\alpha}{2}(y_t, y), \\ H_d(t, z) &= \alpha(F(y(t)), 1) - \frac{\alpha}{2}(f(y(t)), y(t)) \\ &\quad + \frac{\alpha}{2}(g(y(t)), y(t))d(t) + (g(y(t)), y_t(t))d(t). \end{aligned}$$

We write (5.10) for $\bar{z_n}$ and after integrating over $[0, M]$ we get:

$$I(\xi_n) = I(z_n(t_n - M))e^{-\alpha M} + \int_0^M e^{\alpha(p - M)} H_{\bar{\sigma_n}}(p, \bar{z_n}(p))\, dp. \tag{5.11}$$

Applying to $\{\bar{z}_n\}$ arguments (3.17),(3.18), and taking into account strong convergence (5.9), we deduce that $\forall M \geq 0$

$$\int_0^M e^{\alpha(p-M)} H_{\bar{\sigma_n}}(p, \bar{z_n}(p))\, dp \to \int_0^M e^{\alpha(p-M)} H_{\bar{\sigma}}(p, \bar{z}(p))\, dp \text{ as } n \to \infty.$$

From estimate (3.7) $\exists c > 0\ \ \forall t \geq 0,\ \forall n \geq 1$

$$|I(z_n(t))| \leq c, \tag{5.12}$$

where c does not depend on M.

Then from (5.11), (5.12) we conclude that

$$\overline{\lim_{n\to\infty}} I(\xi_n) \le ce^{-\alpha M} + \int_0^M e^{-\alpha(p-M)} H_{\bar\sigma}(\bar z(p))\,dp$$
$$= ce^{-\alpha M} + I(\xi) - I(\xi_M)e^{-\alpha M} \le 2ce^{-\alpha M} + I(\xi).$$

Thus,

$$\overline{\lim_{n\to\infty}} \frac{1}{2}\|\xi_n\|_X^2 \le 2ce^{-\alpha M} + \frac{1}{2}\|\xi\|_X^2.$$

Passing to the limit as $M \to \infty$, we get

$$\overline{\lim_{n\to\infty}} \|\xi_n\|_X \le \|\xi\|_X.$$

Combining this inequality with weak convergence (5.8), we obtain that the sequence $\{\xi_n\}$ is precompact in X. Theorem is proved. $\square$

References

1. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, AMS, **68**, 1997.
2. J.C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge University Press, 2001.
3. V.S. Melnik, *Multivalued dynamics of nonlinear infinite-dimensional*, Kyiv, 1994. (Preprint / Acad. Sci. Ukraine. Inst. Cybernetics, # 94–17).
4. V.S. Melnik, J. Valero, *On attractors of multivalued semi-flows and differential inclusions*, Set-Valued Analysis, **6**, 1998, 83–111.
5. J. Ball, *Global attractors for damped semilinear wave equations*, Discrete and Continuous Dynamical Systems, **10**, 2004, 31–52.
6. V.V. Chepyzhov, M.I.. Vishik, *Attractors for equations of mathematical physics*, AMS, **49**, Providence, RI, 2002.
7. O.V. Kapustyan, P.O. Kasyanov, J. Valero, M.Z. Zgurovsky, *Structure of Uniform Global Attractor for General Non-Autonomous Reaction-Diffusion System*, Continuous and Distributed Systems: Theory and Applications, Solid Mechanics and Its Applications, **211**, 2014, 251–264.
8. O.V. Kapustyan, M.O. Perestyuk, *Global Attractors in Impulsive Infinite-Dimensional Systems*, Ukrainian Mathematical Journal, **68(4)**, 2016, 583–597.
9. S. Dashkovskiy, P. Feketa, O. Kapustyan, I. Romaniuk, *Invariance and stability of global attractors for multi-valued impulsive dynamical systems*, Journal of Mathematical Analysis and Applications, **458(1)**, 2018, 193–218.
10. S. Dashkovskiy, A. Mironchenko, *Input-to-state stability of infinite-dimensional control systems*, Mathematics of Control, Signals and Systems, **25(1)**, 2013, 1–35.
11. E.D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Transactions on Automatic Control, **34(4)**, 1989, 435–443.
12. E.D. Sontag, Y. Wang, *On characterizations of the input-to-state stability property*, Systems & Control Letters, **24(5)**, 1995, 351–359.
13. A. Mironchenko, *Local input-to-state stability: Characterizations and counterexamples*, Systems & Control Letters, **87**, 2016, 23–28.

14. S. Dashkovskiy, O. Kapustyan, J. Schmid, *A local input-to-state stability result w.r.t. attractors of nonlinear reaction?diffusion equations*, Mathematics of Control, Signals and Systems, **32(3)**, 2020, 309–326.
15. J. Schmid, O. Kapustyan, S. Dashkovskiy, *Asymptotic gain results for attractors of semilinear systems*, Mathematical Control and Related Fields, 2021.
16. S. Dashkovskiy, O.V. Kapustyan, *Robustness of global attractors: abstract framework and application to dissipative wave equations*, Evolution Equations and Control Theory, **11(5)**, 2022, 1565—1577.
17. O.V. Kapustyan, V.V. Sobchuk, T.V. Yusypiv, A.V. Pankov, *Robust stability of global attractors for evolutionary systems without uniqueness*, Journal of optimization, differential equations and their applications (JODEA), **30(2)**, 2022, 49–61.
18. G.R. Sell, Y. You, *Dynamics of Evolutionary Equations*, Springer, 2002.

Received 13.04.2023

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **31**, Issue **1**, June **2023**, pp. 125–142, DOI 10.15421/142307

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

ON THE ASYMPTOTIC EQUIVALENCE OF ORDINARY AND FUNCTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

Olexandr M. Stanzhytskyi*, Grygoriy O. Petryna†, Maryna V. Hrysenko‡

Abstract. This paper studies the asymptotic behavior of solutions of linear stochastic functional-differential equations. This behavior is investigated using the method of asymptotic equivalence, according to which an ordinary system of linear differential equations is constructed based on the initial stochastic system, and the asymptotic behavior of the solutions of this system is analogous to the behavior of the solutions of the initial system.

Key words: asymptotic behavior, delay, Cauchy matrix, probability.

2010 Mathematics Subject Classification: 60H15, 60H30, 35R60.

Communicated by Prof. P. Kogut

1. Introduction

The work is dedicated to the study of the asymptotic behavior of solutions in linear systems of stochastic functional-differential equations. Functional-differential equations model evolutionary processes in which the future depends not only on the current state but also on the system's past state (delay effect). The presence of delays significantly influences the qualitative behavior of the system. The right-hand side of such mathematical models is a functional of a segment of the solution, which complicates the research object and requires the development and application of methods of infinite-dimensional analysis. The wide application of such models has led to a rapid development of the theory of functional-differential equations. Its foundations for deterministic functional-differential equations in the finite-dimensional case are thoroughly presented in the monograph [1], and for deterministic equations in the infinite-dimensional case in the monograph [2]. As for stochastic functional-differential equations in finite-dimensional spaces, the monograph [3] provides a detailed bibliography and presents elements of the asymptotic and qualitative theory of such equations. Regarding stochastic functional-differential equations in infinite-dimensional spaces, the monograph [4] is noteworthy. The existence of invariant measures in shift spaces for stochastic functional-differential equations with partial derivatives is addressed in works [5–8]. In this

*Department of General Mathematics, Taras Shevchenko National University of Kyiv, 4e, Akademika Hlushkova av., Kyiv, 03127, Ukraine, ostanzh@gmail.com
†Department of General Mathematics, Taras Shevchenko National University of Kyiv, 4e, Akademika Hlushkova av., Kyiv, 03127, Ukraine, grpetryna@gmail.com
‡Department of General Mathematics, Taras Shevchenko National University of Kyiv, 4e, Akademika Hlushkova av., Kyiv, 03127, Ukraine, magryss@gmail.com

work, the asymptotic behavior of solutions at infinity is investigated using a well-known method in the theory of differential equations called the method of asymptotic equivalence. According to this method, a simpler system is constructed based on the original system, and the behavior of solutions at infinity of the simpler system is equivalent to the behavior of solutions of the original system. The classical result by Levinson [9] is relevant in the linear case. For stochastic systems without delay, this approach is further developed in works [10, 11]. The article is structured as follows: Section 2 introduces the notation and formulates the main results. Section 3 is devoted to proving the main results of the study. Finally, an illustrative example is provided at the end of the work.

2. Preliminaries

For $h > 0$ we define a function space $C_h = C([-h, 0]; \mathbf{R}^d)$ of continuous functions with a norm $||\phi||_C = \sup_{\theta\in[-h,0]} |\phi(\theta)|$. We denote the norm of a vector in $\mathbf{R}^d$ space using the symbol $|\cdot|$ and the norm of a $(d \times d)$ matrix, consistent with a vector norm, using $||\cdot||$ throughout this paper. Consider the system of ordinary differential equations (ODE) in the following form

$$dx = Ax\,dt, \tag{2.1}$$

with the initial conditions $x(t_0) = x_0,\ \ t \geq t_0 \geq 0,\ \ x \in \mathbf{R}^d$, and A be a constant deterministic matrix. Along with system (2.1), we consider the system of functional stochastic differential equations (FSDE)

$$dy = \left(Ay + \int_{-h}^{0} B(t,\theta) y(t+\theta)\,d\theta \right) dt + \left(\int_{-h}^{0} D(t,\theta) y(t+\theta) d\theta \right) dW(t), \tag{2.2}$$

where $B(t,\theta), D(t,\theta)$ are continuous deterministic matrices for $t \geq 0,\ \theta \in [-h, 0]$, integrable with respect to θ. $W(t)$ is a Wiener process on a probability space $(\Omega, \mathbf{F}, P)$ with filtration $\{\mathcal{F}_t, t \geq 0\} \subset \mathbf{F}$, and there exist such $b(t)$ and $d(t)$

$$\left\| \int_{-h}^{0} B(t,\theta)\phi(\theta)\,d\theta \right\| \leq b(t)\|\phi\|_C, \tag{2.3}$$

$$\left\| \int_{-h}^{0} D(t,\theta)\phi(\theta)\,d\theta \right\| \leq d(t)\|\phi\|_C. \tag{2.4}$$

We introduce the definition of asymptotic equivalence, which is a generalization of the classical definition of asymptotic equivalence for systems of ordinary differential equations to the stochastic case.

Definition 2.1. If for each solution $y(t)$ of system (2.2) there corresponds a solution $x(t)$ of (2.1) such that

$$\lim_{t\to\infty} \mathbf{E}|x(t) - y(t)|^2 = 0,$$

then system (2.2) is called asymptotically mean square equivalent to system (2.1). In case when for each solution $y(t)$ of system (2.2) there corresponds a solution $x(t)$ of system (2.1) such that

$$P\left\{\lim_{t\to\infty}|x(t)-y(t)|=0\right\}=1,$$

then system 2.2 is called asymptotically equivalent to system 2.1 with probability 1.

Now let us formulate the main result of our work.

Theorem 2.1. *Let all solutions of system (2.1) be bounded on $t\in[0,\infty)$. If*

$$\int_0^\infty |b(t)|\,dt \le K_1 < \infty, \tag{2.5}$$

$$\int_0^\infty |d(t)|^2\,dt \le K_1 < \infty, \tag{2.6}$$

Then system (2.2) is asymptotically equivalent to the system (2.1) in the mean square sense. Also, if we change (2.6) on

$$\int_0^\infty t d(t)^2\,dt \le K_1, \tag{2.7}$$

then, (2.2) is asymptotically equivalent to the system (2.1) with the probability 1.

3. Proof of the main result

Proof. This theorem consists of two parts the following proof will deal with them sequentially. We will start with the first part.
By our conditions, the solutions of system (2.1) are bounded, hence the eigen values $\lambda(A)$ of the matrix A satisfy the inequality $Re\lambda(A)$, also the values which real part equals to zero have simple elementary divisors. We can assume that matrix A has a quasi-diagonal form,

$$A=\operatorname{diag}(A_1,A_2), \tag{3.1}$$

where A_1 and A_2 are $(p\times p)$ and $(q\times q)$ - matrices, $p+q=d$, such that

$$Re\lambda(A_1)\le-\alpha<0, \quad Re\lambda(A_2)=0. \tag{3.2}$$

Let

$$X(t)=\operatorname{diag}(e^{tA_1},e^{tA_2}), \tag{3.3}$$

be a fundamental matrix of system (2.1), normalized in zero, $X(0)=E_d$, and

$$I_1=\operatorname{diag}(E_p,0), \quad I_2=\operatorname{diag}(0,E_q),$$

where E_p and E_q are the identity matrices of order p and q, and $I_1 + I_2 = E_d$. Let's define

$$\begin{aligned} X(t) &= X_1(t) + X_2(t) = X(t)I_1 + X(t)I_2 \\ &= \mathrm{diag}(e^{tA_2}, 0) + \mathrm{diag}(0, e^{tA_2}). \end{aligned} \tag{3.4}$$

Therefore, the Cauchy matrix can be written in the following way

$$\begin{aligned} \tilde{X} &= X(t)X^{-1}(\tau) = X(t-\tau) \\ &= X_1(t-\tau) + X_2(t-\tau). \end{aligned} \tag{3.5}$$

Using our previous estimates we get

$$\|X_1(t)\| = \left\|e^{tA_1}\right\| \le a_1 e^{-\alpha t}, \quad t \ge t_0 \ge 0, \tag{3.6}$$

$$\|X_2(t)\| = \left\|e^{tA_2}\right\| \le a_2, \quad t \in \mathbb{R}. \tag{3.7}$$

Where a_1, a_2, α are some positive constants. Let us write a solution of system (2.2) with the initial conditions $y(t_0) = y_0$ in terms of a Cauchy matrix for the deterministic differential system (2.1).

$$\begin{aligned} y(t) = X(t-t_0)y(t_0) &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{t} X_2(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau) \\ &+ \int_{t_0}^{t} X_2(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau), \end{aligned} \tag{3.8}$$

for $t \ge t_0 \ge 0$ and $\theta \in [-h, 0]$. Using the evolution properties of the matriciant

$$X_2(t-\tau) = X(t-\tau)I_2 = X(t-t_0)X(t_0-\tau)I_2 = X(t-t_0)X_2(t_0-\tau), \tag{3.9}$$

we can rewrite (3.8) in the following way:

$$\begin{aligned} y(t) = X(t-t_0)\{y(t_0) &+ \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau)\} \\ &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &+ \int_{t_0}^{t} X_1(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau) \\ &- \int_{t}^{\infty} X_2(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ &- \int_{t}^{\infty} X_2(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau). \end{aligned} \tag{3.10}$$

Let $y(t) \equiv y(t,\omega)$ be a solution of system (2.2) with the initial condition $y(t_0+\theta) = \phi(\theta),\ \theta \in [-h,0]$, which correspond to a solution $x(t)$ of system (2.1) with the initial condition

$$\begin{aligned} x(t_0) = y(t_0) + \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ + \int_{t_0}^{\infty} X_2(t_0-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau). \end{aligned} \tag{3.11}$$

For every solution of system (2.2) with the initial condition $y(t_0+\theta) = \phi(\theta)$ by formula (3.11) we define correspondence between the set of solutions $\{y(t) \equiv y(t,\omega)\}$ of system (2.2) and the set of solutions $\{x(t)\}$ of system (2.1)
Now we can start proving our first statement. We know that

$$\begin{aligned} y(t) = X(t-t_0)y(t_0) + \int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \\ + \int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau), \end{aligned} \tag{3.12}$$

Hence, using the stochastic integral properties and the above equality we obtain

$$\begin{aligned} \mathbf{E}|y(t)|^2 \le 3||X(t-t_0)||^2 \mathbf{E}|y(t_0)|^2 \\ + 3\mathbf{E}\left| \int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \right|^2 \\ + 3\mathbf{E}\left| \int_{t_0}^{t} X(t-\tau) \int_{-h}^{0} [D(\tau,\theta)y(\tau+\theta)]\, d\theta\, dW(\tau) \right|^2, \end{aligned} \tag{3.13}$$

For simplicity, let us explicitly consider each term in the above inequality:

$$\|X(t-t_0)\|^2 \mathbf{E}|y(t_0)|^2 \le \max(a_1^2, a_2^2)\mathbf{E}|y(t_0)|, \tag{3.14}$$

$$\begin{aligned} \mathbf{E}\Big| \int_{t_0}^{t} X(t_0-\tau) \int_{-h}^{0} [B(\tau,\theta)y(\tau+\theta)]\, d\theta\, d\tau \Big|^2 \\ \le \mathbf{E}\left(\int_{t_0}^{t} \sqrt{||X(t_0-\tau)||} \sqrt{\|X(t_0-\tau)\|} \sqrt{b(\tau)} \sqrt{b(\tau)} \|y_\tau\|_C \right)^2 \\ \le \int_{t_0}^{t} \|X(t-\tau)\| b(\tau) \mathbf{E}\|y_\tau\|_C^2\, d\tau \int_{t_0}^{t} ||X(t-\tau)|| b(\tau)\, d\tau \\ \le \max(a_1^2, a_2^2) \int_{0}^{\infty} b(\tau)\, d\tau \int_{t_0}^{t} b(\tau) E\|y_\tau\|_C^2\, d\tau, \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&\mathbf{E}\left|\int_{t_0}^{t} X(t_0-\tau)\int_{-h}^{0}[D(\tau,\theta)y(\tau+\theta)]\,d\theta\,dW(\tau)\right|^2\\
&\le \int_{t_0}^{t} E\left|X(t-\tau)\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right|^2 d\tau\\
&\le \int_{t_0}^{t}\|X(t-\tau)\|^2\mathbf{E}\left\|\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right\|^2 d\tau\\
&\le \int_{t_0}^{t}\|X(t-\tau)\|^2 d^2(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau\\
&\le \max(a_1^2,a_2^2)\int_{t_0}^{t} d(\tau)^2\mathbf{E}\|y_\tau\|_C^2\,d\tau.
\end{aligned}
\tag{3.16}
$$

Now we can substitute our estimates into the (3.13).

$$
\begin{aligned}
\mathbf{E}|y(t)|^2 \le{}& 3\max(a_1^2,a_2^2)(\mathbf{E}|y(t_0)|^2+\int_0^\infty b(\tau)\,d\tau\int_{t_0}^{t} b(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau\\
&+\int_{t_0}^{t} d^2(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau),
\end{aligned}
\tag{3.17}
$$

It is clear that

$$
\max_{s\in[0,t]}\mathbf{E}||y_s||_C^2 \le \max_{s\in[-h,0]}\mathbf{E}|\phi(s)|^2+\max_{s\in[0,t]}\mathbf{E}|y(s)|^2, \tag{3.18}
$$

hence,

$$
\begin{aligned}
\max_{s\in[0,t]}\mathbf{E}|y(s)|^2 \le{}& 3\max(a_1^2,a_2^2)\Big(\mathbf{E}\|\phi(\theta)\|_C^2\\
&+\int_0^\infty b(\tau)\,d\tau\int_{t_0}^{t} b(\tau)\max_{s\in[0,\tau]}\mathbf{E}|y(s)|^2\,d\tau\\
&+\int_{t_0}^{t} d^2(\tau)\max_{s\in[0,\tau]}\mathbf{E}|y(s)|^2\,d\tau\Big),
\end{aligned}
\tag{3.19}
$$

Using the Gronwall-Bellman inequality we get:

$$
\begin{aligned}
\max_{s\in[0,t]}\mathbf{E}|y(s)|^2 &\le 3\max(a_1^2,a_2^2)\mathbf{E}\|\phi(\theta)\|_C^2\\
&\quad\times\exp\left(3\max(a_1^2,a_2^2)\int_{t_0}^{t}(K_1b(\tau)+d^2(\tau))\,d\tau\right)\\
&\le 3\max(a_1^2,a_2^2)\mathbf{E}\|\phi(\theta)\|_C^2\\
&\quad\times\exp\left(3\max(a_1^2,a_2^2)\int_0^\infty(K_1b(\tau)+d^2(\tau))\,d\tau\right)\\
&\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2,\\
\tilde{K} &= 3\max(a_1^2,a_2^2)\exp\left(3\max(a_1^2,a_2^2)\int_0^\infty(K_1b(\tau)+d^2(\tau))\,d\tau\right).
\end{aligned}
\tag{3.20}
$$

The latter inequality indicates that the integrals in equation (3.11) exhibit mean square convergence. Next, we will assess the expected difference in square norms between the respective solutions $x(t)$ and $y(t)$. Since

$$x(t) = X(t-t_0)x(t_0), \tag{3.21}$$

where $x(t_0)$ is defined in (3.11), using (3.10) we obtain

$$\begin{aligned}
\mathbf{E}|x(t)-y(t)|^2 &= \mathbf{E}\Big|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau \\
&\quad + \int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau) \\
&\quad - \int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau \\
&\quad - \int_{t}^{\infty} X_2(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta]\,dW(\tau)|^2 \\
&\le 4\mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|^2 \\
&\quad + 4\mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|^2 \\
&\quad + 4\mathbf{E}|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau|^2 \\
&\quad + 4\mathbf{E}|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)|^2.
\end{aligned} \tag{3.22}$$

Using (3.20), let's estimate each term of the last inequality:

$$\begin{aligned}
&\mathbf{E}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|^2 \\
&\le \mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\|\,\left\|\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right\|\,d\tau\right)^2 \\
&\le \mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\|\|b(\tau)\|\|y_\tau\|_C\,d\tau\right)^2 \\
&= \mathbf{E}\left(\int_{t_0}^{t} \sqrt{\|X_1(t-\tau)\|\|b(\tau)}\sqrt{\|X_1(t-\tau)\|\|b(\tau)\|}\|y_\tau\|_C\,d\tau\right)^2 \\
&\le \mathbf{E}\left(\int_{t_0}^{t} \|X_1(t-\tau)\|b(\tau)d\tau \int_{t_0}^{t} ||X_1(t-\tau)||b(\tau)\|y_\tau\|_C^2\,d\tau\right)
\end{aligned}$$

$$\begin{aligned}
&\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\left(\int_{t_0}^{t}\|X_1(t-\tau)\|b(\tau)\,d\tau\right)^2 \\
&\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\left(\int_{t_0}^{t}a_1e^{-\alpha(t-\tau)}b(\tau)\,d\tau\right)^2,
\end{aligned} \tag{3.23}$$

Since $b(t)$ is absolutely integrable for $t \ge 2t_0$:

$$\begin{aligned}
\int_{t_0}^{t}e^{-\alpha(t-\tau)}b(\tau)\,d\tau &= \int_{t_0}^{\frac{t}{2}}e^{-\alpha(t-\tau)}b(\tau)\,d\tau + \int_{\frac{t}{2}}^{t}e^{-\alpha(t-\tau)}b(\tau)\,d\tau \\
&\le e^{-\frac{\alpha t}{2}}\int_{t_0}^{\frac{t}{2}}b(\tau)\,d\tau + \int_{\frac{t}{2}}^{t}b(\tau)\,d\tau \\
&\le e^{-\frac{\alpha t}{2}}\int_{0}^{\infty}b(\tau)\,d\tau + \int_{\frac{t}{2}}^{t}b(\tau)\,d\tau,
\end{aligned}$$

From the last inequality, it becomes evident that the first term on the right-hand side of (3.22) tends to 0, as $t \to \infty$.

$$\begin{aligned}
&\mathbf{E}\Big|\int_{t_0}^{t}X_1(t-\tau)\left[\int_{-h}^{0}D(\tau,\theta)y(\tau+\theta)\,d\theta\right]dW(\tau)\Big|^2 \\
&\qquad\le \int_{t_0}^{t}\mathbf{E}|X_1(t-\tau)\left[\int_{-h}^{0}D(\tau,\theta)y(\tau+\theta)\,d\theta\right]|^2\,d\tau \\
&\qquad\le \int_{t_0}^{t}\|X_1(t-\tau)\|^2d^2(\tau)\mathbf{E}\|y_\tau\|_C^2d\tau \\
&\qquad\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\int_{t_0}^{t}a_1^2e^{-2\alpha(t-\tau)}d^2(\tau)d\tau,
\end{aligned} \tag{3.24}$$

From $d^2(\tau)$ integrability and the previous term we can conclude that the second term in (3.22) also tends to 0 as $t \to \infty$.

$$\begin{aligned}
&\mathbf{E}\Big|\int_{t}^{\infty}X_2(t-\tau)\left[\int_{-h}^{0}B(\tau,\theta)y(\tau+\theta)\,d\theta\right]d\tau\Big|^2 \\
&\qquad\le \mathbf{E}\left(\int_{t}^{\infty}\|X_2(t-\tau)\|\left\|\int_{-h}^{0}B(\tau,\theta)y(\tau+\theta)\,d\theta\right\|d\tau\right)^2 \\
&\qquad\le \mathbf{E}\left(\int_{t}^{\infty}\|X_2(t-\tau)\|b(\tau)\|y_\tau\|_C\,d\tau\right)^2 \\
&\qquad\le \int_{t}^{\infty}\|X_2(t-\tau)\|b(\tau)E\|y_\tau\|_C^2\,d\tau\int_{t}^{\infty}\|X_2(t-\tau)\|b(\tau)\,d\tau \\
&\qquad\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\left(\int_{t}^{\infty}\|X_2(t-\tau)\|b(\tau)\,d\tau\right)^2 \\
&\qquad\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2a_2^2\left(\int_{t}^{\infty}b(\tau)\,d\tau\right)^2,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\mathbf{E}\Big|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|^2&\\
\le \mathbf{E}\int_t^\infty \|X_2(t-\tau)\|^2 d^2(\tau)\mathbf{E}\|y\|_C^2\,d\tau&\qquad(3.26)\\
\le \tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 a_2^2\int_t^\infty d^2(\tau)\,d\tau.&
\end{aligned}$$

Both (3.25, 3.26) tend to 0 as $t\to\infty$. These results prove the first part of our theorem,

$$\lim_{t\to\infty}\mathbf{E}|x(t)-y(t)|^2=0.$$

Next, we will proceed with proving the second part of the theorem. We start with introducing a sequence denoted as $\{n_k|k\ge 1\}$, satisfying the condition $n_k>k$, $k\ge 1$, such that

$$\int_{n_k}^\infty b(\tau)\,d\tau\le\frac{1}{2^k},\quad k\ge 1,$$

and a sequence $m_k|k\ge 1$, where $m_k>k,\ k\ge 1$ and,

$$\int_{m_k}^\infty \tau d^2(\tau)\,d\tau\le\frac{1}{2^k},\quad k\ge 1.$$

Now we use these sequences in order to construct l_k:

$$l_k=2\max\{n_k,m_k\},\quad k\ge 1.$$

Using (3.20), where $x(t_0)$ is defined in (3.11), from (3.10) we know that arbitrary solutions $x(t)$ and $y(t)$ satisfy the following:

$$\begin{aligned}
&P\Big\{\sup_{t\ge l_k}|x(t)-y(t)|\ge\frac{1}{k}\Big\}\\
&\quad=P\Big\{\sup_{t\ge l_k}\int_{t_0}^t X_1(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\\
&\qquad+\int_{t_0}^t X_1(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)|\\
&\qquad-\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\\
&\qquad-\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)|\ge\frac{1}{k}\Big\}\\
&\quad\le P\left\{\sup_{t\ge l_k}\left|\int_{t_0}^t X_1(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\ge\frac{1}{4k}\right\}
\end{aligned}$$

$$
\begin{aligned}
&+P\left\{\sup_{t\ge l_k}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{4k}\right\}\\
&+P\left\{\sup_{t\ge l_k}\left|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\ge\frac{1}{4k}\right\}\\
&+P\left\{\sup_{t\ge l_k}\left|\int_{t}^{\infty} X_2(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{4k}\right\},
\end{aligned}
\tag{3.27}
$$

$k\in\mathbf{N}$.

Similar to the approach taken in the first part of the theorem, we shall now proceed to estimate each term present on the right-hand side of the aforementioned inequality. We start from the first term. By applying Chebyshev's inequality, we obtain the following expression:

$$
\begin{aligned}
&P\Big\{\sup_{t\ge l_k}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\ge\frac{1}{4k}\Big\}\\
&\quad\le 4k\mathbf{E}\sup_{t\ge l_k}\left|\int_{t_0}^{t} X_1(t-\tau)\left[\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\\
&\quad\le 4k\mathbf{E}\sup_{t\ge l_k}\int_{t_0}^{t}\|X_1(t-\tau)\|\left\|\int_{-h}^{0} B(\tau,\theta)y(\tau+\theta)\,d\theta\right\|\,d\tau\\
&\quad\le 4k\mathbf{E}\sup_{t\ge l_k}\int_{t_0}^{t}\|X_1(t-\tau)\|b(\tau)\|y_\tau\|_C\,d\tau\\
&\quad\le 4k\mathbf{E}\sup_{t\ge l_k}\int_{t_0}^{t} a_1e^{-\alpha(t-\tau)}b(\tau)\|y_\tau\|_C\,d\tau\\
&\quad= 4ka_1\mathbf{E}\sup_{t\ge l_k}\left(\int_{t_0}^{\frac{t}{2}} e^{-\alpha(t-\tau)}b(\tau)\|y_\tau\|_C\,d\tau+\int_{\frac{t}{2}}^{t} e^{-\alpha(t-\tau)}b(\tau)\|y_\tau\|_C\,d\tau\right)\\
&\quad\le 4ka_1\sqrt{\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2}(e^{\frac{-\alpha l_k}{2}}\int_{t_0}^{\infty} b(\tau)\,d\tau+\int_{\frac{l_k}{2}}^{\infty} b(\tau)\,d\tau)\\
&\quad\le 4ka_1\sqrt{\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2}(e^{\frac{-\alpha l_k}{2}}K_1+\frac{1}{2^k})=:I_k^{(1)}.
\end{aligned}
\tag{3.28}
$$

In order to estimate the second term on the right-hand side of the inequality (3.27), let's consider the sequence of random events

$$
A_N=\{\omega|\sup_{l_k\le t\le N}|\int_{t_0}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)d\theta]\,dW(\tau)|\ge\frac{1}{4k}\}.
$$

For an arbitrary $K_1\le K_2$ we have $A_{K_1}\subset A_{K_2}$. Therefore A_N is a monotone sequence of sets, and

$$
\begin{aligned}
A = \lim_{N\to\infty} A_N &= \bigcup_{N=0}^{\infty} A_N \\
&= \{\omega | \sup_{l_k \le t} | \int_{t_0}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta]\, dW(\tau)| \ge \frac{1}{4k}\},
\end{aligned}
$$

so that

$$P\{A\} = \lim_{N\to\infty} P\{A_N\}.$$

Hence, for $N \ge l_k$,

$$
\begin{aligned}
&\sup_{l_k \le t \le N} \left| \int_{t_0}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta]\, dW(\tau) \right| \\
&\quad \le \sup_{l_k \le t \le N} \left| \int_{t_0}^{l_k} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta]\, dW(\tau) \right| \qquad (3.29) \\
&\quad + \sup_{l_k \le t \le N} \left| \int_{l_k}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta]\, dW(\tau) \right|.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&P\Big\{ \sup_{l_k \le t \le N} \left| \int_{t_0}^{t} X_1(t-\tau) \left[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta\right] dW(\tau) \right| \ge \frac{1}{4k} \Big\} \\
&\le P\left\{ \sup_{l_k \le t \le N} \left| \int_{t_0}^{l_k} X_1(t-\tau) \left[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta\right] dW(\tau) \right| \ge \frac{1}{8k} \right\} \\
&\quad + P\left\{ \sup_{l_k \le t \le N} \left| \int_{l_k}^{t} X_1(t-\tau) \left[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta\right] dW(\tau) \right| \ge \frac{1}{8k} \right\}.
\end{aligned} \tag{3.30}
$$

Let us start with the first term of the last inequality

$$
\begin{aligned}
&P\Big\{ \sup_{l_k \le t \le N} \left| \int_{t_0}^{l_k} X_1(t-\tau) \left[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta\right] dW(\tau) \right| \ge \frac{1}{8k} \Big\} \\
&\le 64k^2 \mathbf{E} \left(\sup_{l_k \le t \le N} \left| \int_{t_0}^{l_k} X_1(t-\tau) \left[\int_{-h}^{0} D(\tau,\theta) y(\tau+\theta)\, d\theta\right] dW(\tau) \right|^2 \right) \\
&\le 64k^2 \tilde{K} \mathbf{E} \|\phi(\theta)\|_C^2 \left(e^{-\alpha l_k} \int_{t_0}^{\frac{l_k}{2}} d^2(\tau)\, d\tau + \int_{\frac{l_k}{2}}^{l_k} d^2(\tau)\, d\tau \right) \\
&\le 64k^2 \tilde{K} \mathbf{E} \|\phi(\theta)\|_C^2 \left(e^{-\alpha l_k} \int_{t_0}^{\infty} d^2(\tau)\, d\tau + \int_{\frac{l_k}{2}}^{\infty} d^2(\tau)\, d\tau \right) \\
&\le 64k^2 \tilde{K} \mathbf{E} \|\phi(\theta)\|_C^2 \left(e^{-\alpha l_k} K_1 + \frac{1}{2^k} \right) =: I_k^{(2)}.
\end{aligned} \tag{3.31}
$$

Now we can move to the second term on the right-hand side of the inequality (3.30)

$$\begin{aligned}
&P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t} X_1(t-\tau)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{8k}\right\}\\
&\quad= P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)+X_1(t-l_k)\\
&\qquad - X_1(t-l_k)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|\ge\frac{1}{8k}\Big\}\\
&\quad\le P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)-X_1(t-l_k)\\
&\qquad\times\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|\ge\frac{1}{16k}\Big\}\\
&\qquad+P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t} X_1(t-l_k)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{16k}\right\},
\end{aligned}\tag{3.32}$$

Next, we can estimate each of the terms in the above inequality. Let us start with the second term.

$$\begin{aligned}
&P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t} X_1(t-l_k)\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{16k}\right\}\\
&\quad\le P\left\{\sup_{l_k\le t\le N}\|X_1(t-l_k)\|\sup_{l_k\le t\le N}\left|\int_{l_k}^{t}\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\,dW(\tau)\right|\ge\frac{1}{16k}\right\}\\
&\quad\le 256k^2a_1^2\mathbf{E}\left(\sup_{l_k\le t\le N}(|\int_{l_k}^{t}\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\,dW(\tau)|^2)\right)\\
&\quad\le 1024k^2a_1^2\int_{l_k}^{N} d^2(\tau)\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\,d\tau\\
&\quad\le 1024k^2a_1^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\int_{l_k}^{\infty} d^2(\tau)\tau\,d\tau\\
&\quad\le 1024k^2a_1^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\frac{1}{2^k}.
\end{aligned}\tag{3.33}$$

The following estimations are essential in order to deal with the first term of the inequality we are considering at this step.

$$\begin{aligned}
P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)+X_1(t-l_k)-X_1(t-l_k)\\
\times\left[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\Big|\ge\frac{1}{8k}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{l_k}^{t}\Big(\int_{l_k}^{\tau} X_1(t-s)A\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\\
&= -\int_{l_k}^{t}\Big(\int_{l_k}^{t} X_1(t-s)AI_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big) \qquad (3.34)\\
&= -\int_{l_k}^{t} X_1(t-s)A\Big(\int_{l_k}^{t} I_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\,ds.
\end{aligned}
$$

From this, we obtain

$$
\begin{aligned}
&P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-\tau)-X_1(t-l_k)\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big|\ge\frac{1}{16k}\Big\}\\
&\le P\Big\{\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-s)A\Big(\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\,ds\Big|\ge\frac{1}{16k}\Big\}\\
&\le 256k^2\mathbf{E}\Big(\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} X_1(t-s)A\Big(\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\,ds\Big|\Big)^2\\
&\le 256k^2\mathbf{E}\Big(\sup_{l_k\le t\le N}\Big(\int_{l_k}^{t} a_1e^{-\alpha(t-s)}\|A\|\\
&\qquad\times\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big)\Big|\,ds\Big)^2\\
&\le 256k^2E\Big(\sup_{l_k\le t\le N}\Big(\int_{l_k}^{t} a_1^2e^{-2\alpha(t-s)}\|A\|^2\\
&\qquad\times\int_{l_k}^{t}\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)d\theta\Big]\,dW(\tau))\Big|^2\,ds\Big)\\
&\le \frac{256k^2a_1^2\|A\|^2}{2\alpha}\mathbf{E}\Big(\sup_{l_k\le t\le N}\int_{l_k}^{t}\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big|^2\,ds\Big)\\
&\le \frac{256k^2a_1^2\|A\|^2}{2\alpha}\int_{l_k}^{N}\mathbf{E}\sup_{l_k\le t\le N}\Big|\int_{l_k}^{t} I_{\{s\le\tau\}}\\
&\qquad\times\Big[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta\Big]\,dW(\tau)\Big|^2\,ds\\
&\le \frac{1024^2a_1^2\|A\|^2}{2\alpha}\int_{l_k}^{N}\Big(\int_{l_k}^{N} d^2(\tau)\mathbf{E}\|y_\tau\|_C^2\,d\tau\,ds\Big)
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1024^2 a_1^2\|A\|^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha}\int_{l_k}^{N}\left(\int_{l_k}^{N} d^2(\tau)\,d\tau\right)ds\\
&\le \frac{1024^2 a_1^2\|A\|^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha}\int_{l_k}^{N}\left(d^2(\tau)\int_{l_k}^{\tau} ds\right)d\tau\\
&\le \frac{1024^2 a_1^2\|A\|^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha}\int_{l_k}^{N}\tau d^2(\tau)\,d\tau\\
&\le \frac{1024^2 a_1^2\|A\|^2\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2 k^2}{2\alpha 2^k}.
\end{aligned} \tag{3.35}$$

According to (3.33) and (3.35)

$$\begin{aligned}
P\Big\{\sup_{l_k\le t\le N}\left|\int_{l_k}^{t} X_1(t-\tau)[\int_{-h}^{0} D(\tau,\theta)y(\tau+\theta)\,d\theta]\,dW(\tau)\right|\ge\frac{1}{8k}\Big\}\\
\le 256\tilde{K}\mathbf{E}||\phi(\theta)||_C^2(1+4a_1^2\|A\|^2)2^{-k} =: I_k^{(3)}.
\end{aligned}$$

Let us now estimate the third term on the right-hand side of the inequality (3.27).

$$\begin{aligned}
&P\Big\{\sup_{t\ge l_k}\left|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]d\tau\right|\ge\frac{1}{4k}\Big\}\\
&\quad\le P\left\{\sup_{t\ge l_k}\left|\int_t^\infty \|X_2(t-\tau)\|\left\|\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right\|d\tau\right|\ge\frac{1}{4k}\right\}\\
&\quad\le P\left\{\sup_{t\ge l_k}\left|\int_{l_k}^\infty \|X_2(t-\tau)\|\left\|\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right\|d\tau\right|\ge\frac{1}{4k}\right\}\\
&\quad\le 16k^2\sup_{t\ge l_k}\left|\int_{l_k}^\infty \|X_2(t-\tau)\|b(\tau)\|y_\tau\|_C\,d\tau\right|\\
&\quad\le 16k^2a_2^2\sqrt{\tilde{K}\mathbf{E}\|\phi(\theta)\|^2}\int_{l_k}^\infty b(\tau)\,d\tau\\
&\quad\le 6k^2a_2^2\sqrt{\tilde{K}\mathbf{E}\|\phi(\theta)\|^2}\frac{1}{2^k} =: I_k^{(4)}.
\end{aligned} \tag{3.36}$$

We shall now proceed to estimate the final term located on the right-hand side of inequality (3.27). Let us consider the random events sequence

$$A_N=\left\{\omega\Big|\sup_{l_k\le t\le N}\left|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]dW(\tau)\right|\ge\frac{1}{4k}\right\}.$$

By definition, A_N is a monotone sequence of sets, therefore

$$\begin{aligned}
A&=\lim_{N\to\infty}A_N=\bigcup_{N=0}^{\infty}A_N\\
&=\left\{\omega\Big|\sup_{l_k\le t}\left|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]dW(\tau)\right|\ge\frac{1}{4k}\right\}.
\end{aligned}$$

so that
$$P\{A\} = \lim_{N\to\infty} P\{A_N\}.$$
Since $l_k \le t$, the following inequality holds
$$\begin{aligned}
&P\left\{\sup_{l_k\le t\le N}\left|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{4k}\right\}\\
&\quad\le P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{8k}\right\}\\
&\qquad+P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^t X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{8k}\right\}.
\end{aligned}\tag{3.37}$$
Let's proceed with each term of the latter inequality
$$\begin{aligned}
&P\left\{\sup_{l_k\le t\le N}\left|\int_{l_k}^\infty X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{8k}\right\}\\
&\le P\left\{\sup_{l_k\le t\le N}\|X_2(\tau)\|\,\left|\int_{l_k}^\infty X_2^{-1}(t)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{8k}\right\}\\
&\le 64k^2a_2^4\tilde{K}\mathbf{E}\|\phi(\theta)\|_C^2\frac{1}{2^k}=:I_k^{(5)}.
\end{aligned}\tag{3.38}$$
Now we move to the second term,
$$\begin{aligned}
&P\Big\{\sup_{l_k\le t\le N}\left|\int_{l_k}^t X_2(t-\tau)\left[\int_{-h}^0 D(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,dW(\tau)\right|\ge\frac{1}{8k}\Big\}\\
&\qquad\le 256k^2a_2^4\tilde{K}\mathbf{E}||\phi(\theta)||_C^2\frac{1}{2^k}=:I_k^{(6)}.
\end{aligned}\tag{3.39}$$
By taking the limit $N\to\infty$ in (3.37), we get
$$P\left\{\sup_{t\ge l_k}\left|\int_t^\infty X_2(t-\tau)\left[\int_{-h}^0 B(\tau,\theta)y(\tau+\theta)\,d\theta\right]\,d\tau\right|\ge\frac{1}{4k}\right\}\le I_k^{(5)}+I_k^{(6)}.$$
Finally
$$P\left\{\sup_{t\ge l_k}|x(t)-y(t)|\ge\frac{1}{k}\right\}\le I_k^{(1)}+I_k^{(2)}+I_k^{(3)}+I_k^{(4)}+I_k^{(5)}+I_k^{(6)}=I_k.$$
The convergence of the series $\sum_{k=1}^\infty I_k$ is evident and according to the Borel-Cantelli lemma there exists a positive integer $M=M(\omega)$ such that, for arbitrary $k\ge M(\omega)$ for arbitrary $k\ge M(\omega)$
$$\sup_{t\ge l_k}|x(t)-y(t)|\ge\frac{1}{k},$$

with probability 1. Hence, for almost all ω and arbitrary $\epsilon > 0$ there exists $T = T(\epsilon, \omega) = l_{k_0}$, where $k_0 = \max\{[\frac{1}{\epsilon}], M(\omega)\}$, such that the following inequality holds for all $t \geq T$:

$$|x(t) - y(t)| \leq \sup_{t \geq T} |x(t) - y(t)| \geq \frac{1}{k_0} \leq \epsilon.$$

Theorem is proved. □

4. Example

Let us give an application of our theorem.

Example 4.1. We consider a system of ordinary differential equations,

$$d \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt, \tag{4.1}$$

Together with the system (4.1), consider the following system of functional stochastic differential equations

$$\begin{aligned} d \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dt + \int_{-h}^{0} B(t, \theta) \begin{bmatrix} y_1(t+\theta) \\ y_2(t+\theta) \end{bmatrix} d\theta \, dt \\ + \int_{-h}^{0} D(t, \theta) \begin{bmatrix} y_1(t+\theta) \\ y_2(t+\theta) \end{bmatrix} d\theta \, dW(t), \end{aligned} \tag{4.2}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \tag{4.3}$$

$$B(t, \theta) = \begin{bmatrix} \frac{b_1(\theta)}{(t+1)^3} & 0 \\ 0 & \frac{b_1(\theta)}{(t+1)^3} \end{bmatrix}, \tag{4.4}$$

$$D(t, \theta) = \begin{bmatrix} 0 & \frac{d_1(\theta)}{(t+1)^2} \\ \frac{d_1(\theta)}{(t+1)^2} & 0 \end{bmatrix}, \tag{4.5}$$

here $b_1(\theta), d_1(\theta)$ are continuous functions on $[-h, 0]$. Then

$$\begin{aligned} \left\| \int_{-h}^{0} B(t, \theta) \phi(\theta) \, d\theta \right\| &\leq \int_{-h}^{0} \|B(t, \theta)\| \, d\theta \|\phi\|_C \\ &= \frac{\sqrt{2}}{(t+1)^2} \int_{-h}^{0} b_1(\theta) \, d\theta \|\phi\|_C. \end{aligned} \tag{4.6}$$

Therefore, $b(t) = \frac{\sqrt{2}}{(t+1)^2} \int_{-h}^{0} b_1(\theta) \, d\theta$ and $\int_0^\infty b(t) \, dt < \infty$

$$\begin{aligned} \left\| \int_{-h}^{0} D(t, \theta) \phi(\theta) \, d\theta \right\| &\leq \int_{-h}^{0} \|D(t, \theta)\| \, d\theta \|\phi\|_C \\ &= \frac{\sqrt{2}}{(t+1)^2} \int_{-h}^{0} d_1(\theta) \, d\theta \|\phi\|_C. \end{aligned} \tag{4.7}$$

Therefore, $d(t) = \frac{\sqrt{2}}{(t+1)^2}\int_{-h}^{0} d_1(\theta)\,d\theta$ and $\int_0^\infty td^2(t)\,dt < \infty$. From this follows that system (4.2) is asymptotically equivalent to the system (4.1) in the mean square sense and with probability 1.

5. Conclusions

This work proposes a new method for studying the asymptotic behavior at infinity of solutions to linear stochastic functional-differential equations. According to this method, the problem is reduced to investigating a much simpler object: a system of ordinary linear equations with constant coefficients. This system is constructed in such a way that for every solution of the original system, a corresponding solution of the deterministic system is assigned, and the difference between them tends to zero at infinity, either in mean square or with probability one. Naturally, some smallness of the stochastic perturbation at infinity is required in terms of the convergence of integrals of the noise intensity.

6. Acknowledgments

The research of Oleksandr Stanzhytskyi was partially supported by the National Research Foundation of Ukraine No. F81/41743 and by the Ukrainian Government Scientific Research Grant No. 210BF38-01. ‘

References

1. J.K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York, Heidelberg, 1977.
2. Jian. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences,**119**, Springer-Verlag, New York, Inc., 1996.
3. E.F. Carkov, *Random perturbations of functional-differential equations*, Zinatne, Riga, 1989.
4. S.-E.A. Mohammed, *Stochastic functional differential equations*, Pitman (Advanced Publishing Program), Boston, MA, 1984.
5. A.M. Samoilenko, N.I. Mahmudov, A.N. Stanzhitskii, *Existence, uniqueness, and controllability results for neutral FSDEs in Hilbert spaces*,Dynam. Systems & Appl., **17** (1), 2008, 53–70.
6. O M. Stanzhytskyi, V.V. Mogilova, A.O. Tsukanova, *On Comparison Results for Neutral Stochastic Differential Equations of Reaction-Diffusion Type in* $L^2(\mathbb{R}^d)$, in Modern Mathematics and Mechanics, Underst. Complex Syst., Springer, Cham, 2019, 351–395.
7. A.O. Stanzhytskyi, *On Weak and Strong Solutions of Paired Stochastic Functional Differential Equations in Infinite-Dimensional Spaces*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **29** (2), 2021, 48–76.
8. A. Stanzhytsky, O. Misiats, O. Stanzhytskyi, *Invariant Measure for Neutral Stochastic Functional Differential Equations with Non-Lipchitz Coefficients*, Evolution Equations and Control Theory, **11** (6), 2022, 1929–1953.
9. N. Levinson, *The Asymptotic Behavior of Systems of Linear Differential Equations*, Amer. J. Math., **68**, 1946, 1–6.

10. O.M. Stanzhytskyi, *Investigation of Exponential Dichotomy of Ito Stochastic Systems by Using Quadratic Forms*, Ukr. Mat. J., **53** (11), 2001, 1882–1894.
11. O.M. Stanzhytskyi, A.P. Krenevich, I.G. Novak, *Asymptotic Equivalence of Linear Stochastic Ito Systems and Oscillation of Solutions of Linear Second-Order Equations*, Differ. Equ., **47** (6), 2011, 799–813.

Received 21.05.2023

Journal of Optimization, Differential Equations and Their Applications

Volume 31 **Issue 1** **June 2023**

JODEA will publish carefully selected, longer research papers on mathematical aspects of optimal control theory and optimization for partial differential equations and on applications of the mathematic theory to issues arising in the sciences and in engineering. Papers submitted to this journal should be correct, innovative, non-trivial, with a lucid presentation, and of interest to a substantial number of readers. Emphasis will be placed on papers that are judged to be specially timely, and of interest to a substantial number of mathematicians working in this area.

Instruction to Authors:

Manuscripts should be in English and submitted electronically, pdf format to the member of the Editorial Board whose area, in the opinion of author, is most closely related to the topic of the paper and the same time, copy your submission email to the Managing Editor. Submissions can also be made directly to the Managing Editor.

Submission of a manuscript is a representation that the work has not been previously published, has not been copyrighted, is not being submitted for publication elsewhere, and that its submission has been approved by all of the authors and by the institution where the work was carried out. Furthermore, that any person cited as a source of personal communications has approved such citation, and that the authors have agreed that the copyright in the article shall be assigned exclusively to the Publisher upon acceptance of the article.

Manuscript style: Number each page. Page 1 should contain the title, authors names and complete affiliations. Place any footnote to the title at the bottom of Page 1. Each paper requires an abstract not exceeding 200 words summarizing the techniques, methods and main conclusions. AMS subject classification must accompany all articles, placed at Page 1 after Abstract. E-mail addresses of all authors should be placed together with the corresponding affiliations. Each paper requires a running head (abbriviated form of the title) of no more than 40 characters.

Equations should be centered with the number placed in parentheses at the right margin.

Figures must be drafted in high resolution and high contrast on separate pieces of white paper, in the form suitable for photographic reproduction and reduction.

References should be listed alphabetically, typed and punctuated according to the following examples:

1. S. N. Chow, J. K. Hale, *Methods od Bifurcation Theory*, *Springer-Verlad*, New York, 1982.
2. J. Serrin, *Gradient estimates for solutions of nonlinear elliptic and parabolic equations*, in "Contributions to Nonlinear Functional Analysis," (ed. E.H. Zarantonello), *Academic Press* (1971).
3. S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, *Ann. Scuola Norm. Sup. Pisa Cl.Sci.*, **18** (1963), 97–116.

For journal abbreviations used in bibliographies, consult the list of serials in the latest *Mathematical Reviews* annual index.

Final version of the manuscript should be typeset using LaTeX which can shorten the production process. Files of sample papers can be downloaded from the Journal's home page, where more information on how to prepare TeX files can be found.

Journal of Optimization, Differential Equations and Their Applications

Volume 31 **Issue 1** **June 2023**

ISSN 2617-0108

9 772617 010000 03101

CONTENTS