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ON AN INITIAL BOUNDARY-VALUE PROBLEM FOR 1D HYPERBOLIC EQUATION WITH INTERIOR DEGENERACY: SERIES SOLUTIONS WITH THE CONTINUOUSLY DIFFERENTIABLE FLUXES

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Abstract. A 1-parameter initial boundary value problem for the linear homogeneous degenerate wave equation $u_{tt}(t,x;\alpha)-(a(x;\alpha)\,u_x(t,x;\alpha))_x=0$ (*JODEA*, **27**(2), 29 – 44), where: 1) $(t,x)\in[0,T]\times[-l,+l]$; 2) the weight function $a(x;\alpha)$: *a*) $a_0\left|\dfrac{x}{c}\right|^{\alpha}$, $0\leqslant|x|\leqslant c$; *b*) a_0, $c\leqslant|x|\leqslant l$; *c*) a_0 is a constant reference value; and 3) the parameter $\alpha\in(0,+\infty)$; is considered. Using a string analogy, the IBVP can be treated as an attempt to set an initially fixed 'string' in motion, the left end of the 'string' being fixed, whereas the right end being forced to move.

It has been proved, using the methods of Frobenius and separation of variables, that: 1) there exist 6 series solutions $u(t,x;\alpha)$, $(t,x)\in[0,T]\times[-c,+c]$, of the degenerate wave equation; 2) the only series solution, having continuous and continuously differentiable flux $f(a,u)=-au_x$, reads $u(t,x;\alpha)=U_{\alpha,0}(t)+U_{\alpha,1}(t)|x|^{\theta}+U_{\alpha,2}(t)|x|^{2\theta}+\ldots$, where *a*) $\theta=2-\alpha$ is a derived parameter; *b*) the coefficient functions obey the following linear recurrence relations: $U''_{\alpha,\mu-1}(t)=\mu\theta\,[(\mu-1)\,\theta+1]\,c^{-\alpha}a_0\,U_{\alpha,\mu}(t)$, $\mu\in\mathbb{N}$.

It has been revealed that a nonlinear change of the independent variables $(t,x)\to(\tau,\xi)$ transforms: 1) the degenerate wave equation to the wave equation $\upsilon_{\tau\tau}-\upsilon_{\xi\xi}=\xi\rho$, or rewritten as the balance law $\pi_\tau+\varphi_\xi=\rho$, where $\pi=\upsilon_\tau$, $-\varphi(\upsilon;\alpha)=\upsilon_\xi+\xi\rho$, $\rho(\upsilon;\alpha)=\dfrac{\alpha}{\theta}\,\dfrac{\upsilon}{\xi^2}$, having: *a*) no singularity in its principal part (due to inflation of the degeneracy), and *b*) the only series solution of the form $\upsilon(\tau,\xi;\alpha)=V_{\alpha,0}(\tau)+V_{\alpha,1}(\tau)\,\xi^2+V_{\alpha,2}(\tau)\,\xi^4+\ldots$ (out of 5 existing and found similarly to those of the degenerate wave equation), leading to the continuous and continuously differentiable regularized flux $\varphi(\mathring{\upsilon};\alpha)$ and the continuous regularized source term $\rho(\mathring{\upsilon};\alpha)$, where $\mathring{\upsilon}(\tau,\xi;\alpha)=\upsilon(\tau,\xi;\alpha)-\upsilon(\tau,0;\alpha)$; 2) the IBVP for the degenerate wave equation to the IBVP for the transformed wave equation.

It has been shown, that if $\alpha\in(0,2)$: 1) the above results are valid; 2) the state of being fixed for the 'string' is not necessary for $(t,x)\in[0,T]\times[-l,0]$, that is a traveling wave could pass the degeneracy and excite vibrations of the 'string' between its fixed end and the point of degeneracy.

Key words: degenerate wave equation, series solutions, the Frobenius method, separation of variables, inflation of singularity, exact solutions, the Bessel functions, conservation and balance laws, the flux, regularization of the flux.

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1. Introduction

1.1. Presentation of the problem

The current study is a continuation of that started and shortly reported in our pilot publication [4] on the subject and dealing with the following spatially 1D degenerate wave equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x;\alpha)\,\frac{\partial u(t,x)}{\partial x}\right) = 0\,, \tag{1.1}$$

where $t\in(0,T)$, $x\in(-l,+l)$ are the independent variables, $a(\cdot;\alpha)\colon[-l,+l]\to\mathbb{R}_+$ is a given 1-parameter weight function which is assumed to be continuous, piecewise smooth and to vanish in the midpoint of the segment $[-l,+l]$ following a power law inside a subsegment $[-c,+c]\subset[-l,+l]$. For instance,

$$a(x;\alpha) = \begin{cases} a_0\left(\dfrac{|x|}{c}\right)^{\alpha}, & 0\leqslant\dfrac{|x|}{l}\leqslant\dfrac{c}{l}\,, \\[2ex] a_0\,, & \dfrac{c}{l}\leqslant\dfrac{|x|}{l}\leqslant 1\,, \end{cases}$$

where $\alpha\in(0,+\infty)$ is a parameter, and $a_0>0$ is a given constant.

Using the following non-dimensional variables and quantities

$$t = \frac{l\,\underline{t}}{\sqrt{a_0}}\,, \qquad x = l\,\underline{x}\,, \qquad u = l\,\underline{u}\,, \qquad a = a_0\,\underline{a}\,, \qquad c = l\,\underline{c}\,,$$

the degenerate wave equation and the power law can be rewritten in a non-dimensional forms. To simplify notation we drop hereafter the bars under the non-dimensional variables and quantities. As a result, the non-dimensional degenerate wave equation reads exactly as (1.1), whereas the non-dimensional power law reduces to

$$a(x;\alpha) = \begin{cases} a_*\,|x|^{\alpha}, & 0\leqslant|x|\leqslant c\,, \\ 1\,, & c\leqslant|x|\leqslant 1\,, \end{cases} \tag{1.2}$$

where the derived quantity a_* is such that $a_* c^{\alpha}=1$. Hereinafter, the space-time segment $[0,T]\times\{x=0\}$, where the degeneracy is located, is referred to as the degeneracy segment.

Another form of the degenerate wave equation (1.1), used in the current study, is known as a conservation law

$$\frac{\partial p}{\partial t} + \frac{\partial f}{\partial x} = 0\,, \tag{1.3}$$

where $f:=-aq$ is the flux, and p, q are auxiliary dependent variables

$$p := \frac{\partial u}{\partial t}\,, \qquad q := \frac{\partial u}{\partial x}\,\cdot$$

Our concern, as in [4], relates to the study of the following initial boundary value problem

$$\begin{cases} \dfrac{\partial^2 u(t,x;\alpha)}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a(x;\alpha)\,\dfrac{\partial u(t,x;\alpha)}{\partial x}\right), & (t,x) \in (0,T]\times(-1,+1)\,, \\ u(t,-1;\alpha) = 0\,, & t \in [0,T]\,, \\ u(t,+1;\alpha) = h(t)\,, & t \in [0,T]\,, \\ u(0,x;\alpha) = 0\,, & x \in [-1,+1]\,, \\ \dfrac{\partial u(0,x;\alpha)}{\partial t} = 0\,, & x \in [-1,+1)\,, \end{cases} \tag{1.4}$$

where $h(t)$ is a given control function.

For the sake of convenience, we interpret the function $u(t,x)$ as the distributed over the segment $[-1,+1]$ displacements of a ‘string’, though properties of the weight function $a(x;\alpha)$ have little in common with those of weight functions being admissible in ‘genuine’ wave equations for string vibrations. So, we deal with a hyperbolic system subject to the action of a control $h(t)$, imposed as the Dirichlet boundary condition at the right end $x = +1$. In contrast to the widely studied standard case, for instance, see [2, 3], we assume that the string has a defect or a damage at the interior point $x=0$, where $a(0;\alpha)=0$. Loosely speaking, the loss of string elasticity at the interior point, which turns into a swivel, is accompanied by the cease of resisting rotations and flexions.

The main question that we are going to discuss in this article is how the defect at the interior point $x=0$ affects the solution of the system (1.4) and its properties. The second point that should be clarified here is about the consistency of this problem and properties of its solutions in a neighborhood of 'the damage point' $x = 0$. It seems that such analysis for the indicated class of degenerate systems can be important for applications, with respect, for instance, the cloaking problem [10] (building of devices that lead to invisibility properties from observation), the evolution of damage in materials, optimization problems for elastic bodies arising, e.g. in contact mechanics, coupled systems, composite materials, where 'life-cycle-optimization' appears as a challenge.

The indicated type of degeneracy raises many new and open questions related to the well-posedness of the hyperbolic equations in suitable functional spaces. It should be emphasized here that boundary value problems for degenerate elliptic and parabolic equations have received a lot of attention in the last years (see, for instance, [5, 6, 16–18]). In the meantime, as for the control issue for degenerate wave equations, we can mention only a few recent publications [2,3,11], where the authors mainly deal with weakly or strongly degenerate wave equations for which the degeneracy zone is located at a boundary point. This analysis shows that because of the rate of 'degree of degeneracy' in the diffusion coefficient $a(x;\alpha)$, especially when $a(x;\alpha)$ degenerations too severely, the new tools are necessary for the analysis of the corresponding initial boundary value problems.

In contrast to the above mentioned results, where the authors mainly deal with the degenerate equation of the form (1.1) with the degeneracy at the boundaries $x=\pm 1$, we focus on the case where the 'damaged' point is interior. From a physical point of view, if we look at the wave equation (1.1) as at the equation of vibrating string for which its the point $x=0$ works like a swivel, then we should allow for the possibility of big angles at the profile of the string around $x=0$. Needless to say that in this case the mathematical substantiation of the wave equation for vibrating string becomes nontrivial. On the other hand, the coefficient $a(x;\alpha)$ in (1.1) can be interpreted as the stiffness of the string. The fact that this coefficient vanishes at $x = 0$ means that the string is getting very weak at this point. So, our core idea is to apply the methods of Frobenius and separation of variables in order to obtain the exact representation for solutions of the original degenerate wave equation in the form of 1-parameter power series. Such analysis allows to find out what kind of compatibility conditions we should impose at the 'damaged' point in order to pass from the original initial boundary value problem (1.4) to its equivalent version in the form of some transmission problem. Therefore, the purpose of this paper is to provide a qualitative analysis of system (1.4), obtain an exact representation for its solution, and find out how the degree of degeneracy α in the principle coefficient $a(x;\alpha)$ affects the system (1.4) and its solution.

1.2. The plan of the article

The article is organized as follows.

Section 2 is devoted to the well-posedness issues for the initial boundary value problem (1.4) provided $\alpha \in (0,2)$. Admitting only two types of degeneracy for $a(x;\alpha)$, namely the so-called weak and strong degeneracy, we prove the existence and uniqueness results for the weak and strong solutions to the problem. We also discuss the transmission conditions at the damage point $x = 0$ and show that, in general, in the framework of functional setting, the continuity of the weak and strong solutions at the point of interior degeneracy and the smoothness of the corresponding fluxes remain open questions.

In Section 3 we construct 1-parameter power series solutions of the original degenerate wave equation in one-sided vicinities of the degeneracy segment. Then, in Section 4, we study continuous matching of the obtained one-sided series solutions. Finally, in Section 5 we obtain the exact solutions of the original degenerate wave equation using the separation of variables. Among all continuous series solutions obtained in Sections 4 and 5, we find those being required in some sense, and possessing the so-called property Z.

Definition 1.1. We say that a solution to the initial boundary value problem (1.4) *possesses property Z* if it vanishes in the left space-time rectangle $[0,T]\times[-1,0]$.

A more precise formulation of our concern in terms of Definition 1.1 is to divide the required solutions to the problem (1.4) into those possessing and violating property Z. Keeping in mind the string analogy, we are interested in choosing: 1) such functions $h(t)$, specifying a motion of the right end of the string, and eventually the right part of the string (segment $(0, +1]$), and 2) such values of the parameter α, specifying a sort of degeneration (in the midpoint of the segment $[-1, +1]$), to make the left part of the string (segment $[-1, 0)$) vibrate.

In Section 6 we first introduce new independent variables, 'inflating' the degeneracy, and then transform the original degenerate wave equation to a wave equation (referred to as the transformed one), having no singularity in its principal part. After, we reformulate the original initial boundary value problem for the transformed wave equation. Then, in Sections 7 − 9, we apply the approaches used in Sections 3 − 5, but for the transformed wave equation and the transformed initial boundary value problem. As for continuous matching to be implemented to the transformed wave equation, it is rewritten as a balance law.

1.3. Short announce of the main results

In the current study, solving the initial boundary value problem (1.4) for the degenerate wave equation (1.1), supplemented with the power law (1.2) of degeneracy $a(x;\alpha) \sim |x|^{\alpha}$, $\alpha \in (0, +\infty)$, has been discussed as it concerns continuity of power series solutions, treated where it is necessary, by analogy, as vibrations of an initially fixed 'string'.

1. We have introduced the definitions of required solutions to the degenerate wave equation and the initial boundary value problem (1.4), both solutions having the continuous and continuously differentiable flux.

2. We have introduced the definition of property Z for solutions to the initial boundary value problem (1.4) to remain trivial between the fixed end of the 'string' and the point of degeneracy, for $t > 0$.

3. We have succeeded in finding power series solutions using the methods of: 1) Frobenius and 2) separation of variables, for the parameter of degeneracy $\alpha \in (0, +\infty)$.

4. We have proved that among the series solutions obtained, there is the only one required, being a power series of the terms $|x|^{\mu\theta}$, $\theta = 2 - \alpha$, $\mu \in \mathbb{Z}_+$, for the parameter of degeneracy $\alpha \in (0, 2)$.

$$u(t, x; \alpha) = U_{\alpha,0}(t) + U_{\alpha,1}(t)\,|x|^{\theta} + U_{\alpha,2}(t)\,|x|^{2\theta} + \dots,$$

where the coefficient functions obey the following recurrence relations

$$U''_{\alpha,\mu-1}(t) = \mu\theta\,[(\mu-1)\,\theta + 1]\,a_* U_{\alpha,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,.$$

5. We have proved that property Z is not necessary for the only required series solution. Physically, not possessing property Z means that a traveling wave could pass the degeneracy and excite vibrations of the 'string' between its fixed

end and the point of degeneracy (see a brief introductory discussion of physical formulations of the initial boundary value problem (1.4) in Section 1 of [4]).

2. Functional setting and well-posedness issues

In this Section we are going to dwell at the well-posedness of the original initial-boundary value problem (1.4) provided $\alpha \in (0,2)$. We will admit only two types of degeneracy for $a(x;\alpha)$ (1.2), namely weak and strong degeneracy. By analogy with [2], we say that the problem (1.4) is weakly degenerate (WDP) if $\alpha \in (0,1]$, and it is strongly degenerate (SDP), if $\alpha \in (1,2)$. So, each type of degeneracy is associated with the corresponding range of the exponent α.

Let us introduce some weighted Sobolev spaces naturally associated with the system (1.4). We denote by $H^1_a(-1,+1)$ the space of all functions $u \in L^2(-1,+1)$ such that

$$\begin{cases} u \text{ is locally absolutely continuous in } [-1,0) \bigcup (0,+1]\,, \\ \sqrt{a}\, u_x \in L^2(-1,+1). \end{cases} \tag{2.1}$$

It is easy to see that $H^1_a(-1,+1)$ is a Hilbert space with respect to the scalar product

$$(u,v)_{H^1_a(-1,+1)} = \int_{-1}^{1} [uv + a\, u_x v_x]\, \mathrm{d}x, \quad \forall\, u,v \in H^1_a(-1,+1)\,,$$

and associated norm

$$\|u\|_{H^1_a(-1,+1)} = \left(\int_{-1}^{1} \left[u^2 + a\, u_x^2\right] \mathrm{d}x \right)^{\frac{1}{2}}, \quad \forall\, u \in H^1_a(-1,+1)\,.$$

We also introduce the closed subspace $H^1_{a,0}(-1,+1)$ of $H^1_a(-1,+1)$ defined as

$$H^1_{a,0}(-1,+1) = \Big\{ u \in H^1_a(-1,+1) \colon\; u(-1) = 0 = u(+1) \Big\}.$$

Arguing as in [13, Theorem 3.1], it can be shown that $H^1_{a,0}(-1,+1)$ is a Banach space with respect to the norm

$$\|u\|_{H^1_{a,0}(-1,+1)} = \left(\int_{-1}^{1} a\, u_x^2\, \mathrm{d}x \right)^{\frac{1}{2}}$$

provided $\alpha \in (0,2)$.

Starting with the weak degenerate case, we have the following result (we refer to [14, Theorem 2.3] for the details).

Theorem 2.1. *Let $a(x;\alpha)\colon [-1,+1] \to \mathbb{R}$ be a weight function defined by* (1.2) *with $\alpha \in (0,1]$. Then $H^1_a(-1,1) \hookrightarrow L^1(-1,+1)$ compactly, and $H^1_a(-1,+1)$ is continuously embedded into the class of absolutely continuous functions on $[-1,+1]$, so*

$$\lim_{x\nearrow 0} u(x) = \lim_{x\searrow 0} u(x), \quad |u(0)| < +\infty, \quad \forall\, u \in H^1_a(-1,+1). \tag{2.2}$$

If, in addition, u belongs to the space

$$H^2_a(-1,+1) := \left\{ u \in H^1_a(-1,+1) \ : \ a u_x \in W^{1,2}(-1,+1) \right\}, \tag{2.3}$$

then the following transmission condition

$$\lim_{x\nearrow 0} a(x)\, u_x(x) = \lim_{x\searrow 0} a(x)\, u_x(x) = L, \quad \textit{with } |L| < +\infty, \tag{2.4}$$

holds true.

Remark 2.1. It is worth to emphasize that if $\alpha \in (0,1]$ and $u \in H^1_a(-1,+1)$ is an arbitrary element, then $u(x)$ is continuous at the damage point $x = 0$. However, the situation changes drastically if we deal with the strong degeneration in (1.4). Indeed, let us consider the following example. Let $\alpha = 7/4$, $c = 1$, $a_* = 1$ in (1.2), and let

$$u(x) = \begin{cases} |x|^{-\frac{1}{4}} - 1, & \text{if } x \in (-1,0), \\[2mm] |x|^{+\frac{1}{2}} - 1, & \text{if } x \in [0,+1). \end{cases}$$

Then, the function $u\colon (-1,+1) \to \mathbb{R}$ has a discontinuity of the second kind at $x = 0$, $u \in H^1_{a,0}(-1,1)$, and

$$a(x;\alpha)\, u_x(x) = \begin{cases} \frac{1}{4}\, |x|^{\frac{1}{2}}, & \text{if } x \in (-1,0), \\[2mm] \frac{1}{2}\, |x|^{\frac{5}{4}}, & \text{if } x \in [0,+1). \end{cases}$$

So, instead of the transmission condition (2.4), we have

$$\lim_{x\nearrow 0} a(x)\, u_x(x) = \lim_{x\searrow 0} a(x)\, u_x(x) = 0. \tag{2.5}$$

In fact, in the case of strong degeneration, the transmission conditions at the damage point $x = 0$ can be specified as follows (see [14, Theorem 2.4]).

Theorem 2.2. *If $\alpha \in (1,2)$ then for any element $u \in H^1_a(-1,+1)$ the following assertions hold true:*

$$\lim_{x\nearrow 0} \sqrt{a(x)}\, u(x) = 0 = \lim_{x\searrow 0} \sqrt{a(x)}\, u(x), \tag{2.6}$$

$$\lim_{x\nearrow 0} a(x)\, u_x(x) = \lim_{x\searrow 0} a(x)\, u_x(x) = 0 \quad \textit{provided } u \in H^2_a(-1,+1), \tag{2.7}$$

$$\lim_{x\nearrow 0} a(x)\, \varphi_x(x)\, u(x) = 0 = \lim_{x\searrow 0} a(x)\, \varphi_x(x)\, u(x), \quad \forall\, \varphi \in H^2_a(-1,+1). \tag{2.8}$$

In order to proceed further, we recall the main results of semi-group theory concerning weak and strong of solutions for differential operator equation. With that in mind, we introduce the Hilbert space $\mathcal{H}_a := H^1_{a,0}(-1,+1) \times L^2(-1,+1)$ and endow it with the scalar product

$$\left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \int_{-1}^{1} v(x)\, \widetilde{v}(x)\, \mathrm{d}x + \int_{-1}^{1} a(x)\, u_x(x)\, \widetilde{u}_x(x)\, \mathrm{d}x.$$

We also define the unbounded operator $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H}_a \to \mathcal{H}_a$, associated with the problem (1.4), as follows

$$\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ (au_x)_x \end{bmatrix}, \tag{2.9}$$

and

Case (WDP):

$$\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A}) \quad \textbf{if} \quad \left\{ \begin{array}{c} u \in H^2_a(-1,+1),\ v \in H^1_{a,0}(-1,+1), \\ \lim\limits_{x\nearrow 0} u(x) = \lim\limits_{x\searrow 0} u(x), \\ \lim\limits_{x\nearrow 0} a(x)\, u_x(x) = \lim\limits_{x\searrow 0} a(x)\, u_x(x), \\ u(-1) = u(+1) = 0; \end{array} \right\} \tag{2.10}$$

Case (SDP):

$$\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A}) \ \textbf{if} \left\{ \begin{array}{c} u \in H^2_a(-1,+1),\ v \in H^1_{a,0}(-1,+1), \\ \lim\limits_{x\nearrow 0} a\, \varphi_x u = 0 = \lim\limits_{x\searrow 0} a\, \varphi_x u,\ \forall\, \varphi \in H^2_a(-1,+1), \\ \lim\limits_{x\nearrow 0} a(x)\, u_x(x) = 0 = \lim\limits_{x\searrow 0} a(x)\, u_x(x), \\ u(-1) = u(+1) = 0. \end{array} \right\} \tag{2.11}$$

Arguing as in [8, Section II.2], it can be shown that, in both cases, $D(\mathcal{A})$ is a dense subset of $\mathcal{H}_a$.

Lemma 2.1. *$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}_a \to \mathcal{H}_a$ is the generator of a contraction semi-group in $\mathcal{H}_a$.*

Proof. It is well-known that if H is a Hilbert space and $B : D(B) \subset H \to H$ is a densely defined linear operator such that both B and B^* are dissipative, i. e.,

$$\langle Bu, u\rangle_H \leqslant 0 \quad \text{and} \quad \langle u, B^* u\rangle_H \leqslant 0 \quad \forall\, u \in D(B),$$

then B generates a strongly continuous semi-group of contraction operators [15, p. 686]. Let us show that $\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}_a$ for all $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A})$, and this operator satisfies the above mentioned properties.

Since the inclusion $\mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}_a$ is obvious for each $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A})$, it remains to check the properties

$$\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} \leqslant 0, \quad \text{and} \quad \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, (\mathcal{A})^* \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} \leqslant 0 \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A}). \tag{2.12}$$

We do it for the case (SDP), because the case (WDP) can be considered in a similar manner. Then the first inequality in (2.12) immediately follows from the definition of the set $D(\mathcal{A})$, transmission conditions (2.8), and the relations

$$\begin{aligned}
\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} &= \left\langle \begin{bmatrix} v \\ (au_x)_x \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \int_{-1}^{1} (au_x)_x \, v \, \mathrm{d}x + \int_{-1}^{1} a v_x u_x \, \mathrm{d}x \\
&= \lim_{x \nearrow 0} \left[\int_{-1}^{x} (au_s)_s \, v \, \mathrm{d}s + \int_{-1}^{x} a v_s u_s \, \mathrm{d}s \right] \\
&\quad + \lim_{x \searrow 0} \left[\int_{x}^{1} (au_s)_s \, v \, \mathrm{d}s + \int_{x}^{1} a v_s u_s \, \mathrm{d}s \right] \\
&= \left[\lim_{x \nearrow 0} a(x)\, u_x(x) v(x) \right] - \left[\lim_{x \searrow 0} a(x)\, u_x(x) v(x) \right] = 0,
\end{aligned} \tag{2.13}$$

which hold true for all $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\mathcal{A})$. Taking into account the equality

$$\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \mathcal{A}^* \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a}, \quad \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \in D(\mathcal{A}),$$

we see that

$$\begin{aligned}
\left\langle \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} &= \left\langle \begin{bmatrix} v \\ (au_x)_x \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} \right\rangle_{\mathcal{H}_a} = \int_{-1}^{1} (au_x)_x \, \widetilde{v} \, \mathrm{d}x + \int_{-1}^{1} a v_x \widetilde{u}_x \, \mathrm{d}x \\
&= \lim_{x \nearrow 0} \left[\int_{-1}^{x} (au_s)_s \, \widetilde{v} \, \mathrm{d}s + \int_{-1}^{x} a v_s \widetilde{u}_s \, \mathrm{d}s \right] \\
&\quad + \lim_{x \searrow 0} \left[\int_{x}^{1} (au_s)_s \, \widetilde{v} \, ds + \int_{x}^{1} a v_s \widetilde{u}_s \, \mathrm{d}s \right] \\
&= \lim_{x \nearrow 0} \left[-\int_{-1}^{x} a u_s \widetilde{v}_s \, \mathrm{d}s - \int_{-1}^{x} v \, (a\widetilde{u}_s)_s \, \mathrm{d}s \right] \\
&\quad + \lim_{x \searrow 0} \left[-\int_{x}^{1} a u_s \widetilde{v}_s \, \mathrm{d}s - \int_{x}^{1} v \, (a\widetilde{u}_s)_s \, \mathrm{d}s \right] \\
&\quad + \left[\lim_{x \nearrow 0} a(x)\, u_x(x)\, \widetilde{v}(x) - \lim_{x \searrow 0} a(x)\, u_x(x)\, \widetilde{v}(x) \right] \\
&\quad + \left[\lim_{x \nearrow 0} a(x)\, \widetilde{u}_x(x)\, v(x) - \lim_{x \searrow 0} a(x)\, \widetilde{u}_x(x)\, v(x) \right] \\
&= -\int_{-1}^{1} (a\widetilde{u}_x)_x \, v \, \mathrm{d}x - \int_{-1}^{1} a \widetilde{v}_x u_x \, \mathrm{d}x = \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} -\widetilde{v} \\ -(a\widetilde{u}_x)_x \end{bmatrix} \right\rangle_{\mathcal{H}_a}.
\end{aligned}$$

Hence, $\mathcal{A}^* \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \end{bmatrix} = \begin{bmatrix} -\widetilde{v} \\ -\left(a\widetilde{u}_x\right)_x \end{bmatrix}$, and arguing as in (2.13), we see that $\mathcal{A}^*$ is a dissipative operator as well. Thus, $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H}_a \to \mathcal{H}_a$ generates a strongly continuous semi-group of contraction operators. □

We denote this semi-group by $e^{\mathcal{A}t}$. Then for any $U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathcal{H}_a$, the representation $U(t) = e^{\mathcal{A}t}\, U_0$ gives the so-called mild solution of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \mathcal{A}\, U(t), \quad t > 0, \\ U(0) = U_0\,. \end{cases} \tag{2.14}$$

When $U_0 \in D(\mathcal{A})$, the solution $U(t) = e^{\mathcal{A}t} U_0$ is strong in the sense that

$$U(\cdot) \in C^1\left([0,\infty);\mathcal{H}_a\right) \bigcap C\left([0,\infty);D(\mathcal{A})\right)$$

and equation (2.14) holds everywhere in $(0,\infty)$.

In view of the above consideration, we adopt the following concept.

Definition 2.1. We say that, for a given control $h(t)$, a function $u = u(t,x;\alpha)$ is the weak solution to the problem (1.4) if

$$u \in C^1([0,\infty);L^2(-1,+1)) \bigcap C([0,\infty);H^1_{a,0}(-1,+1)), \tag{2.15}$$

$$u(t,x;\alpha) = y(t,x;\alpha) + G(t,x), \tag{2.16}$$

and $U(t) := \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$ is the mild solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \mathcal{A}U(t) + F(t), \quad t > 0, \\ U(0) = \begin{bmatrix} -G(0,\cdot) \\ -G_t(0,\cdot) \end{bmatrix}, \end{cases} \tag{2.17}$$

where

$$F(t) = \begin{bmatrix} 0 \\ \left(aG_x\right)_x - G_{tt} \end{bmatrix}, \tag{2.18}$$

and $G \in W^{2,2}(0,T;H^2_a(-1,+1)) \bigcap C^2(0,T;L^2(-1,+1))$, $\forall\, T > 0$, is an arbitrary function such that

$$\begin{gathered} G(t,-1) = 0,\ G(t,1) = h(t),\ G(0,x) \in H^1_{a,0}(-1,+1), \\ \text{and } G_t(0,x) \in L^2(-1,+1) \text{ for a.a. } t \geqslant 0 \text{ and } x \in [-1,+1]. \end{gathered} \tag{2.19}$$

Definition 2.2. We say that a function $u = u(t, x; \alpha)$ is the strong solution to the problem (1.4) if each relation in (1.4) is satisfied for all $t \in [0, \infty)$ and a.a. $x \in [-1, +1]$,

$$u \in C^2([0,\infty); L^2(-1,+1)) \bigcap C^1([0,\infty); H^1_{a,0}(-1,+1)) \cap C([0,\infty); H^2_a(-1,+1)),$$

the representation (2.16) holds with a function G that, in addition to (2.19), satisfies

$$G(0,x) \in H^2_a(-1,+1),\ G_t(0,x) \in H^1_{a,0}(-1,+1) \text{ for } t \geqslant 0 \text{ and } x \in [-1,+1], \tag{2.20}$$

and the function y such that $U(t) := \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$ is the strong solution of the problem (2.17). If, in addition, all relations in (1.4) are satisfied for for all $t \in [0, \infty)$ and $x \in [-1, +1]$, then $u = u(t, x; \alpha)$ is the classical solution of the original initial-boundary value problem (1.4).

Our further intention is to examine the well-posedness of the problem (1.4) in the weak and strong degenerate cases.

Theorem 2.3. *Let $a(x; \alpha)\colon [-1, +1] \to \mathbb{R}$ be a weight function defined by* (1.2) *with $\alpha \in (0, 2)$. Assume that for a given control $h(t)$ there exists function $G = G(t, x)$ satisfying properties* (2.19). *Then initial boundary value problem*(1.4) *admits a unique weak solution $u = u(t, x; \alpha)$ for which representation* (2.16) *holds.*

Proof. Let $G \in W^{2,2}(0,T; H^2_a(-1,+1)) \bigcap C^2(0,T; L^2(-1,+1))$ be a function with properties (2.19). Then $(aG_x)_x - G_{tt} \in C([0,T]; L^2(-1,+1))$ and $U(0) \in \mathcal{H}_a$. Hence, by the Duhamel principle, we deduce that there exists a unique mild solution of the problem (2.17) and it can be represented as follows

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = e^{\mathcal{A}t} U(0) + \int_0^t e^{\mathcal{A}(t-s)} F(s)\, \mathrm{d}x \quad \forall\, t \in [0,T]. \tag{2.21}$$

As immediately follows from (2.16), the function $u(t, x)$ satisfies both initial and boundary conditions in (1.4), and its functional properties (2.15) easily follow from the semi-group properties of $e^{\mathcal{A}t}$ and (2.21). Thus, $u(t, x; \alpha)$ is a weak solution to the problem (1.4) in the sense of Definition 2.1.

As for uniqueness of the weak solution, it is a direct consequence of representation (2.16) and formula (2.21). □

Arguing in a similar manner, it can be established the following result.

Theorem 2.4. *Let $a(x; \alpha)\colon [-1, +1] \to \mathbb{R}$ be a weight function defined by* (1.2) *with $\alpha \in (0, 2)$. Assume that for a given control $h(t)$ there exists function $G = G(t, x)$ satisfying properties* (2.19)–(2.20). *Then initial boundary value problem*(1.4) *admits a unique strong solution.*

To prove this assertion, it is enough to notice that due to the properties (2.19)–(2.20), the function $G(t,x)$ is sufficiently smooth and the vector of initial data $U(0)$ belongs to the set $D(\mathcal{A})$ both in the weak and strong degenerate cases.

Remark 2.2. In view of the transmission conditions (see Theorem 2.2), it is still unknown whether the strong solution to the problem (1.4) preserves its continuity at the damage point $x = 0$ in the case of strong degeneration. The second point that should be clarified, is about existence of a classical solution to the problem (1.4) for different range of parameter $\alpha > 0$ provided $h \in C([0,T])$. To the best knowledge of authors, this question remains arguably open for nowadays.

For our further analysis, in order to specify the notion of solution to the problem (1.4) possessing the continuously differentiable flux for a wide range of parameter α, we adopt the following concept.

Definition 2.3. A function $u(t,x;\alpha)$ is called a solution with the continuously differentiable flux (or, shortly the required solution) to the initial boundary value problem (1.4) if it: 1) is continuous in the space-time rectangle $[0,T] \times [-1,+1]$ and is twice continuously differentiable in the variables t and x inside the space-time rectangle, except for the degeneracy segment; 2) has the one-sided derivative in the variable x, bounded or integrable, and the flux, continuously differentiable in the variable x, both on the degeneracy segment; 3) satisfies the degenerate wave equation inside the space-time rectangle; and 4) satisfies the initial and the boundary conditions of the problem.

3. Series solutions of the original wave equation

In this Section we construct power series solutions $u^{\mp}(t,x;\alpha)$ of the original degenerate wave equation in one-sided vicinities $[0,T]\times(-\epsilon,0)$ and $[0,T]\times(0,+\epsilon)$, $\epsilon \leqslant c$, of the degeneracy segment. These solutions are further referred to as one-sided and constitute a pair for the ϵ-band $\subseteq c$-band (with the degeneracy segment removed or not removed). To construct such pairs, we introduce the set of rational numbers

$$\mathbb{Q}_o := \left\{ \frac{m}{n} :\ m \in \mathbb{Z}\,,\ \ n \in \mathbb{N}\,,\ \ m \neq 0\,,\ \ n > 1\,,\ \ n \equiv 1 \ (\mathtt{mod}\, 2) \right\}, \tag{3.1}$$

where m, n are coprime numbers, and those derived from subsets of $\mathbb{R}$, for instance, $(0,2)_o := (0,2) \bigcap \mathbb{Q}_o$, etc., to distinguish between the exponents σ of those power monomials x^{σ}, $x > 0$, extendable straightforwardly to $x < 0$ and not extendable. Let $\mathbb{Q}_{o,e}$ be the subset of $\mathbb{Q}_o$, where $m \equiv 0 \ (\mathtt{mod}\, 2)$, and $\mathbb{Q}_{o,o}$ be the subset of $\mathbb{Q}_o$, where $m \equiv 1 \ (\mathtt{mod}\, 2)$, then both subsets partition the set $\mathbb{Q}_o$: $\mathbb{Q}_{o,e} \bigcup \mathbb{Q}_{o,o} = \mathbb{Q}_o$, $\mathbb{Q}_{o,e} \bigcap \mathbb{Q}_{o,o} = \varnothing$. It is clear that monomials x^{σ} are extendable to $x < 0$ evenly, if $\sigma \in \mathbb{Q}_{o,e}$, and oddly, if $\sigma \in \mathbb{Q}_{o,o}$.

Initially (A), we present our attempts to find pairs of one-sided series solutions, substituting pairs of trial one-sided power series, or pairs of ansatze, into the original degenerate wave equation, and finally (B), we find the above pairs of one-sided series solutions (A) once again using the Frobenius method [7, 12].

A) Let the pair of trial one-sided power series solutions of the degenerate wave equation be of the form

$$u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\sigma^{\mp}_{\alpha,1}} + U^{\mp}_{\alpha,2}(t)\,|x|^{\sigma^{\mp}_{\alpha,2}} + \ldots,$$

where $U^{\mp}_{\alpha,0}(t)$, $U^{\mp}_{\alpha,1}(t)$, $U^{\mp}_{\alpha,2}(t)$, etc., are unknown coefficient functions of variable t; $\sigma^{\mp}_{\alpha,1}$, $\sigma^{\mp}_{\alpha,2}$, etc., are unknown real exponents. Differentiating the above pair of the ansatze with respect to t and x (and dropping for a while hereafter the argument t of the functions and some of the lower and upper indices of the functions and the exponents, to simplify the notation where this will not lead to confusion)

$$\begin{aligned}
\frac{\partial p^{\mp}}{\partial t} &= U''_{\alpha,0} + U''_{\alpha,1}\,|x|^{\sigma_1} + U''_{\alpha,2}\,|x|^{\sigma_2} + \ldots,\\
q^{\mp} &= \mp\Big(\sigma_1\,U_{\alpha,1}\,|x|^{\sigma_1-1} + \sigma_2\,U_{\alpha,2}\,|x|^{\sigma_2-1} + \ldots\Big),\\
-f^{\mp} &= \mp a_*\Big(\sigma_1\,U_{\alpha,1}\,|x|^{\omega_1} + \sigma_2\,U_{\alpha,2}\,|x|^{\omega_2} + \ldots\Big),\\
-\frac{\partial f^{\mp}}{\partial x} &= a_*\Big(\omega_1\,\sigma_1\,U_{\alpha,1}\,|x|^{\omega_1-1} + \omega_2\,\sigma_2\,U_{\alpha,2}\,|x|^{\omega_2-1} + \ldots\Big),
\end{aligned}$$

where $\omega_1 = \sigma_1 + \alpha - 1$, $\omega_2 = \sigma_2 + \alpha - 1$, etc., and substituting the obtained pairs of the series for the derivatives into the degenerate wave equation (1.3) yields to the following pair of the one-sided series identities

$$\begin{aligned}
&\underbrace{U''_{\alpha,0}}_{1} + \underbrace{U''_{\alpha,1}\,|x|^{\sigma_1}}_{2} + \underbrace{U''_{\alpha,2}\,|x|^{\sigma_2}}_{3} + \ldots\\
&= a_*\Big(\underbrace{\sigma_1\,\omega_1\,U_{\alpha,1}\,|x|^{\omega_1-1}}_{1} + \underbrace{\sigma_2\,\omega_2\,U_{\alpha,2}\,|x|^{\omega_2-1}}_{2} + \underbrace{\sigma_3\,\omega_3\,U_{\alpha,3}\,|x|^{\omega_3-1}}_{3} + \ldots\Big),
\end{aligned} \tag{3.2}$$

being used for non-unique determining all unknown functions and exponents.

i) Let terms 1, 2, 3, etc., in the pair of the one-sided series identities (3.2) (i. e. the couples of summands marked with the same numbers) be of like powers and cancel respectively each other, then, after applying a little portion of algebra, we obtain: 1) the explicit expressions for the exponents $\sigma^{\mp}_{\alpha,\mu} = \mu\theta$, $\mu \in \mathbb{N}$, where $\theta = 2 - \alpha$ is a derived parameter, leading to the following pairs of the one-sided series solutions

$$u^{\mp}(t,x;\alpha) = \begin{cases} U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\theta} + U^{\mp}_{\alpha,2}(t)\,|x|^{2\theta} + \ldots\,,\\ U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\;x^{\,\theta} + U^{\mp}_{\alpha,2}(t)\;x^{\,2\theta} + \ldots\,, \end{cases} \tag{3.3}$$

valid respectively for $\alpha \in (0,2) \bigcup (2,+\infty)$ and $(0,2)_o \bigcup (2,+\infty)_o$; and 2) the following pairs of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,\mu-1}(t)}{\mathrm{d}t^2} = \mu\theta\,[(\mu-1)\,\theta+1]\,a_*\,U^{\mp}_{\alpha,\mu}(t)\,, \qquad \mu\in\mathbb{N}\,. \tag{3.4}$$

ii) Let each term in the pair of the one-sided series identities (3.2) vanish separately, then: 1) the functions $U^{\mp}_{\alpha,0}(t)$, $U^{\mp}_{\alpha,\mu}(t)$, $\mu\in\mathbb{N}$, are linear; 2) the exponents are equal each other: $\sigma^{\mp}_{\alpha,\mu}=1-\alpha$; and 3) the resulting pairs of the one-sided series solutions reduce to the following pairs of the one-sided binomials

$$u^{\mp}(t,x;\alpha) = \begin{cases} U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{1-\alpha}, & \alpha\in(0,+\infty)\,,\\ U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\;\;x^{\,1-\alpha}, & \alpha\in(0,+\infty)_o. \end{cases} \tag{3.5}$$

iii) Let terms 1 in the pair of the one-sided series identities (3.2) vanish separately, then: 1) $U^{\mp}_{\alpha,0}(t)$ are linear functions; 2) $\omega^{\mp}_{\alpha,1}=0$, from where $\sigma^{\mp}_{\alpha,1}=1-\alpha$. Applying the procedure from item *i*) to terms 2, 3, 4, etc., we find: 1) the explicit expressions for the exponents $\sigma^{\mp}_{\alpha,\mu+1}=\mu\theta+1-\alpha$, $\mu\in\mathbb{N}$, leading to the following pairs of the one-sided series solutions

$$\begin{cases} u^{\mp}(t,x;\alpha)=U^{\mp}_{\alpha,0}(t)+|x|^{1-\alpha}\Big(U^{\mp}_{\alpha,1}(t)+U^{\mp}_{\alpha,2}(t)\,|x|^{\theta}+U^{\mp}_{\alpha,3}(t)\,|x|^{2\theta}+\ldots\Big),\\ u^{\mp}(t,x;\alpha)=U^{\mp}_{\alpha,0}(t)+\;x^{\,1-\alpha}\Big(U^{\mp}_{\alpha,1}(t)+U^{\mp}_{\alpha,2}(t)\;\;x^{\,\theta}+U^{\mp}_{\alpha,3}(t)\;\;x^{\,2\theta}+\ldots\Big), \end{cases} \tag{3.6}$$

valid respectively for $\alpha\in(0,+\infty)$ and $\alpha\in(0,+\infty)_o$; and 2) the following pairs of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,\mu}(t)}{\mathrm{d}t^2} = \mu\theta\,[(\mu+1)\,\theta-1]\,a_*\,U^{\mp}_{\alpha,\mu+1}(t)\,, \qquad \mu\in\mathbb{N}\,. \tag{3.7}$$

B) Let the pair of trial one-sided power series solutions of the degenerate wave equation, following the Frobenius method, be of the form

$$u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha}(t) + |x|^{\omega^{\mp}_{\alpha}}\Big(U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\sigma^{\mp}_{\alpha,1}} + U^{\mp}_{\alpha,2}(t)\,|x|^{\sigma^{\mp}_{\alpha,2}} + \ldots\Big)\,,$$

where $U^{\mp}_{\alpha}(t)$, $U^{\mp}_{\alpha,0}(t)$, $U^{\mp}_{\alpha,1}(t)$, $U^{\mp}_{\alpha,2}(t)$, etc., are unknown coefficient functions of variable t; $\omega^{\mp}_{\alpha}$, $\sigma^{\mp}_{\alpha,1}$, $\sigma^{\mp}_{\alpha,2}$, etc., are unknown real exponents. Differentiating the above pair of the ansatze with respect to t and x

$$\begin{aligned} \frac{\partial p^{\mp}}{\partial t} &= U''_{\alpha} + |x|^{\omega}\Big(U''_{\alpha,0} + U''_{\alpha,1}\,|x|^{\sigma_1} + U''_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big),\\ q^{\mp} &= \mp\quad |x|^{\omega-1}\quad\Big(\omega\,U_{\alpha,0} + \omega_1\,U_{\alpha,1}\,|x|^{\sigma_1} + \omega_2\,U_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big),\\ -f^{\mp} &= \mp a_*|x|^{\omega-1+\alpha}\Big(\omega\,U_{\alpha,0} + \omega_1\,U_{\alpha,1}\,|x|^{\sigma_1} + \omega_2\,U_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big),\\ -\frac{\partial f^{\mp}}{\partial x} &= \quad a_*|x|^{\omega-2+\alpha}\Big(\omega\,(\omega-1+\alpha)\,U_{\alpha,0} + \omega_1\,(\omega-1+\alpha+\sigma_1)\,U_{\alpha,1}\,|x|^{\sigma_1}\\ &\qquad\qquad + \omega_2\,(\omega-1+\alpha+\sigma_2)\,U_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big), \end{aligned}$$

where $\omega_1 = \omega + \sigma_1$, $\omega_2 = \omega + \sigma_2$, etc., and substituting the obtained pairs of the series for the derivatives into the degenerate wave equation, we obtain the following pair of the one-sided series identities

$$\begin{aligned} &\underbrace{U''_\alpha}_{1} + |x|^\omega \Big(\underbrace{U''_{\alpha,0}}_{2} + \underbrace{U''_{\alpha,1}\,|x|^{\sigma_1}}_{3} + U''_{\alpha,2}\,|x|^{\sigma_2} + \ldots\Big) \\ &= a_*|x|^{o-1}\Big(\underbrace{\omega o\, U_{\alpha,0}}_{1} + \underbrace{\omega_1\,(o+\sigma_1)\, U_{\alpha,1}\,|x|^{\sigma_1}}_{2} + \underbrace{\omega_2\,(o+\sigma_2)\, U_{\alpha,2}\,|x|^{\sigma_2}}_{3} + \ldots\Big), \end{aligned} \tag{3.8}$$

where $o = \omega - 1 + \alpha$. The identities are to be used for determining all unknown functions and exponents, but in a non-unique way.

i) Let terms 1 in (3.8) be of like powers (accounting for the multiplier $a_*|x|^{o-1}$), then we obtain: *a*) $o_\alpha^\mp = 1$, $\omega_\alpha^\mp = \theta$; and *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_\alpha^\mp(t)}{\mathrm{d}t^2} = \omega_\alpha^\mp o_\alpha^\mp\, a_* U_{\alpha,0}^\mp(t) = \theta\, a_* U_{\alpha,0}^\mp(t)\,.$$

Let now terms 2 in (3.8) be of like powers, then we obtain: *a*) $\sigma_{\alpha,1}^\mp = \omega_\alpha^\mp = \theta$; and *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_{\alpha,0}^\mp(t)}{\mathrm{d}t^2} = \omega_{\alpha,1}^\mp\big(o_\alpha^\mp + \sigma_{\alpha,1}^\mp\big)\, a_* U_{\alpha,1}^\mp(t) = 2\theta\,(\theta+1)\, a_* U_{\alpha,1}^\mp(t)\,.$$

Then, repeating the above procedure for terms 3 in (3.8), we obtain: *a*) $\sigma_{\alpha,2}^\mp = \omega_\alpha^\mp + \sigma_{\alpha,1}^\mp = 2\theta$, and *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_{\alpha,1}^\mp(t)}{\mathrm{d}t^2} = \omega_{\alpha,2}^\mp\big(o_\alpha^\mp + \sigma_{\alpha,2}^\mp\big)\, a_* U_{\alpha,2}^\mp(t) = 3\theta\,(2\theta+1)\, a_* U_{\alpha,2}^\mp(t)\,,$$

etc. That is, we obtain nothing but the pairs of the one-sided series solutions (3.3) again.

ii) Let in (3.8): *a*) the functions $U_\alpha^\mp(t)$ and $U_{\alpha,0}^\mp(t)$ be linear; *b*) $\omega_\alpha^\mp = 1-\alpha$; and *c*) $\sigma_{\alpha,\mu}^\mp = 0$, $\mu \in \mathbb{N}$, then we obtain the pairs of the one-sided binomial solutions (3.5) again.

iii) Let terms 1 in (3.8) vanish separately, then: *a*) the functions $U_\alpha^\mp(t)$ are linear; *b*) $o_\alpha^\mp = 0$, $\omega_\alpha^\mp = 1-\alpha$. Let now terms 2 in (3.8) be of like powers, then we obtain: *a*) $\sigma_{\alpha,1}^\mp = \theta$; *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U_{\alpha,0}^\mp(t)}{\mathrm{d}t^2} = \omega_{\alpha,1}^\mp\big(o_\alpha^\mp + \sigma_{\alpha,1}^\mp\big)\, a_* U_{\alpha,1}^\mp(t) = \theta\,(2\theta-1)\, a_* U_{\alpha,1}^\mp(t)\,.$$

Treating terms 3 in (3.8) in the same way we obtain: *a*) $\sigma_{\alpha,2}^\mp = 2\theta$; *b*) the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,1}(t)}{\mathrm{d}t^2} = \omega^{\mp}_{\alpha,2}\big(o^{\mp}_{\alpha} + \sigma^{\mp}_{\alpha,2}\big)\, a_* U^{\mp}_{\alpha,2}(t) = 2\theta\,(3\theta - 1)\, a_* U^{\mp}_{\alpha,2}(t)\,.$$

Repeating the above procedure, we eventually obtain: 1) the explicit expressions for the exponents $\sigma^{\mp}_{\alpha,\mu} = \mu\theta$, leading to the following pairs of the one-sided series solutions

$$\begin{cases} u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha}(t) + |x|^{1-\alpha}\Big(U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\,|x|^{\theta} + U^{\mp}_{\alpha,2}(t)\,|x|^{2\theta} + \dots\Big), \\ u^{\mp}(t,x;\alpha) = U^{\mp}_{\alpha}(t) + \ x^{\ 1-\alpha}\Big(U^{\mp}_{\alpha,0}(t) + U^{\mp}_{\alpha,1}(t)\ \ x^{\ \theta} + U^{\mp}_{\alpha,2}(t)\ \ x^{\ 2\theta} + \dots\Big), \end{cases} \tag{3.9}$$

valid respectively for $\alpha \in (0,+\infty)$ and $(0,+\infty)_o$; and 2) the following pairs of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 U^{\mp}_{\alpha,\mu-1}(t)}{\mathrm{d}t^2} = \mu\theta\,[(\mu+1)\,\theta - 1]\, a_* U^{\mp}_{\alpha,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,. \tag{3.10}$$

That is, we obtain nothing but the pairs of the the one-sided series solutions (3.6) again.

C) When constructing the pairs of the one-sided series solutions (3.3), (3.5) and (3.9) of the degenerate wave equation, we have not accounted for the following (1-parameter families of the) pairs of the binomial solutions

$$u^{\mp}(t,x;2) = \begin{cases} U^{\mp}_{2,0}(t) + U^{\mp}_{2,\sigma}(t)\,|x|^{\sigma^{\mp}}, & \sigma^{\mp} \in \mathbb{R}\,, \\ U^{\mp}_{2,0}(t) + U^{\mp}_{2,\sigma}(t)\ \ x^{\ \sigma^{\mp}}, & \sigma^{\mp} \in \mathbb{R}_o\,, \end{cases} \tag{3.11}$$

where $\sigma^{\mp} \in \mathbb{R}$, $U^{\mp}_{2,0}(t)$ are linear functions, and the functions $U^{\mp}_{2,\sigma}(t)$ satisfy the following ordinary linear homogeneous differential equations

$$\frac{\mathrm{d}^2 U^{\mp}_{2,\sigma}(t)}{\mathrm{d}t^2} - \sigma\,(\sigma+1)\, a_* U^{\mp}_{2,\sigma}(t) = 0\,. \tag{3.12}$$

Now we are to follow Definition 2.3 and to select the required series solutions out of those obtained in this Section.

4. Series based analysis of the original problem

In this Section we turn out to possessing or violating property Z by solutions to the initial boundary value problem (1.4) using the series solutions of the degenerate wave equation. On the one hand, the one-sided series solutions constituting a pair are independent, on the other hand, possessing property Z implies integrity of the ‘string’ on the degeneracy segment and, more generally, fulfilling the conditions of the Definition 2.3, except perhaps the last one.

To gain the required properties for the series solutions, we follow the three-step procedure applied on the degeneracy segment: 1) continuous matching the one-sided series solutions by setting $U^-_\alpha(t) = U^+_\alpha(t)$, $U^-_{\alpha,\mu}(t) = U^+_{\alpha,\mu}(t)$ (provided all exponents of the one-sided series solutions are non-negative for the resulting continuous, or two-sided, series solutions to be bounded); 2) verifying continuity of the flux; 3) verifying differentiability of the flux.

First, we apply the procedure to the pairs of the one-sided series solutions (3.3), (3.5), and (3.9).

1) Continuous matching the above one-sided series solutions formally yields to the following bounded and continuous, or two-sided, nontrivial series solutions (trivial solutions, valid if $\alpha \in [2, +\infty)$, are discussed in the proof of Proposition 4.2)

$$\begin{aligned}
u_1(t,x;\alpha) &= U_{\alpha,0}(t) + \sum_{\mu=1}^{\infty} U_{\alpha,\mu}(t)\,|x|^{\mu\theta}, && \alpha \in (0,2)\,,\\
u_2(t,x;\alpha) &= U_{\alpha,0}(t) + \sum_{\mu=1}^{\infty} U_{\alpha,\mu}(t)\;\, x^{\,\mu\theta}, && \alpha \in (0,2)_o\,,\\
u_3(t,x;\alpha) &= U_{\alpha,0}(t) + U_{\alpha,1}(t)\,|x|^{1-\alpha}, && \alpha \in (0,1)\,,\\
u_4(t,x;\alpha) &= U_{\alpha,0}(t) + U_{\alpha,1}(t)\;\, x^{\,1-\alpha}, && \alpha \in (0,1)_o\,,\\
u_5(t,x;\alpha) &= U_{\alpha}(t) + |x|^{1-\alpha} \sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\,|x|^{\mu\theta}, && \alpha \in (0,1)\,,\\
u_6(t,x;\alpha) &= U_{\alpha}(t) + \;x^{\,1-\alpha} \sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\;\, x^{\,\mu\theta}, && \alpha \in (0,1)_o\,.
\end{aligned} \tag{4.1}$$

2) The respective one-sided values $-f^{\mp} = aq^{\mp}$ of the fluxes for the above two-sided series solutions (for $x < 0$ and $x > 0$) are given by the following expressions

$$\begin{aligned}
-f_1^{\mp}(t,x;\alpha) &= \quad a_*\theta\, x \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{(\mu-1)\theta},\\
-f_2^{\mp}(t,x;\alpha) &= \mp a_*\theta\, x \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\;\, x^{\,(\mu-1)\theta},\\
-f_3^{\mp}(t,x;\alpha) &= \mp a_*(1-\alpha)\, U_{\alpha,1}(t)\,,\\
-f_4^{\mp}(t,x;\alpha) &= \mp a_*(1-\alpha)\, U_{\alpha,1}(t)\,,\\
-f_5^{\mp}(t,x;\alpha) &= \mp a_* \sum_{\mu=0}^{\infty} (1-\alpha+\mu\theta)\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta},\\
-f_6^{\mp}(t,x;\alpha) &= \mp a_* \sum_{\mu=0}^{\infty} (1-\alpha+\mu\theta)\, U_{\alpha,\mu}(t)\;\, x^{\,\mu\theta}.
\end{aligned} \tag{4.2}$$

To complete the current step of the procedure we calculate the one-sided values of the fluxes on the degeneracy segment (by substituting in (4.2) zero value instead of x) and conclude that the series solutions 1 and 2 produce zero one-sided fluxes: $f_{1,2}(t,0-;\alpha)=0=f_{1,2}(t,0+;\alpha)$, whereas the series solutions $3-6$ produce non-zero one-sided fluxes of opposite signs: $f_{3-6}(t,0-;\alpha)=-f_{3-6}(t,0+;\alpha)$. Therefore, we retain the series solutions 1 and 2 with the continuous fluxes to implement the next step.

3) It is clear that the flux of the series solution 1 is continuously differentiable on the degeneracy segment, whereas the flux of the series solution 2 is not. Therefore, the only required series solution, we retain for further studying, reads

$$u(t,x;\alpha) = U_{\alpha,0}(t) + \sum_{\mu=1}^{\infty} U_{\alpha,\,\mu}(t)\,|x|^{\mu\theta}, \qquad \alpha \in (0,2)\,, \tag{4.3}$$

where the coefficient functions obey the following recurrence relations

$$U''_{\alpha,\mu-1}(t) = \mu\theta\,[(\mu-1)\,\theta+1]\,a_* U_{\alpha,\,\mu}(t)\,, \qquad \mu \in \mathbb{N}\,. \tag{4.4}$$

Second, we apply the procedure to the pairs of the binomial solutions (3.11).

1) Implementing continuous matching yields to the following bounded and continuous solutions (1-parameter families of solutions)

$$\begin{cases} u_7(t,x;2,\sigma) = U_{2,0}(t) + U_{2,\sigma}(t)\,|x|^{\sigma}, & \sigma \in (0,+\infty)\,,\\ u_8(t,x;2,\sigma) = U_{2,0}(t) + U_{2,\sigma}(t)\ \ x^{\ \sigma}, & \sigma \in (0,+\infty)_o\,, \end{cases} \tag{4.5}$$

where the functions $U_{2,0}(t)$ are linear, and the functions $U_{2,\sigma}(t)$ satisfy the following ordinary linear homogeneous differential equations

$$U''_{2,\sigma}(t) - a_*\sigma\,(\sigma+1)\,U_{2,\sigma}(t) = 0\,. \tag{4.6}$$

2) The one-sided values of the fluxes for the above solutions

$$\begin{aligned} -f_7(t,x;2,\sigma) &= \mp a_*\sigma\,U_{2,\sigma}(t)\,|x|^{\sigma+1}\,,\\ -f_8(t,x;2,\sigma) &= \quad a_*\sigma\,U_{2,\sigma}(t)\ \ x^{\ \sigma+1}\,, \end{aligned}$$

vanish on the degeneracy segment, proving continuity of both fluxes.

3) Continuous differentiability of both fluxes is evident. Completing the procedure, we conclude that both binomial solutions (4.5) are required.

Proposition 4.1. *Possessing property Z is not necessary for the required solution to the 1-parameter initial boundary value problem* (1.4) *if* $\alpha \in (0,2)$.

Proof. Applying the required series solution (4.3) exactly on the degeneracy segment, we find that $u(t,0;\alpha)=U_{\alpha,0}(t)$, where the undetermined function $U_{\alpha,0}(t)$ is a solution to the Cauchy problem

$$U''_{\alpha,0}(t) = \theta\, a_* U_{\alpha,1}(t)\,, \quad t \in (0,T]\,, \qquad U_{\alpha,0}(0) = 0\,, \quad U'_{\alpha,0}(0) = 0\,,$$

assembled from the recurrence relations (4.4) and the initial conditions of the problem (1.4). Non-uniqueness of $U_{\alpha,0}(t)$ stems from the undetermined function $U_{\alpha,1}(t)$.

Assumption $U_{\alpha,1}(t) \equiv 0$ yields to a linear function $U_{\alpha,0}(t)$, and from the above initial conditions we conclude that $U_{\alpha,0}(t) \equiv 0$. But assuming that $U_{\alpha,1}(t)$ is not identically zero, we conclude that $U_{\alpha,0}(t)$ is not identically equal to zero as well, though the initial conditions it satisfies are all zero. Hence, property Z is not necessary for the required solution to the initial boundary value problem (1.4).

It should be noted, that continuity of the required solution is essentially used to prove the proposition. Indeed, let the 'string' lose its integrity, then problem (1.4) immediately splits into two quite independent subproblems referred to as the left one

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial u}{\partial x}\right), & (t,x) \in (0,T)\times(-l,0)\,, \\ u(t,-l;\alpha) = 0\,, & t \in [0,T]\,, \\ u(t,\ \ 0;\alpha) = 0\,, & t \in [0,T]\,, \\ u(0,x;\alpha) = 0\,, & x \in [-l,0]\,, \\ \dfrac{\partial u(0,x;\alpha)}{\partial t} = 0\,, & x \in [-l,0]\,, \end{cases} \tag{4.7}$$

and the right one

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial u}{\partial x}\right), & (t,x) \in (0,T)\times(0,+l)\,, \\ u(t,\ \ 0;\alpha) = 0\,, & t \in [0,T]\,, \\ u(t,+l;\alpha) = h(t)\,, & t \in [0,T]\,, \\ u(0,x;\alpha) = 0\,, & x \in [0,+l]\,, \\ \dfrac{\partial u(0,x;\alpha)}{\partial t} = 0\,, & x \in [0,+l)\,. \end{cases} \tag{4.8}$$

Both problems are evidently solvable, whereas the unique solution to the left subproblem is trivial and property Z necessarily holds. □

Proposition 4.2. *Possessing property Z is necessary for the required solutions to the 1-parameter initial boundary value problem* (1.4) *if* $\alpha \in [2,+\infty)$.

Proof. Let $\alpha = 2$, then the required series solutions to the problem are those given by (4.5), where the linear functions $U_{2,0}(t) \equiv 0$, due to the initial conditions of the problem. Therefore, $u(t,0;2)\equiv 0$, and the unique solution to the left subproblem (4.7) is trivial. Hence, property Z necessarily holds.

Let $\alpha\in(2,+\infty)$, then the required series solutions (3.3), (3.5), and (3.9) to the problem are those where all functions $U_{\alpha,\mu}(t)$ preceding the power monomials are identically equal to zero. Therefore, the series solutions reduce to the leading terms $U_\alpha(t)$ or $U_{\alpha,0}(t)$, being zero due to the initial conditions of the problem. The remaining part of the proof is the same as for $\alpha=2$. □

5. Separation of variables applied to the original equation

In this Section we find the series solutions to the original degenerate wave equation that can be rewritten as

$$\frac{\partial^2 u(t,x)}{\partial t^2} - a(x;\alpha)\,\frac{\partial^2 u(t,x)}{\partial x^2} = a'(x;\alpha)\,\frac{\partial u(t,x)}{\partial x}\,, \tag{5.1}$$

in a way different to that used in Section 3.

First, we assume that the independent variables in (5.1) are separable, hence the trial solution reads

$$u(t,x;\alpha) = O(t;\alpha)\,X(x;\alpha)\,, \tag{5.2}$$

and substitute the above representation into (5.1) to obtain the following system of two ordinary differential equations of the second order

$$\begin{cases} O''(t;\alpha) \mp \lambda^2\, O\,(t;\alpha) = 0\,, \\ a(x;\alpha)\,X''(x;\alpha) + a'(x;\alpha)\,X'(x;\alpha) \mp \lambda^2 X(x;\alpha) = 0\,, \end{cases} \tag{5.3}$$

where $\pm\lambda^2$ is the unknown parameter of separation of the variables ($\lambda\geqslant 0$).

Second, we: *a*) substitute the power law (1.2) into the system (5.3)

$$a_* x^\alpha\, X''(x;\alpha) + a_*\alpha\, x^{\alpha-1}\, X'(x;\alpha) \mp \lambda^2 X(x;\alpha) = 0\,,$$

assuming initially that $x>0$; *b*) introduce a new dependent variable

$$X(x;\alpha) = x^\beta w(x;\alpha)\,, \tag{5.4}$$

where β is the undetermined exponent; and *c*) find the relations between the first and the second derivatives of X and w as follows

$$X' = \beta\, x^{\beta-1}\, w + x^\beta w', \qquad X'' = \beta\,(\beta-1)\, x^{\beta-2}\, w + 2\beta\, x^{\beta-1} w' + x^\beta w'';$$

d) then, a 4-parameter $(\alpha,\beta,\lambda,\mp)$ family of ordinary differential equations of the second order for the required function w reads

$$a_* x^\alpha\, w'' + a_* x^{\alpha-1}\,(2\beta+\alpha)\, w' + a_* x^{\alpha-2}\beta\,(\beta-1+\alpha)\, w' \pm \lambda^2 w = 0\,. \tag{5.5}$$

Third, we a) introduce a new independent variable: $\bar{x} = \chi(x)$, where function $\chi(x)$ is invertible and differentiable; and b) obtain the relations between the first and the second derivatives of $w(x;\alpha)$ and $\bar{w}(\bar{x};\alpha) := w(\chi^{-1}(\bar{x});\alpha)$

$$w' = \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}}\,\frac{\mathrm{d}\chi}{\mathrm{d}x}\,, \qquad w'' = \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2}\left(\frac{\mathrm{d}\chi}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}}\,\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2}\,.$$

Imposing constraint $a_* x^\alpha \left(\dfrac{\mathrm{d}\chi}{\mathrm{d}x}\right)^2 = 1$ on the required function $\chi(x)$ yields to

$$\frac{\mathrm{d}\chi}{\mathrm{d}x} = \frac{1}{\sqrt{a_* x^\alpha}}\,, \qquad a_* x^\alpha\,\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} = -\frac{\alpha}{2}\,\frac{1}{x}\,\sqrt{a_* x^\alpha}\,,$$

then assuming that $\bar{x}=0$ when $x=0$, we obtain by integration the explicit relation between x and $\bar{x}$

$$\bar{x} = \frac{2}{\theta}\,\frac{x}{\sqrt{a_* x^\alpha}} = \Omega\, x^{\frac{\theta}{2}} \qquad \Leftrightarrow \qquad \frac{\sqrt{a_* x^\alpha}}{x} = \frac{2}{\theta}\,\frac{1}{\bar{x}}\,,$$

where an α-dependent auxiliary quantity is used

$$\Omega = \frac{2}{\theta}\,\frac{1}{\sqrt{a_*}}\,. \tag{5.6}$$

Replacing the couple of the variables (x, w) with that of the variables $(\bar{x}, \bar{w})$ in (5.5) yields to a new 4-parameter $(\alpha, \beta, \lambda, \mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} + \left(2\beta + \frac{\alpha}{2}\right)\frac{2}{\theta}\,\frac{1}{\bar{x}}\,\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} + \beta\,(\beta-1+\alpha)\left(\frac{2}{\theta}\,\frac{1}{\bar{x}}\right)^2 \bar{w} \mp \lambda^2\bar{w} = 0\,.$$

Setting the value of the first coefficient

$$\left(2\beta + \frac{\alpha}{2}\right)\frac{2}{\theta} = 1\,,$$

we find: a) the exponent in the transformation (5.4)

$$\beta = \frac{1-\alpha}{2}\,,$$

b) the value of the second coefficient in the above ordinary differential equation

$$\beta\,(\beta-1+\alpha)\left(\frac{2}{\theta}\right)^2 = -\beta^2\left(\frac{2}{\theta}\right)^2 = -\left(\frac{1-\alpha}{\theta}\right)^2 \equiv -\varrho^2,$$

and c) the resulting 3-parameter $(\varrho(\alpha), \lambda, \mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} + \frac{1}{\bar{x}}\,\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} - \frac{\varrho^2}{\bar{x}^2}\,\bar{w} \mp \lambda^2\bar{w} = 0\,. \tag{5.7}$$

I) Assuming that $\lambda > 0$ and introducing a new couple of the independent and the dependent variables once again: $s = \lambda\bar{x}$, $W(s;\alpha) := \bar{w}(\lambda^{-1}s;\alpha)$, we eventually obtain a 2-parameter $(\varrho(\alpha), \mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\varrho^2}{s^2} \pm 1\right) W = 0\,.$$

The choice of the lower sign in the obtained family leads to the 1-parameter $(\varrho(\alpha))$ Bessel equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\varrho^2}{s^2} - 1\right) W = 0\,, \tag{5.8}$$

whereas the choice of the upper sign leads to the 1-parameter $(\beta(\alpha))$ modified Bessel equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\varrho^2}{s^2} + 1\right) W = 0\,. \tag{5.9}$$

i) A 3-parameter family of solutions of the ordinary differential equation (5.8), when $\varrho \notin \mathbb{Z}$ (i. e. $\alpha \in (0,1) \bigcup (1,2)$) , is known [9, 19] to be

$$W(s;\alpha) = A_1\,\mathtt{J}_{-\varrho}(s) + A_2\,\mathtt{J}_{+\varrho}(s)\,, \tag{5.10}$$

where $A_{1,2}$ are arbitrary constants (parameters), and $\mathtt{J}_{\mp\varrho}(s)$ are the Bessel functions of the first kind of orders $\mp\varrho$

$$\mathtt{J}_{\mp\varrho}(s) = \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{\mu!\,\Gamma(\mu \mp \varrho + 1)} \left(\frac{s}{2}\right)^{2\mu\mp\varrho} = \left(\frac{s}{2}\right)^{\mp\varrho} \sum_{\mu=0}^{\infty} A_{\mp\varrho,\,\mu} \left(\frac{s}{2}\right)^{2\mu}. \tag{5.11}$$

whereas in case $\varrho \in \mathbb{Z}$ (i. e. $\alpha = 1$) a 2-parameter family of solutions of the ordinary differential equation (5.8) is known [9, 19] to be

$$W(s;\alpha) = B_1\,\mathtt{J}_0(s) + B_2\,\mathtt{N}_0(s)\,, \tag{5.12}$$

where $B_{1,2}$ are arbitrary constants (parameters), $\mathtt{J}_0(s)$ and $\mathtt{N}_0(s)$ are respectively the Bessel and the Neumann functions of the first kind, both of order zero

$$\mathtt{J}_0(s) = \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{(\mu!)^2} \left(\frac{s}{2}\right)^{2\mu} = \sum_{\mu=0}^{\infty} A_{0,\,\mu} \left(\frac{s}{2}\right)^{2\mu}, \tag{5.13}$$

$$\mathtt{N}_0(s) = \frac{2}{\pi}\left(C + \mathtt{ln}\,\frac{s}{2}\right)\mathtt{J}_0(s) - \frac{1}{\pi}\sum_{\mu=0}^{-1} \frac{(-\mu-1)!}{\mu!} \left(\frac{s}{2}\right)^{2\mu} - \frac{2}{\pi}\sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}\,\Phi(\mu)}{(\mu!)^2} \left(\frac{s}{2}\right)^{2\mu},$$

where $C = 0.5772\ldots$ is the Euler constant and $\Phi(\mu) = \sum_{\rho=1}^{\mu} \frac{1}{\rho}$, $\Phi(0) = 0$.

ii) A 3-parameter family of solutions of the ordinary differential equation (5.9), when $\varrho \notin \mathbb{Z}$ (i. e. $\alpha \in (0,1) \bigcup (1,2)$), is known [9,19] to be

$$W(s;\alpha) = C_1 \mathtt{I}_{-\varrho}(s) + C_2\, \mathtt{I}_{+\varrho}(s)\,, \tag{5.14}$$

where $C_{1,2}$ are arbitrary constants (parameters), and $\mathtt{I}_{\mp\varrho}(s)$ are the modified Bessel functions of the first kind of orders $\mp\varrho$

$$\mathtt{I}_{\mp\varrho}(s) = \sum_{\mu=0}^{\infty} \frac{1}{\mu!\,\Gamma(\mu \mp \varrho + 1)} \left(\frac{s}{2}\right)^{2\mu\mp\varrho} = \left(\frac{s}{2}\right)^{\mp\varrho} \sum_{\mu=0}^{\infty} B_{\mp\varrho,\,\mu} \left(\frac{s}{2}\right)^{2\mu}, \tag{5.15}$$

whereas in case $\varrho \in \mathbb{Z}$ (i. e. $\alpha = 1$) a 2-parameter family of solutions of the ordinary differential equation (5.9) is known [9,19] to be

$$W(s;\alpha) = D_1 \mathtt{I}_0(s) + D_2\, \mathtt{K}_0(s)\,, \tag{5.16}$$

where $D_{1,2}$ are arbitrary constants (parameters), $\mathtt{I}_0(s)$ and $\mathtt{K}_0(s)$ are respectively the modified Bessel function of the first kind and the modified Bessel function of the second kind, both of order zero

$$\mathtt{I}_0(s) = \sum_{\mu=0}^{\infty} \frac{1}{(\mu!)^2} \left(\frac{s}{2}\right)^{2\mu} = \sum_{\mu=0}^{\infty} B_{0,\,\mu} \left(\frac{s}{2}\right)^{2\mu}, \tag{5.17}$$

i) Tracing backwards all the transformations of the independent and the dependent variables we find from (5.10), (5.12) the following family of solutions of the second equation of the system (5.3)

$$X(x;\alpha) = \begin{cases} x^{\frac{1-\alpha}{2}} \left[A_1\, \mathtt{J}_{-\varrho}\left(\lambda\Omega\, x^{\frac{\theta}{2}}\right) + A_2\, \mathtt{J}_{+\varrho}\left(\lambda\Omega\, x^{\frac{\theta}{2}}\right) \right], & \alpha \in (0,1) \bigcup (1,2)\,, \\ B_1\, \mathtt{J}_0 \left(\lambda\Omega\, x^{\frac{\theta}{2}}\right) + B_2\, \mathtt{N}_0 \left(\lambda\Omega\, x^{\frac{\theta}{2}}\right), & \alpha = 1\,. \end{cases}$$

Substituting the respective power series instead of $\mathtt{J}_{\mp\varrho}$, $\mathtt{J}_0$, $\mathtt{N}_0$ and retaining only the bounded terms we obtain

$$X(x;\alpha) = \begin{cases} \underbrace{\overbrace{A_1\, \Lambda^{-\frac{1-\alpha}{\theta}} \sum_{\mu=0}^{\infty} A_{-\varrho,\mu}\, \Lambda^{2\mu}\, x^{\mu\theta}}^{\alpha\in(0,2)} + x^{1-\alpha}\, A_2\, \Lambda^{+\frac{1-\alpha}{\theta}} \sum_{\mu=0}^{\infty} A_{+\varrho,\mu}\, \Lambda^{2\mu}\, x^{\mu\theta}}_{\alpha\in(0,1)}, \\ B_1 \sum_{\mu=0}^{\infty} A_{0,\mu}\, \Lambda^{2\mu}\, x^{\mu\theta}, \qquad \alpha = 1\,, \end{cases}$$

where $2\Lambda = \lambda\,\Omega$. Then, performing the even extension of the above function $X(x;\alpha)$ to $x<0$, leading to the continuous and continuously differentiable flux, similarly to Section 4, we obtain the following required composite solution of the second equation of the system (5.3)

$$X(x;\alpha) = \sum_{\mu=0}^{\infty} A_{-\varrho,\mu}\,\Lambda^{2\mu}\,|x|^{\mu\theta}, \tag{5.18}$$

where the coefficients $A_{-\varrho,\mu}$ are taken from the series (5.11), if $\alpha\in(0,1)$ ($\varrho>0$) and $\alpha\in(1,2)$ ($\varrho<0$), and are taken from the series (5.13), if $\alpha=1$ ($\varrho=0$).

ii) Tracing backwards all the transformations of the independent and the dependent variables we find from (5.14), (5.16) the following family of solutions of the second equation of the system (5.3)

$$X(x;\alpha) = \begin{cases} x^{\frac{1-\alpha}{2}}\left[C_1\,\mathtt{I}_{+\varrho}\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right) + C_2\,\mathtt{I}_{-\varrho}\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right)\right], & \alpha\in(0,1)\bigcup(1,2)\,, \\ D_1\,\mathtt{I}_0\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right) + D_2\,\mathtt{K}_0\left(\lambda\Omega\,x^{\frac{\theta}{2}}\right), & \alpha=1\,. \end{cases}$$

Substituting the respective power series instead of $\mathtt{I}_{\mp\varrho}$, $\mathtt{I}_0$, $\mathtt{K}_0$ and retaining only the bounded terms we obtain

$$X(x;\alpha) = \begin{cases} \underbrace{\overbrace{C_1\,\Lambda^{-\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty} B_{-\varrho,\mu}\,\Lambda^{2\mu}\,x^{\mu\theta}}^{\alpha\in(0,2)} + x^{1-\alpha}\,C_2\,\Lambda^{+\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty} B_{+\varrho,\mu}\,\Lambda^{2\mu}\,x^{\mu\theta}}_{\alpha\in(0,1)}, \\ D_1\sum_{\mu=0}^{\infty} B_{0,\mu}\,\Lambda^{2\mu}\,x^{\mu\theta}, \qquad \alpha=1\,, \end{cases}$$

Performing the even extension of the above function $X(x;\alpha)$ to $x<0$ and doing as in case of the solution (5.10), (5.12), we obtain the following required composite solution of the second equation of the system (5.3)

$$X(x;\alpha) = \sum_{\mu=0}^{\infty} B_{-\varrho,\mu}\,\Lambda^{2\mu}|x|^{\mu\theta}, \tag{5.19}$$

where the coefficients $B_{-\varrho,\mu}$ are taken from the series (5.15), if $\alpha\in(0,1)$ ($\varrho>0$) and $\alpha\in(1,2)$ ($\varrho<0$), and are taken from the series (5.17), if $\alpha=1$ ($\varrho=0$).

II) Assuming that $\lambda=0$, we obtain directly from (5.7) a 1-parameter ($\varrho(\alpha)$) family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} + \frac{1}{\bar{x}}\,\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} - \frac{\varrho^2}{\bar{x}^2}\,\bar{w} = 0\,. \tag{5.20}$$

A 3-parameter family of solutions of equation (5.20) reads

$$\bar{w}\,(\bar{x};\alpha) = E_1\,\bar{x}^{-\varrho} + E_2\,\bar{x}^{+\varrho},$$

where $E_{1,2}$ are arbitrary constants. Applying the transformation leading from $(\bar{x},\bar{w})$ to (x,w) backwards we obtain

$$w\,(x;\alpha) = E_1\Big(\Omega\,x^{\frac{\theta}{2}}\Big)^{-\varrho} + E_2\Big(\Omega\,x^{\frac{\theta}{2}}\Big)^{+\varrho} = \ldots = \bar{E}_1\,x^{-\frac{1-\alpha}{2}} + \bar{E}_2\,x^{+\frac{1-\alpha}{2}}.$$

Eventually, applying the transformation (5.4) yields to the solution of the second equation of the system (5.3)

$$X(x;\alpha) = x^{+\frac{1-\alpha}{2}}\left(\bar{E}_1\,x^{-\frac{1-\alpha}{2}} + \bar{E}_2\,x^{+\frac{1-\alpha}{2}}\right) = \bar{E}_1 + \bar{E}_2\,x^{1-\alpha}.$$

The evident extensions of the above function to $x<0$

$$X(x;\alpha) = \begin{cases} \bar{E}_1 + \bar{E}_2\,|x|^{1-\alpha}, \\ \bar{E}_1 + \bar{E}_2\ \ x^{\,1-\alpha}, \end{cases} \tag{5.21}$$

lead, as it is known from Section 4, to the discontinuous flux f, therefore no function (5.21) must be accounted for in the representation (5.2).

Now we turn out to the first ordinary differential equation of the system (5.3); its 2-parameter $((\lambda,\mp),\ \lambda>0)$ family of solutions is known to be

$$O(t;\alpha) = \begin{cases} F_1\,\mathtt{exp}\,(-\lambda t) + F_2\,\mathtt{exp}\,(+\lambda t)\,, \\ G_1\,\mathtt{cos}\,(+\lambda t) + G_2\,\mathtt{sin}\,(+\lambda t)\,, \end{cases} \tag{5.22}$$

where $F_{1,2}$, $G_{1,2}$ are arbitrary constants.

This completes finding solutions of the original wave equation using separation of variables, but we especially note that the spatial parts $X(x;\alpha)$ (5.18), (5.19) of the solutions (5.2) obtained in this Section includes the same terms $|x|^{\,\mu\theta}$, $\mu\in\mathbb{Z}_+$, as those present in the required series solution (4.3).

6. Reformulation of the original problem

The power law (1.2) in the coefficient function $a(x;\alpha)$ produces the degeneracy of the wave equation (1.1). We implement stretching of the spatial independent variable x leading to 'inflation' of the degeneracy. For this we introduce a transformation of the independent variables $(t,x)\to(\tau,\xi)$ using the following system of the first order differential equations

$$\begin{cases} \dfrac{\mathrm{d}\tau}{\mathrm{d}t} = 1\,, \\[2mm] \dfrac{\mathrm{d}\xi}{\mathrm{d}x} = \dfrac{1}{\sqrt{a(x;\alpha)}}\,, \end{cases} \tag{6.1}$$

supplemented with the evident boundary conditions: $\tau = 0$ when $t = 0$ and $\xi = 0$ when $x = 0$. The only non-trivial solution of the Cauchy problem gives the desired 1-parameter transformation of the independent variables

$$\begin{cases} \tau = \vartheta(t)\,, \\ \xi = \psi(x;\alpha)\,, \end{cases} \tag{6.2}$$

where $\vartheta(t) = t$,

$$\psi(x;\alpha) = \mathtt{sign}\,(x) \begin{cases} \Omega\,|x|^{\frac{\theta}{2}}, & 0 \leqslant |x| \leqslant c\,, \\ \xi_c + (|x| - c)\,, & c < |x| \leqslant 1\,, \end{cases} \tag{6.3}$$

and $\xi_c = \psi(c;\alpha)$. The function (6.3) is monotonic differentiable (with the only exception for the point $x=0$, Fig. 6.1), hence the transformation (6.4) is uniquely invertible

$$\begin{cases} t = \bar{\vartheta}(\tau)\,, \\ x = \phi(\xi;\alpha)\,, \end{cases} \tag{6.4}$$

where $\bar{\vartheta}(\tau) = \vartheta^{-1}(\tau) = \tau$,

$$\phi(\xi;\alpha) = \mathtt{sign}\,(\xi) \begin{cases} \left(\Omega^{-1}|\xi|\right)^{\frac{2}{\theta}}\,, & 0 \leqslant |\xi| \leqslant \xi_c\,, \\ c + (|\xi| - \xi_c)\,, & \xi_c < |\xi| \leqslant \xi_l\,, \end{cases} \tag{6.5}$$

$\xi_l = \psi(1;\alpha)$, and Ω is given by (5.6).

The transformation formulas (6.2), (6.4) are equivalent to the following operator identities

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \tau^2}\,, \quad \frac{\partial}{\partial x} = \left(\frac{1}{\sqrt{a}}\right)_{x\to\xi} \frac{\partial}{\partial \xi}\,, \quad \frac{\partial^2}{\partial x^2} = \left(\frac{1}{a}\right)_{x\to\xi} \frac{\partial^2}{\partial \xi^2} - \frac{1}{2}\left(\frac{a'}{a\sqrt{a}}\right)_{x\to\xi} \frac{\partial}{\partial \xi}\,,$$

yielding to the transformed wave equation

$$\frac{\partial^2 \upsilon(\tau,\xi;\alpha)}{\partial \tau^2} - \frac{\partial^2 \upsilon(\tau,\xi;\alpha)}{\partial \xi^2} = g(\xi;\alpha)\,\frac{\partial \upsilon(\tau,\xi;\alpha)}{\partial \xi}\,, \tag{6.6}$$

where $\upsilon(\tau,\xi;\alpha) := u\left(\vartheta(\tau), \phi(\xi;\alpha);\alpha\right)$, $g(\xi;\alpha) := \left(\sqrt{a(x;\alpha)}\right)'_{x\to\xi}$.

There are two ways to rewrite the transformed wave equation purely in the variables (τ,ξ). The first one is straightforward and implies two steps. The first step needs obtaining an explicit dependence of the coefficient function $g(\xi;\alpha)$ in (6.6) on the variable x

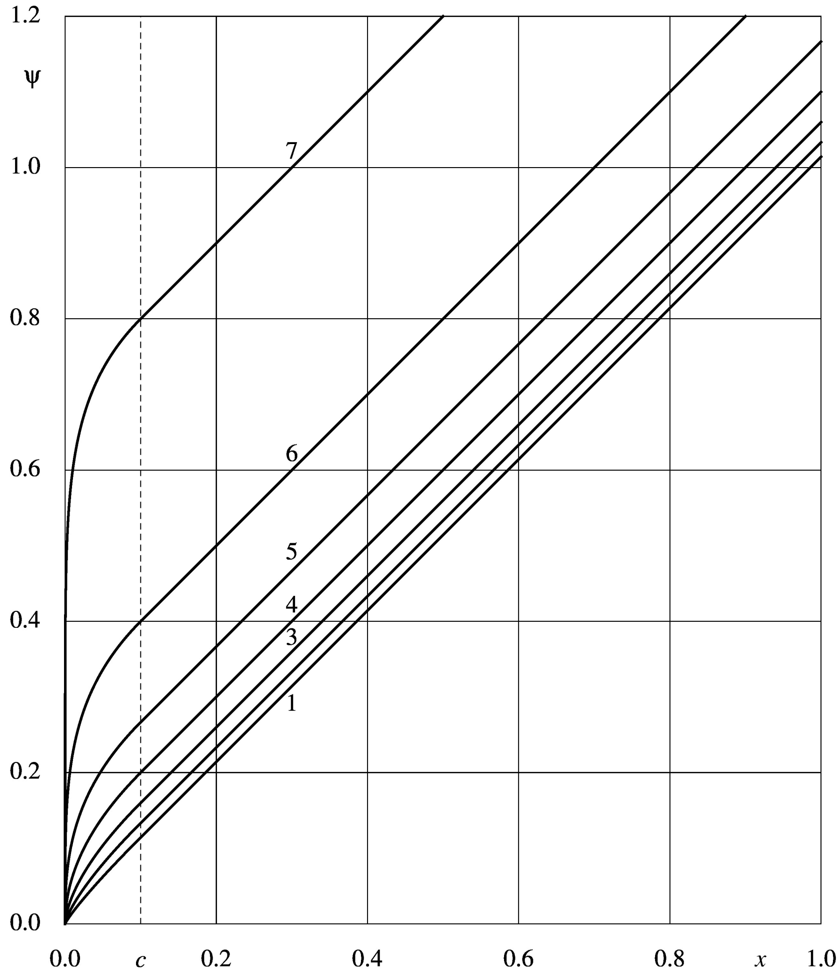

Fig. 6.1. The function $\psi(x;\alpha)$ (6.3) stretches the variable x near the degeneracy segment and 'inflates' the degeneracy of the original wave equation (1.1), (5.1): bold solid curves $1-7$ are drawn for $\alpha=0.25\,(0.25)\,1.75$ respectively ($c=0.1,\ x\geqslant 0$); the thin dashed line shows the right boundary $x=c$ of the segment $[-c,+c]$ in which the power law $a(x;\alpha)=a_*|x|^\alpha$ (1.2) of the degeneracy holds

$$\left(\sqrt{a(x;\alpha)}\right)'\equiv\frac{1}{2}\,\frac{a'(x;\alpha)}{\sqrt{a(x;\alpha)}}=\mathtt{sign}\,(x)\begin{cases}\dfrac{1}{2}\,\dfrac{a_*\alpha\,|x|^{\alpha-1}}{\sqrt{a_*}\,|x|^{\frac{\alpha}{2}}}=\dfrac{\alpha}{\theta}\left(\Omega\,|x|^{\frac{\theta}{2}}\right)^{-1}, & 0<|x|\leqslant c\,,\\[2ex] 0\,, & c<|x|\leqslant 1\,,\end{cases}$$

whereas the second step needs replacing the variable x with formula (6.5) and yields to the desired composite expression

$$
g(\xi;\alpha) \equiv \left(\sqrt{a(x;\alpha)}\right)'_{x\to\xi} = \begin{cases} \dfrac{\alpha}{\theta}\,\dfrac{1}{\xi}\,, & 0\ < |\xi| \leqslant \xi_c\,, \\ 0\,, & \xi_c < |\xi| \leqslant \xi_l\,, \end{cases} \tag{6.7}
$$

where the hyperbolic and identically equaled to zero branches are not continuously matched as shown in Fig. 6.2.

The second way implies two steps to be done as well. The first step needs implementing the transformation of the variable x into the variable ξ

$$
\frac{1}{2}\left(\frac{a'(x;\alpha)}{\sqrt{a(x;\alpha)}}\right)_{x\to\xi} = \frac{1}{2}\,\frac{1}{\sqrt{a(\phi(\xi;\alpha);\alpha)}}\,\frac{\mathrm{d}a(\phi(\xi;\alpha);\alpha)}{\mathrm{d}\xi}\,\underbrace{\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)_{x\to\xi}}_{\frac{1}{\sqrt{a(\phi(\xi;\alpha);\alpha)}}} = \frac{1}{2}\,\frac{b'(\xi;\alpha)}{b(\xi;\alpha)}\,,
$$

where $b(\xi;\alpha)$ is the coefficient function of the original wave equation expressed purely through the variable ξ

$$
b(\xi;\alpha) \equiv a(\phi(\xi;\alpha);\alpha) = \begin{cases} a_*\left(\Omega^{-1}|\xi|\right)^{\frac{2\alpha}{\theta}}\,, & 0\ < |\xi| \leqslant \xi_c\,, \\ 1\,, & \xi_c < |\xi| \leqslant \xi_l\,, \end{cases} \tag{6.8}
$$

whereas the second step involves ‘deciphering’ the coefficient function of the only first derivative of the transformed wave equation in terms of the variable ξ and yields exactly to the previously obtained composite expression

$$
\frac{1}{2}\,\frac{b'(\xi;\alpha)}{b(\xi;\alpha)} = g(\xi;\alpha) \overset{(6.7)}{\equiv} \left(\sqrt{a(x;\alpha)}\right)'_{x\to\xi}\,.
$$

Finally, we obtain the following transformed formulation of the original initial boundary value problem (1.4)

$$
\begin{cases} \dfrac{\partial^2 \upsilon(\xi;\alpha)}{\partial\tau^2} = \dfrac{\partial^2 \upsilon(\xi;\alpha)}{\partial\xi^2} + g(\xi;\alpha)\dfrac{\partial\upsilon(\xi;\alpha)}{\partial\xi}\,, & (\tau,\xi)\in(0,T]\times(-\xi_l,+\xi_l), \\ \upsilon(\tau,-\xi_l;\alpha) = 0\,, & \tau\in[0,T]\,, \\ \upsilon(\tau,+\xi_l;\alpha) = h(\tau)\,, & \tau\in[0,T]\,, \\ \upsilon(0,\xi;\alpha) = 0\,, & \xi\in[-\xi_l,+\xi_l]\,, \\ \dfrac{\partial\upsilon(0,\xi;\alpha)}{\partial\tau} = 0\,, & \xi\in[-\xi_l,+\xi_l)\,, \end{cases} \tag{6.9}
$$

referred to as the transformed initial boundary value problem.

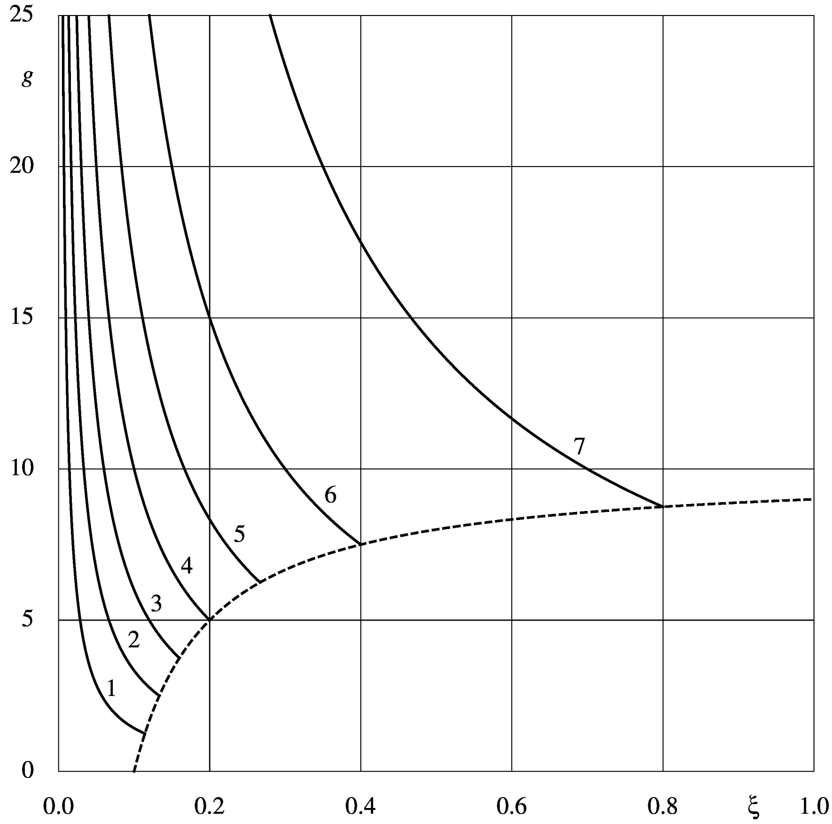

Fig. 6.2. The piece-wise continuous and differentiable coefficient function $g(\xi;\alpha)$ (6.7) of the transformed wave equation: only hyperbolic branches are drawn with bold solid curves $1-7$ for $\alpha = 0.25\,(0.25)\,1.75$ respectively; the bold dashed line joins the right ends of the hyperbolic branches

7. Series solutions of the transformed wave equation

There are two ways, to find one-sided series solution of the transformed wave equation (6.6). The first one uses the transformation of the independent variables (6.2), (6.4), applied to the pairs of the one-sided series solutions (3.3), (3.5), and (3.9) of the original wave equation (1.1), whereas the second one uses the procedures of Sections 3, but as applied to the transformed wave equation.

I) Implementing the first way is straightforward to replace the independent variables $(t,x)\to(\tau,\xi)$, where $t=\bar{\vartheta}(\tau)\equiv\tau$, whereas $|x|, x$ are respectively replaced as

$$\begin{cases} |x| = \left(\Omega^{-1}|\xi|\right)^{\frac{2}{\theta}}, & \alpha \in (0,2)\,, \\ \;x\; = \left(\Omega^{-1}\,\xi\,\right)^{\frac{2}{\theta}}, & \alpha \in (0,2)_o\,. \end{cases} \tag{7.1}$$

Initially, consider case A) of Section 3.

i) Let both pairs of the one-sided series solutions (3.3) be given, then using the independent variables substitution yields to the only pair

$$\upsilon^{\mp}(\tau,\xi;\alpha) \equiv u^{\mp}\big(\bar{\vartheta}(\tau),\phi(\xi;\alpha);\alpha\big) = \sum_{\mu=0}^{\infty} V^{\mp}_{\alpha,\mu}(\tau)\,\xi^{2\mu}, \tag{7.2}$$

where

$$V^{\mp}_{\alpha,\mu}(\tau) = \left(\frac{\theta}{2}\right)^{2\mu} a_*^{\mu}\, U^{\mp}_{\alpha,\mu}\left(\bar{\vartheta}(\tau)\right), \qquad \mu \in \mathbb{Z}_+\,, \tag{7.3}$$

and the following pair of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 V^{\mp}_{\alpha,\mu-1}(\tau)}{\mathrm{d}\tau^2} = 4\mu\frac{(\mu-1)\,\theta+1}{\theta}\,V^{\mp}_{\alpha,\mu}(\tau)\,, \qquad \mu \in \mathbb{N}\,. \tag{7.4}$$

ii) Let both pairs of the one-sided binomial solutions (3.5) be given, then using the independent variables substitution yields to the following two pairs

$$\upsilon^{\mp}(\tau,\xi;\alpha) = \begin{cases} V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,|\xi|^{(1-\alpha)\frac{2}{\theta}}, & \alpha \in (0,+2)\,, \\[2ex] V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\ \ \xi^{\ (1-\alpha)\frac{2}{\theta}}, & \alpha \in (0,+2)_o, \end{cases} \tag{7.5}$$

where functions $V^{\mp}_{\alpha,0}(\tau)$ and $V^{\mp}_{\alpha,1}(\tau)$ are linear and

$$V^{\mp}_{\alpha,0}(\tau) = U^{\mp}_{\alpha,0}\left(\bar{\vartheta}(\tau)\right), \qquad V^{\mp}_{\alpha,1}(\tau) = \left[\left(\frac{\theta}{2}\right)^2 a_*\right]^{\frac{1-\alpha}{\theta}} U^{\mp}_{\alpha,1}\left(\bar{\vartheta}(\tau)\right). \tag{7.6}$$

iii) Let both pairs of the one-sided series solutions (3.6) be given, then using the independent variables substitution yields to the following two pairs

$$\upsilon^{\mp}(\tau,\xi;\alpha) = \begin{cases} V^{\mp}_{\alpha,0}(\tau) + |\xi|^{(1-\alpha)\frac{2}{\theta}} \displaystyle\sum_{\mu=1}^{\infty} V^{\mp}_{\alpha,\mu}(\tau)\,\xi^{2(\mu-1)}, & \alpha \in (0,+2)\,, \\[2ex] V^{\mp}_{\alpha,0}(\tau) +\ \ \xi^{\ (1-\alpha)\frac{2}{\theta}} \displaystyle\sum_{\mu=1}^{\infty} V^{\mp}_{\alpha,\mu}(\tau)\,\xi^{2(\mu-1)}, & \alpha \in (0,+2)_o, \end{cases} \tag{7.7}$$

where

$$V^{\mp}_{\alpha,\mu}(\tau) = \left[\left(\frac{\theta}{2}\right)^2 a_*\right]^{\frac{\theta\mu-1}{\theta}} U^{\mp}_{\alpha,\mu}\big(\bar{\vartheta}(\tau)\big)\,, \qquad \mu \in \mathbb{N}\,, \tag{7.8}$$

and the following pair of the recurrence relations

$$\frac{\mathrm{d}^2 V^{\mp}_{\alpha,\mu}(\tau)}{\mathrm{d}\tau^2} = 4\mu\,\frac{\mu\theta-1}{\theta}\,V^{\mp}_{\alpha,\mu+1}(\tau)\,, \qquad \mu \in \mathbb{N}\,. \tag{7.9}$$

Considering three items of case B) of Section 3 is performed exactly in the same way, therefore we omit case B) and complete implementing the first way.

II) Now we turn out to implementing the second way.

A) Let trial one-sided series solutions of the transformed wave equation (6.6) be of the form

$$\upsilon^{\mp}(\tau,\xi;\alpha) = V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,1}} + V^{\mp}_{\alpha,2}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,2}} + \dots$$

where all undetermined functions and quantities have the same meaning, as in Section 3, then

$$\begin{aligned}
\frac{\partial^2\upsilon^{\mp}}{\partial\tau^2} &= V''_{\alpha,0} + V''_{\alpha,1}\,|\xi|^{\sigma_1} + V''_{\alpha,2}\,|\xi|^{\sigma_2} + \dots,\\
\frac{1}{\xi}\,\frac{\partial\upsilon^{\mp}}{\partial\xi} &= \mp\Big(\quad \sigma_1\,V_{\alpha,1}\,|\xi|^{\sigma_1-2} + \quad \sigma_2\,V_{\alpha,2}\,|\xi|^{\sigma_2-2} + \dots\Big),\\
\frac{\partial^2\upsilon^{\mp}}{\partial\xi^2} &= \quad \omega_1\,\sigma_1\,V_{\alpha,1}\,|\xi|^{\sigma_1-2} + \omega_2\,\sigma_2\,V_{\alpha,2}\,|\xi|^{\sigma_2-2} + \dots \quad ,
\end{aligned}$$

and substituting the above series instead of the respective terms of the transformed wave equation yields to the following pair of the one-sided series identities

$$\underbrace{V''_{\alpha,0}}_{1} + \underbrace{V''_{\alpha,1}\,|\xi|^{\sigma_1}}_{2} + V''_{\alpha,2}\,|\xi|^{\sigma_2} + \dots = \underbrace{\sigma_1\,o_1\,V_{\alpha,1}\,|\xi|^{\sigma_1-2}}_{1} + \underbrace{\sigma_2\,o_2\,V_{\alpha,2}\,|\xi|^{\sigma_2-2}}_{2} + \dots, \tag{7.10}$$

where $\omega_1=\sigma_1-1$, $\omega_2=\sigma_2-1$, etc., $o_1=\omega_1+\theta^{-1}\alpha$, $o_2=\sigma_2+\theta^{-1}\alpha$, etc.

i) Assuming that in (7.10) the terms with the same numbers have the same powers in the variable $|\xi|$, we obtain the only pair of one-sided series solutions (7.2) and the pair of the one-sided recurrence relations (7.4) again.

ii) Assuming that in (7.10) the terms must vanish separately, we obtain two pairs of the binomial solutions (7.5) again.

iii) Assuming that in (7.10) both terms 1 vanish separately, we obtain two pairs of the one-sided series solutions (7.7) and the pair of the one-sided recurrence relations (3.7) again.

B) Let trial one-sided series solutions of the transformed wave equation (6.6) be of the form

$$\upsilon^{\mp}(\tau,\xi;\alpha) = V^{\mp}_{\alpha}(\tau) + |\xi|^{\omega^{\mp}_{\alpha}}\left(V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,1}} + V^{\mp}_{\alpha,2}(\tau)\,|\xi|^{\sigma^{\mp}_{\alpha,2}} + \dots\right),$$

then

$$\begin{aligned}
\frac{\partial^2 \upsilon^{\mp}}{\partial \tau^2} &= V''_\alpha + |\xi|^\omega \Big(V''_{\alpha,0} + V''_{\alpha,1}\,|\xi|^{\sigma_1} + V''_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots \Big),\\
\frac{1}{\xi}\,\frac{\partial \upsilon^{\mp}}{\partial \xi} &= \mp|\xi|^{\omega-2} \Big(\omega\, V_{\alpha,0} + \omega_1\, V_{\alpha,1}\,|\xi|^{\sigma_1} + \omega_2\, V_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots \Big),\\
\frac{\partial^2 \upsilon^{\mp}}{\partial \xi^2} &= \quad |\xi|^{\omega-2} \Big(\omega\,(\omega-1)\, V_{\alpha,0} + \omega_1\,(\omega-1+\sigma_1)\, V_{\alpha,1}\,|\xi|^{\sigma_1} \\
&\qquad\qquad + \omega_2\,(\omega-1+\sigma_2)\, V_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots \Big),
\end{aligned}$$

and substituting the above series instead of the respective terms of the transformed wave equation we obtain the following pair of the one-sided series identities

$$\begin{aligned}
&\underbrace{V''_\alpha}_{1} + |\xi|^\omega \Big(\underbrace{V''_{\alpha,0}}_{2} + \underbrace{V''_{\alpha,1}\,|\xi|^{\sigma_1}}_{3} + V''_{\alpha,2}\,|\xi|^{\sigma_2} + \ldots \Big)\\
&= |\xi|^{\omega-2} \Big(\underbrace{\omega\, o\, V_{\alpha,0}}_{1} + \underbrace{\omega_1\,(o+\sigma_1)\, V_{\alpha,1}\,|\xi|^{\sigma_1}}_{2} + \underbrace{\omega_2\,(o+\sigma_2)\, V_{\alpha,2}\,|\xi|^{\sigma_2}}_{3} + \ldots \Big),
\end{aligned} \tag{7.11}$$

where $o = \omega - 1 + \theta^{-1}\alpha$, $\omega_1 = \omega + \sigma_1$, $\omega_2 = \omega + \sigma_2$, etc.

i) Let the terms in (7.11) with the same numbers be of the same powers, then we obtain: 1) the explicit expressions for the exponents: $\omega^{\mp}_\alpha = 2$, $\sigma^{\mp}_{\alpha,\mu} = 2\mu$, leading to the only pair of the one-sided series solutions

$$\upsilon^{\mp}(\tau, \xi; \alpha) = V^{\mp}_\alpha(\tau) + V^{\mp}_{\alpha,0}(\tau)\,\xi^2 + V^{\mp}_{\alpha,1}(\tau)\,\xi^4 + \ldots;$$

and 2) the pair of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 V^{\mp}_\alpha(\tau)}{\mathrm{d}\tau^2} = \frac{4}{\theta}\, V^{\mp}_{\alpha,0}(\tau)\,, \qquad \frac{\mathrm{d}^2 V^{\mp}_{\alpha,\mu-1}(\tau)}{\mathrm{d}\tau^2} = 4(\mu+1)\,\frac{\mu\theta+1}{\theta}\, V^{\mp}_{\alpha,\mu}(\tau)\,, \qquad \mu \in \mathbb{N}\,.$$

It is evident, that we obtain nothing but the series solution (7.2), (7.4) again.

ii) Let the functions $V^{\mp}_\alpha(\tau)$, $V^{\mp}_{\alpha,0}(\tau)$ be linear, then $\theta\,\omega^{\mp}_\alpha = 2\,(1-\alpha)$, $\sigma^{\mp}_{\alpha,\mu} = 0$, $\mu \in \mathbb{N}$, and we obtain the pairs of the binomial solutions (7.5) again.

iii) Let terms 1 in (7.11) vanish separately, and the other terms with the same numbers be of the same powers, then we obtain: 1) that $V^{\mp}_\alpha(\tau)$ are linear functions; 2) the explicit expressions for the exponents: $\theta\,\omega^{\mp}_\alpha = 2\,(1-\alpha)$, $\sigma^{\mp}_{\alpha,\mu} = 2\mu$, leading to the following pairs of the one-sided series solutions

$$\begin{aligned}
\upsilon^{\mp}(\tau, \xi; \alpha) &= V^{\mp}_\alpha(\tau) + |\xi|^{(1-\alpha)\frac{2}{\theta}} \Big(V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,\xi^2 + V^{\mp}_{\alpha,2}(\tau)\,\xi^4 + \ldots \Big)\,,\\
\upsilon^{\mp}(\tau, \xi; \alpha) &= V^{\mp}_\alpha(\tau) + \ \xi^{\,(1-\alpha)\frac{2}{\theta}} \Big(V^{\mp}_{\alpha,0}(\tau) + V^{\mp}_{\alpha,1}(\tau)\,\xi^2 + V^{\mp}_{\alpha,2}(\tau)\,\xi^4 + \ldots \Big)\,,
\end{aligned} \tag{7.12}$$

valid respectively if $\alpha \in (0,2)$ and if $\alpha \in (0,2)_o$; 3) the following pair of the one-sided recurrence relations

$$\frac{\mathrm{d}^2 V^{\mp}_{\alpha,\mu-1}(\tau)}{\mathrm{d}\tau^2} = 4\mu\,\frac{(\mu+1)\,\theta-1}{\theta}\,V^{\mp}_{\alpha,\mu}(\tau)\,, \qquad \mu \in \mathbb{N}\,. \tag{7.13}$$

This completes implementing the second way of obtaining one-sided series solutions for the transformed wave equation.

Both ways have been proved to give identical one-sided series solutions.

8. Series based analysis of the transformed problem

There are two ways of finding the required continuous, or two-sided, series solutions of the transformed wave equation (6.6). The first way uses the transformation of the independent variables (6.2), (6.4), applied to the only required two-sided series solution (4.3) of the original wave equation (1.1), whereas the second one uses the procedure of continuous matching of one-sided series solutions, but as applied to the transformed wave equation, followed by introducing the proper definition of the required solution and keeping in mind property Z.

I) Implementing the first way immediately gives the following two-sided series solution

$$\upsilon(\tau,\xi;\alpha) = V_{\alpha,0}(\tau) + V_{\alpha,1}(\tau)\,\xi^2 + V_{\alpha,2}(\tau)\,\xi^4 + \ldots = \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, \tag{8.1}$$

where the coefficient functions obey the following recurrence relations

$$V''_{\alpha,\mu-1}(\tau) = 4\mu\,\frac{(\mu-1)\,\theta+1}{\theta}\,V_{\alpha,\mu}(\tau)\,, \qquad \mu \in \mathbb{N}\,. \tag{8.2}$$

II) Implementing the second way follows the three-step procedure of Section 4.

1) Implementing the continuous matching of the one-sided series solutions obtained in Section 7 gives the following two-sided bounded series solutions

$$\begin{aligned}
&\upsilon_1(\tau,\xi;\alpha) = \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, && \alpha \in (0,2)\,,\\
&\upsilon_3(\tau,\xi;\alpha) = V_{\alpha,0}(\tau) + V_{\alpha,1}(\tau)\,|\xi|^{(1-\alpha)\frac{2}{\theta}}, && \alpha \in (0,1)\,,\\
&\upsilon_4(\tau,\xi;\alpha) = V_{\alpha,0}(\tau) + V_{\alpha,1}(\tau)\;\xi^{\,(1-\alpha)\frac{2}{\theta}}, && \alpha \in (0,1)_o\,,\\
&\upsilon_5(\tau,\xi;\alpha) = V_{\alpha}(\tau) + |\xi|^{(1-\alpha)\frac{2}{\theta}} \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, && \alpha \in (0,1)\,,\\
&\upsilon_6(\tau,\xi;\alpha) = V_{\alpha}(\tau) + \;\xi^{\,(1-\alpha)\frac{2}{\theta}} \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu}, && \alpha \in (0,1)_o\,.
\end{aligned} \tag{8.3}$$

2) Implementing continuity of the flux needs, first of all, the definition of the flux, therefore we rewrite the transformed wave equation in a flux form

$$\frac{\partial \pi}{\partial \tau} + \frac{\partial \varphi}{\partial \xi} = \rho\,, \tag{8.4}$$

usually referred to as the balance law, where $\pi(\tau,\xi;\alpha) := \dfrac{\partial \upsilon}{\partial \tau}$,

$$\begin{cases} \varphi(\tau,\xi;\alpha) := -\dfrac{\partial \upsilon}{\partial \xi} - \dfrac{\alpha}{\theta}\,\dfrac{\upsilon}{\xi}\,, & \rho(\tau,\xi;\alpha) := \dfrac{\alpha}{\theta}\,\dfrac{\upsilon}{\xi^2}\,, \qquad 0 \leqslant |\xi| \leqslant \xi_c\,, \\ \varphi(\tau,\xi;\alpha) := -\dfrac{\partial \upsilon}{\partial \xi}\,, & \rho(\tau,\xi;\alpha) := 0\,, \qquad \xi_c < |\xi| \leqslant 1\,, \end{cases} \tag{8.5}$$

φ being the desired flux and ρ being the right side, or the source term.

Calculation of the flux and the right side of the balance law (8.4) for the series solutions 1, 3 − 6 shows that both are unbounded on the degeneracy segment, due to the leading terms $V_{\alpha,0}(\tau)$ or $V_\alpha(\tau)$, therefore we rewrite the balance law (8.4) by introducing regularization

$$\frac{\partial \pi}{\partial \tau} + \frac{\partial \mathring{\varphi}}{\partial \xi} = \mathring{\rho}\,, \tag{8.6}$$

where

$$\mathring{\varphi}(\tau,\xi;\alpha) := -\frac{\partial \upsilon}{\partial \xi} - \frac{\alpha}{\theta}\,\frac{\mathring{\upsilon}}{\xi}\,, \qquad \mathring{\rho}(\tau,\xi;\alpha) := \frac{\alpha}{\theta}\,\frac{\mathring{\upsilon}}{\xi^2}\,, \qquad 0 \leqslant |\xi| \leqslant \xi_c\,, \tag{8.7}$$

and $\mathring{\upsilon}(\tau,\xi;\alpha) := \upsilon(\tau,\xi;\alpha) - \upsilon(\tau,0;\alpha)$.

The above regularization yields to the following fluxes calculated on both sides of the degeneracy segment

$$\begin{aligned} -\mathring{\varphi}_1^{\mp}(\tau,\xi;\alpha) &= \xi \sum_{\mu=1}^{\infty} \Big(\frac{\alpha}{\theta} + 2\mu\Big)\, V_{\alpha,\mu}(\tau)\, \xi^{2\mu-2}, \\ -\mathring{\varphi}_3^{\mp}(\tau,\xi;\alpha) &= \mp V_{\alpha,1}(\tau)\, |\xi|^{-\frac{\alpha}{\theta}}, \\ -\mathring{\varphi}_4^{\mp}(\tau,\xi;\alpha) &= \quad V_{\alpha,1}(\tau)\;\; \xi^{\,-\frac{\alpha}{\theta}}, \\ -\mathring{\varphi}_5^{\mp}(\tau,\xi;\alpha) &= \mp|\xi|^{-\frac{\alpha}{\theta}} \sum_{\mu=0}^{\infty} (1+2\mu)\, V_{\alpha,\mu}(\tau)\, \xi^{2\mu}, \\ -\mathring{\varphi}_6^{\mp}(\tau,\xi;\alpha) &= \quad \xi^{\,-\frac{\alpha}{\theta}} \sum_{\mu=0}^{\infty} (1+2\mu)\, V_{\alpha,\mu}(\tau)\, \xi^{2\mu}, \end{aligned} \tag{8.8}$$

and we immediately find the flux $\mathring{\varphi}_1(\tau,\xi;\alpha)$ for the series solution $\mathring{\upsilon}_1(\tau,\xi;\alpha)$ to be the only continuous and even continuously differentiable on the degeneracy segment, whereas all other fluxes turn out to be unbounded.

We gather below the series solution $\upsilon_1(\tau,\xi;\alpha)$ and the derived source term and the flux

$$\begin{aligned}
\upsilon(\tau,\xi;\alpha) &= \sum_{\mu=0}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu},\\
\mathring{\rho}(\tau,\xi;\alpha) &= \sum_{\mu=1}^{\infty} V_{\alpha,\mu}(\tau)\,\xi^{2\mu-2},\\
-\mathring{\varphi}(\tau,\xi;\alpha) &= \sum_{\mu=1}^{\infty}\left(\frac{\alpha}{\theta}+2\mu\right) V_{\alpha,\mu}(\tau)\,\xi^{2\mu-1},
\end{aligned} \tag{8.9}$$

to introduce the following

Definition 8.1. A function $\upsilon(\tau,\xi;\alpha)$ is called the solution with the continuously differentiable flux (or shortly, the required solution) to the initial boundary value problem (6.9) if inside the space-time rectangle $[0,T]\times[-\xi_l,+\xi_l]$ it: 1) is continuous and twice continuously differentiable in the variables τ and ξ; 2) satisfies the degenerate wave equation; and 3) satisfies the initial and boundary conditions of the problem.

Hence, the only required series solution of the transformed wave equation (introduced by Definition 8.1, in which the last item is removed) out of those obtained in Section 7 and presented in (8.3) is that given by (8.9). Note, that continuity of: 1) the right side of the transformed wave equation and 2) the flux $\mathring{\varphi}$ and the source term $\mathring{\rho}$ of the regularized balance law (8.6) are evident implications of Definition 8.1.

Proposition 8.1. *Possessing property Z is not necessary for the required solution to the 1-parameter initial boundary value problem* (6.9) *if* $\alpha\in(0,2)$.

Proof. Essential part of the proof is a repetition of that for Proposition 4.1 at p. 18. □

9. Separation of variables applied to the transformed equation

Assume that the trial solution of the transformed wave equation is of the form

$$\upsilon(\tau,\xi;\alpha) = \Theta\,(\tau;\alpha)\;\Xi\,(\xi;\alpha)\,, \tag{9.1}$$

where functions $\Theta\,(\tau;\alpha)$ and $\Xi\,(\xi;\alpha)$ are to be determined. Then, substituting the ansatz (9.1) into the transformed wave equation (6.6), (6.7) $(0<|\xi|\leqslant\xi_c)$

$$\frac{\partial^2\upsilon}{\partial\tau^2}-\frac{\partial^2\upsilon}{\partial\xi^2}=\frac{\alpha}{\theta}\,\frac{1}{\xi}\,\frac{\partial\upsilon}{\partial\xi}\,,$$

we find the above equation rewritten as

$$\frac{\Theta''(\tau;\alpha)}{\Theta\,(\tau;\alpha)} = \frac{1}{\Xi\,(\tau;\alpha)}\left(\Xi''(\tau;\alpha) + \frac{\alpha}{\theta}\,\frac{1}{\xi}\,\Xi'(\tau;\alpha)\right) = \pm\lambda^2 = \mathtt{const}\,,$$

where $\pm\lambda^2$ is the unknown parameter of separation of the independent variables ($\lambda\geqslant 0$), whereas the functions $\Theta\,(\tau;\alpha)$, $\Xi\,(\xi;\alpha)$ satisfy the following system of two ordinary differential equations of the second order

$$\begin{cases} \Theta''(\tau;\alpha) \mp \lambda^2\,\Theta\,(\tau;\alpha) = 0\,, \\ \Xi''(\xi;\alpha) + \dfrac{\alpha}{\theta}\,\dfrac{1}{\xi}\,\Xi'(\xi;\alpha) \mp \lambda^2\,\Xi\,(\xi;\alpha) = 0\,. \end{cases} \tag{9.2}$$

We start our solving the system (9.2) from the second equation: 1) assume that $\xi > 0$; 2) introduce a new dependent variable $w(\xi;\alpha)$ using a power substitution (the dependent variable transformation)

$$\Xi\,(\xi;\alpha) = \xi^{\beta} w(\xi;\alpha)\,, \tag{9.3}$$

where the exponent β is to be determined by imposing a proper constraint; and then 3) determine uniquely the ordinary differential equation the function $w\,(\xi;\alpha)$ satisfies. For this, we apply the following three-step procedure.

First, differentiating the substitution (9.3) twice yields to the relations between the first and the second derivatives of the functions $\Xi\,(\xi;\alpha)$ and $w(\xi;\alpha)$

$$\Xi' = \beta\,\xi^{\beta-1}w + x^{\beta}w', \qquad \Xi'' = \beta\,(\beta-1)\,\xi^{\beta-2}w + 2\beta\,\xi^{\beta-1}w' + \xi^{\beta}w''.$$

Second, substituting the obtained relations into the second differential equation of the system (9.2) gives a 4-parameter $(\alpha,\beta,\lambda,\mp)$ family of ordinary differential equations of the second order

$$\xi^{\beta}w'' + \xi^{\beta-1}\left(2\beta + \frac{\alpha}{\theta}\right)w' + \xi^{\beta-2}\beta\left(\beta - 1 + \frac{\alpha}{\theta}\right)w \mp \lambda^2 w = 0\,.$$

Third, imposing the following constraint on the exponent β

$$2\beta + \frac{\alpha}{\theta} = 1 \qquad \Rightarrow \qquad \beta(\alpha) = \frac{1-\alpha}{\theta}$$

leads to a 3-parameter $(\beta(\alpha),\lambda,\mp)$ family of ordinary differential equations of the second order

$$w'' + \frac{1}{\xi}\,w' - \left(\frac{\beta^2}{\xi^2} \pm \lambda^2\right)w = 0\,. \tag{9.4}$$

I) Assuming that $\lambda \neq 0$, we change both dependent and independent variables: $s = \lambda\xi$, $W(s;\alpha) := w\left(\lambda^{-1}s;\alpha\right)$, and obtain a 2-parameter $(\beta(\alpha),\mp)$ family of ordinary differential equations of the second order

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\beta^2}{s^2} \pm 1\right) W = 0\,.$$

The choice of the lower sign in the obtained family leads to the 1-parameter ($\beta(\alpha)$) Bessel equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\beta^2}{s^2} - 1\right) W = 0\,, \tag{9.5}$$

whereas the choice of the upper sign leads to the 1-parameter ($\beta(\alpha)$) modified Bessel equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}s^2} + \frac{1}{s}\,\frac{\mathrm{d}W}{\mathrm{d}s} - \left(\frac{\beta^2}{s^2} + 1\right) W = 0\,. \tag{9.6}$$

i) A 3-parameter family of solutions of (9.5), $\beta \notin \mathbb{Z}$ (i. e. $\alpha \in (0,1) \bigcup (1,2)$), is known [9, 19] to be

$$W(s;\alpha) = A_1\,\mathtt{J}_{-\beta}(s) + A_2\,\mathtt{J}_{+\beta}(s)\,, \tag{9.7}$$

where $A_{1,2}$ are arbitrary constants, and $\mathtt{J}_{\mp\beta}(s)$ are the Bessel functions of the first kind of orders $\mp\beta$ (see Section 5), whereas a 2-parameter family of solutions of the equation (9.5), $\beta \in \mathbb{Z}$, is known [9, 19] to be

$$W(s;\alpha) = B_1\,\mathtt{J}_0(s) + B_2\,\mathtt{N}_0(s)\,, \tag{9.8}$$

where $B_{1,2}$ are arbitrary constants, and $\mathtt{J}_0(s)$ and $\mathtt{N}_0(s)$ are respectively the Bessel and the Neumann functions of the first kind of order zero (see Section 5).

ii) A 3-parameter family of solutions of the equation (9.6), $\beta \notin \mathbb{Z}$, is known [9, 19] to be

$$W(s;\alpha) = C_1\,\mathtt{I}_{-\beta}(s) + C_2\,\mathtt{I}_{+\beta}(s)\,, \tag{9.9}$$

where $C_{1,2}$ are arbitrary constants, and $\mathtt{I}_{\mp\beta}(s)$ are the modified Bessel functions of the first kind of orders $\mp\beta$ (see Section 5), whereas a 2-parameter family of solutions of the equation (9.6), $\beta \in \mathbb{Z}$, is known [9, 19] to be

$$W(s;\alpha) = D_1\,\mathtt{I}_0(s) + D_2\,\mathtt{K}_0(s)\,, \tag{9.10}$$

where $D_{1,2}$ are arbitrary constants, and $\mathtt{I}_0(s)$ and $\mathtt{K}_0(s)$ are respectively the modified Bessel function of the first kind and the modified Bessel function of the second kind of order zero (see Section 5).

i) Similarly to Section 5, we find from (9.7), (9.8) the following families of solutions of the second equation of the system (9.2)

$$\Xi(\xi;\alpha) = \begin{cases} \xi^{\frac{1-\alpha}{\theta}} \Big[A_1\,\mathtt{J}_{-\beta}\,(\lambda\xi) + A_2\,\mathtt{J}_{+\beta}\,(\lambda\xi)\Big], & \alpha \in (0,1) \bigcup (1,2)\,, \\ B_1\,\mathtt{J}_0\ \ (\lambda\xi) + B_2\,\mathtt{N}_0\ \ (\lambda\xi)\ , & \alpha = 1\,, \end{cases}$$

and after retaining only the bounded terms the families read

$$\Xi(\xi;\alpha)=\begin{cases}\underbrace{\overbrace{A_1\,\lambda^{-\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}A_{-\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}^{\alpha\in(0,2)}+\xi^{\frac{1-\alpha}{\theta}}\,A_2\,\lambda^{+\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}A_{+\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}_{\alpha\in(0,1)}, & \\ B_1\sum_{\mu=0}^{\infty}A_{0,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}, \qquad \alpha=1\,, & \end{cases}$$

The straightforward extension of the above bounded families to $\xi<0$ and retaining only those terms leading to the continuous and continuously differentiable flux gives the required composite solution of the second equation of the system (9.2)

$$\Xi(\xi;\alpha)=\sum_{\mu=0}^{\infty}A_{-\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu},\qquad \alpha\in(0,2), \tag{9.11}$$

where the coefficients $A_{-\beta,\mu}$ are taken from the series (5.11), if $\alpha\in(0,1)$ $(\beta>0)$ and $\alpha\in(1,2)$ $(\beta<0)$, and are taken from the series (5.13), if $\alpha=1$ $(\beta=0)$.

ii) Again, doing similarly to Section 5, we find from (9.9), (9.10) the following families of solutions of the second equation of the system (9.2)

$$\Xi(\xi;\alpha)=\begin{cases}\xi^{\frac{1-\alpha}{\theta}}\Big[C_1\,\mathtt{I}_{-\beta}(\lambda\xi)+C_2\,\mathtt{I}_{+\beta}(\lambda\xi)\Big], & \alpha\in(0,1)\bigcup(1,2)\,,\\ D_1\,\mathtt{I}_0\ (\lambda\xi)+D_2\,\mathtt{K}_0\ (\lambda\xi)\ , & \alpha=1\,.\end{cases}$$

Retaining only the bounded terms in the families yields to

$$\Xi(\xi;\alpha)=\begin{cases}\underbrace{\overbrace{C_1\,\lambda^{-\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}B_{-\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}^{\alpha\in(0,2)}+\xi^{\frac{1-\alpha}{\theta}}\,C_2\,\lambda^{+\frac{1-\alpha}{\theta}}\sum_{\mu=0}^{\infty}B_{+\beta,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}}_{\alpha\in(0,1)}, & \\ D_1\sum_{\mu=0}^{\infty}B_{0,\mu}\,\lambda^{2\mu}\,\xi^{2\mu}, \qquad \alpha=1\,. & \end{cases}$$

Eventually, doing exactly as in item *i*), we obtain the following required composite solution of the second equation of the system (9.2)

$$\Xi(\xi;\alpha)=\sum_{\mu=0}^{\infty}B_{-\beta,\mu}\,\lambda^{2\mu}\xi^{2\mu}, \tag{9.12}$$

where the coefficients $B_{-\varrho,\mu}$ are taken from the series (5.15), if $\alpha \in (0,1)$ ($\varrho > 0$) and $\alpha \in (1,2)$ ($\varrho < 0$), and are taken from the series (5.17), if $\alpha = 1$ ($\varrho = 0$).

II) Assuming that $\lambda = 0$, we obtain directly from (9.4) a 1-parameter ($\beta(\alpha)$) ordinary differential equation of the second order

$$\frac{\mathrm{d}^2 w_0}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}w_0}{\mathrm{d}\xi} - \frac{\beta^2 w_0}{\xi^2} = 0\,. \tag{9.13}$$

A 3-parameter family of solutions of the equation (9.13) is as follows

$$w_0\,(\xi;\alpha) = E_1\,\xi^{-\beta} + E_2\,\xi^{+\beta},$$

where $E_{1,2}$ are arbitrary constants, and after applying the inverse transformation (9.3) we obtain the family of solutions of the second equation of the system (9.2)

$$\Xi\,(\xi;\alpha) = \xi^{\beta}\left(E_1\,\xi^{-\beta} + E_2\,\xi^{+\beta}\right) = E_1 + E_2\,\xi^{(1-\alpha)\frac{2}{\theta}},$$

extendable to $\xi < 0$ as follows

$$\Xi\,(\xi;\alpha) = \begin{cases} E_1 + E_2\,|\xi|^{(1-\alpha)\frac{2}{\theta}}, & \alpha \in (0,2), \\[2ex] E_1 + E_2\;\xi^{\,(1-\alpha)\frac{2}{\theta}}, & \alpha \in (0,2)_o. \end{cases} \tag{9.14}$$

Both families (9.14) leads, as it is known from Section 8, to the discontinuous flux $\mathring{\varphi}$ (8.7), therefore no family (9.14) must be accounted for in the ansatz (9.1).

Now we turn to the first ordinary differential equation of the system (9.2); its 3-parameter families of solutions are known to be

$$\Theta(\tau;\alpha) = \begin{cases} F_1\,\mathtt{exp}\,(-\lambda\tau) + F_2\,\mathtt{exp}\,(+\lambda\tau)\,, \\[2ex] G_1\,\mathtt{cos}\,(+\lambda\tau) + G_2\,\mathtt{sin}\,(+\lambda\tau)\,, \end{cases} \tag{9.15}$$

where $F_{1,2}$, $G_{1,2}$ are arbitrary constants.

Combining the families (9.15) for $\Theta(\tau;\alpha)$ and the families (9.11) and (9.12) for $\Xi\,(\xi;\alpha)$ in the ansatz (9.1), we obtain the required solutions of the transformed wave equation (6.6), (6.7). The spatial parts $\Xi\,(\xi;\alpha)$ (9.11), (9.12) of the obtained solutions include the same terms $\xi^{2\mu}$, $\mu \in \mathbb{Z}_+$, as those present in the only required series solution (8.9).

10. Functional properties of the series solutions

In this Section we turn out to our preliminary functional estimations concerning solutions of the degenerate wave equation made in Section 2.

Proposition 10.1. *All bounded series solutions* $u(t,x;\alpha)$ (4.1) *of the original degenerate wave equation* (1.1) *are elements of the functional space* $H^1_a(-c,+c)$ *for all* $t \in [0,T]$.

Proof. The underlying idea of the proof is straightforwardly based on properties of convergent power series [1, 20]. On the one hand, 1) usual convergence implies absolute convergence, then 2) absolute convergence implies uniform convergence, and eventually 3) uniform convergence implies term-by-term differentiation and integration (term-by-term differentiation was, by the way, implied when finding the one-sided series solutions in Section 3 and then matching the obtained one-sided series solutions continuously in Section 4). On the other hand, uniform convergence of two (or more) power series implies their term-by-term product, and the resulting power series is also uniformly convergent. These properties and the definition of the norm of the functional space $H^1_a(-c,+c)$ are quite sufficient to immediately complete the proof, nevertheless we perform careful calculations of the norm for some of the series solutions (4.1). The only thing we need is to assume that all series solutions are convergent in the c-band. Accounting for the exact solutions of the degenerate wave equation found in Section 9 using separation of the variables our assumption seems even more than reasonable.

First, we take the series solution $u_1(t,x;\alpha)$ (4.1), valid if $\alpha \in (0,2)$ and uniformly convergent for all $t \in [0,T]$, then we have

$$u_1^2 = \left(\sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\,|x|^{\mu\theta}\right)^2 = \sum_{\mu=0}^{\infty} \bar{U}_{\alpha,\mu}(t)\,|x|^{\mu\theta},$$

$$q_1 = \mp\theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta-1},$$

$$aq_1 = \mp a_*\theta\,|x|^{\alpha} \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta-1} = \mp a_*\theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{(\mu-1)\theta+1},$$

$$aq_1^2 = a_*\theta^2 \left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{\mu\theta-1}\right)\left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\,|x|^{(\mu-1)\theta+1}\right) = \sum_{\mu=1}^{\infty} \widetilde{U}_{\alpha,\mu}(t)\,|x|^{\mu\theta},$$

where the coefficient functions $\bar{U}_{\alpha,\mu}(t)$, $\mu \in \mathbb{Z}_+$, and $\widetilde{U}_{\alpha,\mu}(t)$, $\mu \in \mathbb{N}$, are determined using the series product rule, the upper sign is taken for $x<0$, whereas the lower sign for $x>0$, and $t \in [0,T]$. It is clear that the derived series $u_1^2 + aq_1^2$ is uniformly convergent for all $t \in [0,T]$, therefore the norm of $u_1(t,x;\alpha)$ is bounded to be

$$\begin{aligned}
\|u_1\|^2_{H^1_a(-c,+c)} &= \int_{-c}^{c} \left[u_1^2 + aq_1^2\right] \mathrm{d}x \\
&= \bar{U}_{\alpha,0}(t) \int_{-c}^{c} \mathrm{d}x + \sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) + \widetilde{U}_{\alpha,\mu}(t)\right) \int_{-c}^{c} |x|^{\mu\theta}\,\mathrm{d}x \\
&= 2c\,\bar{U}_{\alpha,0}(t) + 2\sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) + \widetilde{U}_{\alpha,\mu}(t)\right) \frac{2c^{\mu\theta+1}}{\mu\theta+1}\,.
\end{aligned}$$

Second, we take the series solution $u_2(t,x;\alpha)$ (4.1), valid if $\alpha \in (0,2)_o$ and uniformly convergent for all $t \in [0,T]$, then we have

$$
\begin{aligned}
u_2^2 &= \left(\sum_{\mu=0}^{\infty} U_{\alpha,\mu}(t)\, x^{\mu\theta}\right)^2 = \sum_{\mu=0}^{\infty} \bar{U}_{\alpha,\mu}(t)\, x^{\mu\theta}, \\
q_2 &= \theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{\mu\theta-1}, \\
aq_2 &= a_*\theta\, |x|^{\alpha} \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{\mu\theta-1} = \mp a_*\theta \sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{(\mu-1)\theta+1}, \\
aq_2^2 &= \mp a_*\theta^2 \left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{\mu\theta-1}\right)\left(\sum_{\mu=1}^{\infty} \mu\, U_{\alpha,\mu}(t)\, x^{(\mu-1)\theta+1}\right) = \mp \sum_{\mu=1}^{\infty} \widetilde{U}_{\alpha,\mu}(t)\, x^{\mu\theta},
\end{aligned}
$$

and the norm of $u_2(t,x;\alpha)$ reads

$$
\begin{aligned}
\|u_2\|^2_{H^1_a(-c,+c)} = \int_{-c}^{c} \left[u_2^2 + aq_2^2\right] \mathrm{d}x &= \bar{U}_{\alpha,0}(t) \int_{-c}^{c} \mathrm{d}x \\
&+ \sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) - \widetilde{U}_{\alpha,\mu}(t)\right) \int_{-c}^{0} x^{\mu\theta}\, \mathrm{d}x \\
&+ \sum_{\mu=1}^{\infty} \left(\bar{U}_{\alpha,\mu}(t) + \widetilde{U}_{\alpha,\mu}(t)\right) \int_{0}^{c} x^{\mu\theta}\, \mathrm{d}x < \infty\,.
\end{aligned}
$$

Performing the calculation gives for the norm the following bounded expressions

$$
\begin{aligned}
\|u_2\|^2_{H^1_a(-c,+c)} &= 2c\, \bar{U}_{\alpha,0}(t) \\
&+ \begin{cases}
\displaystyle\sum_{\gamma=1}^{\infty} \bar{U}_{\alpha,2\gamma}(t)\, \frac{2c^{\,2\gamma\theta+1}}{2\gamma\theta+1} + \sum_{\mu=1}^{\infty} \widetilde{U}_{\alpha,\mu}(t)\, \frac{2c^{\,\mu\theta+1}}{\mu\theta+1}\,, & \alpha \in \mathbb{Q}_{o,o}\,, \\
\displaystyle\sum_{\mu=1}^{\infty} \bar{U}_{\alpha,\mu}(t)\, \frac{2c^{\,\mu\theta+1}}{\mu\theta+1}\,, & \alpha \in \mathbb{Q}_{o,e}\,.
\end{cases}
\end{aligned}
$$

Calculating the bounded norms for the binomial solutions 2, 3 (4.1) and the series solutions 5, 6 (4.1) is performed exactly in the same way. □

Proposition 10.1 says that we cannot distinguish between the series solutions (4.1) using the norm of the functional space $H^1_a(-c,+c)$.

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ON EQUIVALENCE OF LINEAR CONTROL SYSTEMS AND ITS USAGE TO VERIFICATION OF THE ADEQUACY OF DIFFERENT MODELS FOR A REAL DYNAMIC PROCESS

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Abstract. A problem of description of algebraic invariants for a linear control system that determine its structure is considered. With the help of these invariants, the equivalence problem of two linear time-invariant control systems with respect to actions of some linear groups on the spaces of inputs, outputs, and states of these systems is solved. The invariants are used to establish the necessary equivalence conditions for two nonlinear systems of differential equations generalizing the well-known Hopfield neural network model. Finally, these conditions are applied to establish the adequacy of two neural network models designed to describe the behavior of a real dynamic process given by two different sets of time series.

Key words: linear time-invariant control system, system of ordinary autonomous differential equations, complete linear group, special linear group, exterior degree, algebraic invariant, null-form, neural network, activation function.

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1. Introduction

In recent decades, researchers have paid much attention to chaotic behavior in many fields, such as meteorology, medicine, economics, signal processing, traffic flow, and many others [16,34,36,49]. They also developed many models describing chaotic time series in order to predict the behavior of these time series. Researchers have found that it is a difficult problem to forecast chaotic time series, which are the evolution of chaotic systems, with the use of traditional time series forecasting methods [16, 36]. Now chaos theory has become an important part of nonlinear science and is used for forecasting chaotic time series. Therefore, modeling of chaotic systems constructed from observed data and predicting multiple future values of the time series has become an important issue [16, 34, 36, 49].

We will assume that we know the dimension n of the real phase space in which the considered dynamic process $\mathbf{P}(t) \in \mathbb{R}^n$ takes place [7]. Further, for modeling of the process $\mathbf{P}(t) = (P_1(t), ..., P_n(t))^T$ neural networks will be used [17, 25, 26]. The motivation for this use is given below.

In the beginning we give a generalization of one well-known result of approximation theory of functions:

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Theorem 1.1. *[17] Let $\phi(z) : \mathbb{R} \to \mathbb{R}$ be a nonconstant, bounded, and monotone-increasing continuous nonlinear function. Let also $f(x_1, ..., x_n) : \mathbb{R}^n \to \mathbb{R}$ be any given continuous function. Then, $\forall \epsilon > 0$ there exist an integer $k > 0$ and sets of real constants α_i, β_i, and γ_{ij}, where $i = 1, ..., k; j = 1, ..., n$ such that we may define the function*

$$F(x_1, ..., x_n) = \sum_{i=1}^{k} \alpha_i \phi\Big(\sum_{j=1}^{n} \gamma_{ij} x_j + \beta_i\Big)$$

as an approximation realization of the function $f(x_1, ..., x_n)$:

$$\forall x_1, ..., x_n \in \mathbb{R} \; |F(x_1, ..., x_n) - f(x_1, ..., x_n)| < \epsilon.$$

The function $\phi(z) : \mathbb{R} \to \mathbb{R}$ is called an activation function [17, 26]. (In most scientific publications is suggested that $\phi(z) : \mathbb{R} \to [0, 1]$ is a sigmoid function. However, in some cases, the condition that the activation function $\phi(x)$ is bounded can be removed [48].)

In the future, we will use the version of Theorem 1.1, in which one activation function $\phi(z)$ will be replaced by several similar functions $\phi_1(z)$,..., $\phi_k(z)$:

$$F(x_1, ..., x_n) = \sum_{i=1}^{k} \alpha_i \phi_i\Big(\sum_{j=1}^{n} \gamma_{ij} x_j + \beta_i\Big).$$

Let

$$x_0 = x(t_0), x_1 = x(t_1), ..., x_N = x(t_N) \tag{1.1}$$

be a finite sequence of numerical values of some scalar dynamical variable $x(t)$ measured with the constant time step Δt in the moments $t_i = t_0 + i\Delta t$; $x_i = x(t_i)$; $i = 0, 1, ..., N$. Sequence (1.1) is called a time series [29] – [31], [34], [49].

Suppose that we know n time series

$$\begin{cases} P_{10}, P_{11}, P_{12}, ..., P_{1N}, \\ P_{20}, P_{21}, P_{22}, ..., P_{2N}, \\ \quad ...., \\ P_{n0}, P_{n1}, P_{n2}, ..., P_{nN} \end{cases} \tag{1.2}$$

that describe the components of the process $\mathbf{P}(t) = (P_1(t), ..., P_n(t))^T$ (here $N >> n$).

Further, we propose an approximate procedure for determining unknown right-hand sides of the differential equations. The procedure is based on the least-squares method and the fact that we know with sufficient precision the values of $\mathbf{x}(t)$ and its derivatives of order equal to the equation order. In this case we avoid considering a possible ill-posed problem by applying consecutive smoothing procedures leading to shortening the given time series (see [16, 36]).

Let $(\mathbf{u},\mathbf{v})$ be a scalar product of real vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$. Introduce the real matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{m\times n}$, and real vectors $\mathbf{d}=(d_1,...,d_n)^T$, $\mathbf{c}_j=(c_{j1},...,c_{jn})\in\mathbb{R}^n$; $j=1,...,m\le n$.

We assume that using suitable methods (based on Theorem 1.1) the system of ordinary autonomous differential equations, the solution $\mathbf{x}(t)\in\mathbb{R}^n$ of which simulates process $\mathbf{P}(t)$ with a given accuracy, was reconstructed [7].

We assume that this system (with the known vector of initial values $\mathbf{x}^T(0)=(x_{10},...,x_{n0})$) has the following form:

$$\dot{\mathbf{x}}(t)=\mathbf{d}+A\mathbf{x}+B\mathbf{\Phi}(C\mathbf{x})\Longleftrightarrow\begin{cases}\dot{x}_1(t)=d_1+\sum\limits_{j=1}^{n}a_{1j}x_j(t)+\sum\limits_{j=1}^{m}b_{1j}\phi_j(\mathbf{c}_j,\mathbf{x}(t)),\\ \dot{x}_2(t)=d_2+\sum\limits_{j=1}^{n}a_{2j}x_j(t)+\sum\limits_{j=1}^{m}b_{2j}\phi_j(\mathbf{c}_j,\mathbf{x}(t)),\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,\\ \dot{x}_n(t)=d_n+\sum\limits_{j=1}^{n}a_{nj}x_j(t)+\sum\limits_{j=1}^{m}b_{nj}\phi_j(\mathbf{c}_j,\mathbf{x}(t)),\end{cases}\tag{1.3}$$

where $\mathbf{\Phi}(\mathbf{u})=(\phi_1(u_1),...,\phi_m(u_m))^T$. (Note the system (1.3) can be interpreted as a generalized Hopfield neural network (see [25, 48]) with activation functions $\phi_1(u_1),...,\phi_m(u_m)$.)

The form of system (1.3) is dictated by the following considerations: the nonlinear parts of system (1.3) are continuous functions, and therefore Theorem 1.1 was used to describe them; the presence of a linear part makes it possible to use linearization methods to study the stability of solutions of system (1.3).

There is only one serious flaw in the research plan outlined above. This disadvantage lies in the insufficient verification of the adequacy of the constructed model and the process under study. In this paper, mathematical tools have been developed to test the adequacy, based on an algebraic theory of invariants. The idea of such verification is based on the following well-known fact: with arbitrary observations of a dynamic process, there are always functions that are independent of the methods of observations, but depend on an internal structure that determines the behavior of the process. Functions describing this structure are called invariants.

Currently, the construction of a complete set of invariants describing arbitrary nonlinear dynamical systems is an unsolved problem. Therefore, in this article, the approach based on obtaining the missing invariants for the studied nonlinear system using known invariants obtained for a special linear system, was developed. (Immediately, we note that the construction of such invariants may also have an independent mathematical interest.)

Suppose that the same dynamic process $\mathbf{P}(t)\in\mathbb{R}^n$ is given by another set of time series

$$\begin{cases} Q_{10}, Q_{11}, Q_{12}, ..., Q_{1N}, \\ Q_{20}, Q_{21}, Q_{22}, ..., Q_{2N}, \\ \quad ...., \\ Q_{n0}, Q_{n1}, Q_{n2}, ..., Q_{nN}. \end{cases} \tag{1.4}$$

Instead of variable $\mathbf{x}$, we introduce a new phase variable $\mathbf{y} \in \mathbb{R}^n$. Introduce also the real matrices $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{m\times n}$, and real vectors $\mathbf{l} = (l_1, ..., l_n)^T, \mathbf{h}_j = (h_{j1}, ..., h_{jn}) \in \mathbb{R}^n; j = 1, ..., m \le n$.

We also assume that in this case other activation functions $\psi_1(u_1), ..., \psi_m(u_m)$ can be used to simulate process $\mathbf{P}(t)$. (Nevertheless, case $\phi_1(u_1) = \psi_1(u_1), ...,$ $\phi_m(u_m) = \psi_m(u_m)$ is not excluded.)

In this case, instead of system (1.3), we get the following system of ordinary autonomous differential equations:

$$\dot{\mathbf{y}}(t) = \mathbf{l} + F\mathbf{y} + G\mathbf{\Psi}(H\mathbf{y}) \Longleftrightarrow \begin{cases} \dot{y}_1(t) = l_1 + \sum\limits_{j=1}^{n} f_{1j} y_j(t) + \sum\limits_{j=1}^{m} g_{1j}\psi_j(\mathbf{h}_j, \mathbf{y}(t)), \\ \dot{y}_2(t) = l_2 + \sum\limits_{j=1}^{n} f_{2j} y_j(t) + \sum\limits_{j=1}^{m} g_{2j}\psi_j(\mathbf{h}_j, \mathbf{y}(t)), \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \\ \dot{y}_n(t) = l_n + \sum\limits_{j=1}^{n} f_{nj} y_j(t) + \sum\limits_{j=1}^{m} g_{nj}\psi_j(\mathbf{h}_j, \mathbf{y}(t)), \end{cases} \tag{1.5}$$

where $\mathbf{\Psi}(\mathbf{v}) = (\psi_1(v_1), ..., \psi_m(v_m))^T$.

Systems (1.3) and (1.5) describe the same dynamic process $\mathbf{P}(t)$. In order to establish (in the sense that will be below) the equivalence of systems (1.3) and (1.5), we introduce the following assumptions:

1) functions $\mathbf{\Phi}(\mathbf{u})$ and $\mathbf{\Psi}(\mathbf{v})$ are continuous in the interval $(-\infty, \infty)$;

2) $\mathbf{\Phi}(\mathbf{0}) = \mathbf{\Psi}(\mathbf{0}) = \mathbf{0}$;

3) there exist constants $L_1 > 0, L_2 > 0$, and neighborhood $\mathbb{O}$ of the origin such that $\forall \mathbf{u}, \mathbf{v} \subset \mathbb{O}$ $\|\mathbf{\Phi}(\mathbf{u}) - \mathbf{\Phi}(\mathbf{v})\|_2 < L_1\|\mathbf{u} - \mathbf{v}\|_2$ and $\|\mathbf{\Psi}(\mathbf{u}) - \mathbf{\Psi}(\mathbf{v})\|_2 < L_2\|\mathbf{u} - \mathbf{v}\|_2$.

Let the conditions 1) – 3) be fulfilled. Then, we can assume that there exist the nondegenerate matrices $S \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times m}$, and $W \in \mathbb{R}^{m\times n}$ such that $\mathbf{x} = S\mathbf{y}$, $\mathbf{d} = S\mathbf{l}$, and

$$\begin{cases} F = S^{-1}AS, G = S^{-1}BT, H = W^{-1}CS, \\ \forall \mathbf{s} \in \mathbb{R}^m \ \mathbf{\Psi}(\mathbf{s}) = T^{-1}\mathbf{\Phi}(W\mathbf{s}). \end{cases} \tag{1.6}$$

If the mentioned matrices exist, then systems (1.3) and (1.5) are called equivalent.

To fulfill all equivalence conditions (especially (1.6)), it is necessary to indicate the class of functions $\mathbf{\Phi}(\mathbf{s})$ (or $\mathbf{\Psi}(\mathbf{s})$) that satisfy these conditions. One of these classes is the class of piecewise linear functions. (In the theory of neural networks, this class is a well-known class of rectified linear units (ReLU)). In the present time ReLU are standard functions to increase the depth of learning of neural networks.

Therefore, we will further assume that either the components of vectors $\mathbf{\Phi}(\mathbf{s})$, $\mathbf{\Psi}(\mathbf{s})$ are piecewise continuous linear functions or these are piecewise continuous nonlinear functions such that $\mathbf{\Phi}(\mathbf{s}) = \mathbf{\Psi}(\mathbf{s})$ (in this case $T = W = E_m$, where E_m is the identity matrix of order m).

A common practice in chaotic time series analysis has been to reconstruct the phase space by utilizing the delay-coordinate embedding technique, and then to compute the dynamical invariants such as fractal dimension of the underlying chaotic set, and its Lyapunov spectrum. As a large body of literature exists on application of the technique of the time series to study chaotic attractors [29] – [31], [34], [38], [49], a relatively unexplored issue is its applicability to dynamical systems of differential equations depending on parameters. Our focus will be concentrated on the analysis of influence of parameters of found dynamic system on the behavior of its solutions. These parameters are determined by the structure of the time series (1.1) and choice of approximating functions in the right-hand sides of the obtained system of differential equations.

1.1. Continuous analog of neural network models

In recent years, an interesting idea has appeared to interpret a system of ordinary differential equations in the form of a suitable neural network (residual network) [11, 27, 51]. The essence of this idea is as follows.

Consider the following neural network

$$\mathbf{x}(t+1) = \mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t), \mathbf{\Omega}), \mathbf{x}(0) = \mathbf{x}_0; t = 1, ..., N. \tag{1.7}$$

Here $\mathbf{h}(\mathbf{u}, \mathbf{v}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a vector of continuous functions, $\mathbf{\Omega} : \mathbb{R}^k \to \mathbb{R}^n$ is a vector of parameters.

Now we rewrite relation (1.7) in the following form:

$$\frac{\mathbf{x}(t+1) - \mathbf{x}(t)}{(t+1) - t} = \mathbf{h}(\mathbf{x}(t), \mathbf{\Omega}).$$

If we consider function $\mathbf{x}(t)$ as a function of a continuous argument on some interval $[\mathbf{x}_0, \mathbf{x}_N]$, then the last equation can be rewritten in the following form:

$$\frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} = \mathbf{h}(\mathbf{x}(t), \mathbf{\Omega}).$$

If now we direct the number of "layers"$N \to \infty$ and we assume $\Delta t \to 0$, then we get the following system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{\Omega}), \mathbf{x}(0) = \mathbf{x}_0, \tag{1.8}$$

So we can say that neural network (1.7) is the well-known Euler discretization procedure of system (1.8):

$$\mathbf{x}(t+\Delta t) - \mathbf{x}(t) = \Delta t \cdot (\mathbf{h}(\mathbf{x}(t), \mathbf{\Omega})), \mathbf{x}(0) = \mathbf{x}_0,$$

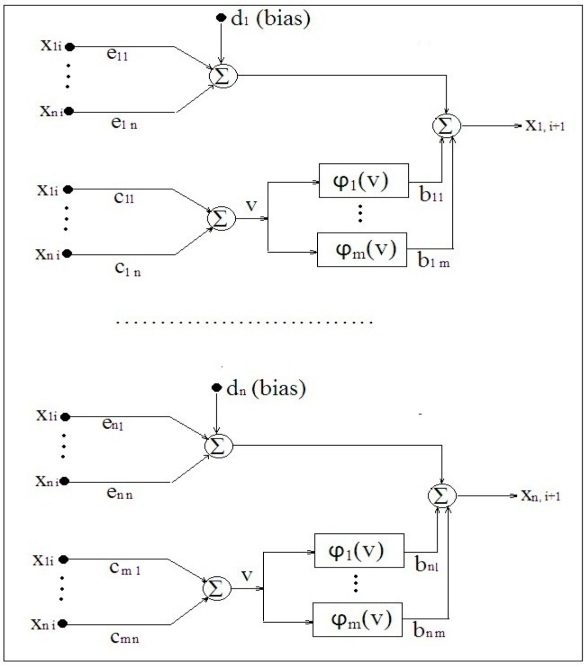


Fig. 1.1. The architecture of the i-th layer of the neural network (1.7) for $\mathbf{h}(\mathbf{x}(t), \boldsymbol{\Omega}) \equiv \mathbf{d} + A\mathbf{x}(t) + B\boldsymbol{\Phi}(C\mathbf{x}(t))$; $i = 0, ..., N$. Here $e_{kj} = a_{kj}$, if $k \neq j$ and $e_{kk} = a_{kk} + 1$; $k, j = 1, ..., n$. All other designations are the same as in (1.3)

where Δt is the discretization step.

Besides, sequence (1.7) can be viewed as a neural network with $N - 1$ hidden layers, input layer $\mathbf{x}_0$ and output layer $\mathbf{x}_N$. The architecture of such neural network is determined by the operator $\mathbf{h}(\mathbf{x}(t), \boldsymbol{\Omega})$, and if $\mathbf{h}(\mathbf{x}(t), \boldsymbol{\Omega}) \equiv \mathbf{d} + A\mathbf{x}(t) + B\boldsymbol{\Phi}(C\mathbf{x}(t))$, then an arbitrary hidden layer of this network will have the structure shown in Fig. 1.

Therefore, sequence (1.7) is a neural network model of the process $\mathbf{P}(t)$. In addition, the number of neurons in an arbitrary hidden layer of this network, in which $\mathbf{h}(\mathbf{x}(t), \boldsymbol{\Omega}) \equiv \mathbf{d} + A\mathbf{x}(t) + B\boldsymbol{\Phi}(C\mathbf{x}(t))$, does not exceed nm [17, 26].

The problem that is usually considered when modeling process $\mathbf{P}(t)$ is as follows: find parameters $\mathbf{d}, A, B, C$ (with the known activation function $\boldsymbol{\Phi}(\mathbf{u})$) of the neural network that minimize the following loss function:

$$\sum_{t=1}^{N} \|\mathbf{P}(t) - \mathbf{x}(t)\|^2.$$

Here $\|\mathbf{u}\|$ is a norm of the vector $\mathbf{u}$.

Since the number n is the dimension of the embedding space and it is completely determined by process $\mathbf{P}(t)$, the numbers N and nm, which determine the depth and number of neurons in any layer of the neural network, can be arbitrarily selected. Thus, there are quite wide opportunities for the neural network modeling of process $\mathbf{P}(t)$.

Now, by analogy with (1.3), we assume that process $\mathbf{P}(t)$ is modeled by neural network (1.7), whose operator has the form $\mathbf{h}(\mathbf{x}(t), \mathbf{\Omega}) \equiv \mathbf{l} + F\mathbf{y}(t) + G\mathbf{\Psi}(H\mathbf{y}(t))$. (In this case, we are already talking about defining parameters $\mathbf{l}, F, G, H$ with the known activation function $\mathbf{\Psi}(\mathbf{u})$.) Obviously, the equivalence conditions for such neural networks coincide with conditions (1.6). Therefore, the problems of establishing the equivalence of two differential or two neural network models do not differ from each other. In both cases, conditions (1.6) should be used to confirm equivalence. (However, it should be noted that differential and neural network models of the same process, generally speaking, are not equivalent [51].)

In conclusion of this section, we note that the question of which of the modeling methods either using differential equations or neural networks, is more effective as long as it remains open.

For simplicity, we put in system (1.3) $\mathbf{d} = 0$. Now we introduce the following control system [50]:

$$\dot{\mathbf{x}}(t) = A\mathbf{x} + B\mathbf{u}, \mathbf{z} = C\mathbf{x}; \mathbf{u} = (u_1, ..., u_m)^T, \mathbf{z} = (z_1, ..., z_m)^T. \tag{1.9}$$

Then system (1.3) can be viewed as the linear system (1.9) closed by nonlinear feedback

$$\mathbf{u} = \mathbf{\Phi}(\mathbf{z}) = \mathbf{\Phi}(C\mathbf{x}) = (\phi_1(c_{11}x_1 + ... + c_{1n}x_n), ..., \phi_m(c_{m1}x_1 + ... + c_{mn}x_n))^T.$$

Our main goal is to show how the problem of reconstructing differential equations from known time series can be reduced to the problem of computing invariants for the linear control system (1.9).

Thus, the main postulate that we will implement in this work can be formulated as follows: *two different sets of time series (1.2) and (1.4) describe the same dynamic process* $\mathbf{P}(t)$ *in two different bases of phase space* $\mathbb{R}^n$*; this assumption ensures that the invariants of systems (1.3) and (1.5) are the same.*

This article is organized as follows. Section 2 describes the mathematical concepts necessary to solve the equivalence problem. Sections 3 – 4 study the actions of algebraic groups on varieties of linear systems. Section 5 gives an algebraic description of the key concept that is used to search for invariants; this is the concept of null-forms. Section 6 is devoted to the analysis of the structural stability of linear systems (small changes in the parameters of the system should not influence on the composition of the invariants of this system). Sections 7 – 9 describe rings of invariants and equivalence conditions for linear control systems. The whole Section 10 is devoted to the application of the theory of invariants to the problem of reconstruction of ordinary differential equations from known measurements of the dynamic characteristics of these equations. Here is a practical example of the reconstruction of equations describing the behavior of current and voltage in the contact electric network [7,43]. Finally, some results are summarized in Section 11.

2. Mathematical preliminaries

Consider a linear time-invariant control system whose state equation is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t) \in \mathbb{R}^m, \tag{2.1}$$

and an output equation has the form

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \mathbf{y}(t) \in \mathbb{R}^p. \tag{2.2}$$

Here $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^p$ are real linear spaces of vector-columns of dimensionalities n, m, p; $\mathbf{x}(t) = (x_1(t), ..., x_n(t))^T$, $u(t) = (u_1(t), ..., u_m(t))^T$, and $\mathbf{y}(t) = (y_1(t), ..., y_p(t))^T$ are vectors of states, inputs, and outputs; $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{B} : \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^p$ are real linear maps of appropriate spaces.

Fix any bases in spaces $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$; then the triple of operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$ will be represented in the chosen bases by matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. Further, we will adhere only to these denotations.

Denote by $\mathbb{S} = \mathbb{R}^{p\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ a direct product of spaces $\mathbb{R}^{p\times n}$, $\mathbb{R}^{n\times n}$, and $\mathbb{R}^{n\times m}$. Then system (2.1),(2.2) can be uniquely represented by the triple of matrices $(C, A, B) \in \mathbb{S}$.

Let $\mathbb{G}L(q, \mathbb{R})$ be a complete linear group of all square invertible matrices of sizes $q \times q$ with elements from the field of real numbers $\mathbb{R}$.

Introduce the direct product

$$\mathbb{G}L = \mathbb{G}L(n, \mathbb{R}) \times \mathbb{G}L(m, \mathbb{R}) \times \mathbb{G}L(p, \mathbb{R})$$

$$= \{(S, T, W) | S \in \mathbb{G}L(n, \mathbb{R}), T \in \mathbb{G}L(m, \mathbb{R}), W \in \mathbb{G}L(p, \mathbb{R})\}.$$

We also introduce into system (2.1),(2.2) new variables $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$, and $\mathbf{h} \in \mathbb{R}^p$ under the formulas: $\mathbf{x}(t) = S\mathbf{z}(t)$, $\mathbf{u}(t) = T\mathbf{v}(t)$, and $\mathbf{y}(t) = W\mathbf{h}(t)$, where $S \in \mathbb{G}L(n, \mathbb{R})$, $T \in \mathbb{G}L(m, \mathbb{R})$, and $W \in \mathbb{G}L(p, \mathbb{R})$. Then we obtain the following operation $\mathbb{G}L : \mathbb{S} \to \mathbb{S}$ of indicated group on space $\mathbb{S}$:

$$\forall\ (C, A, B) \in \mathbb{S} \text{ and } \forall (S, T, W) \in \mathbb{G}L,$$

$$(S, T, W) \circ (C, A, B) = (W^{-1}CS, S^{-1}AS, S^{-1}BT) \in \mathbb{S}. \tag{2.3}$$

Let $\mathbf{s} = (C, A, B) \in \mathbb{S}$ and $\mathbf{g} = (S, T, W) \in \mathbb{G}L$ be an arbitrary elements of corresponding sets.

Definition 2.1. A polynomial $f(\mathbf{s})$ is called an invariant of weight $\mathbf{l} = (l_S, l_T, l_W)$ of group $\mathbb{G}L$ for system (2.1),(2.2) if

$$f(\mathbf{g} \circ \mathbf{s}) = (\det S)^{l_S} (\det T)^{l_T} (\det W)^{l_W} \times f(\mathbf{s}), \quad \forall \mathbf{g} \in \mathbb{G}L \text{ and } \forall \mathbf{s} \in \mathbb{S},$$

where l_S, l_T, and l_W are some integers. The invariant $f(\mathbf{s})$ of weight $\mathbf{l} = (l_S, l_T, l_W) = (0, 0, 0)$ is called absolute; otherwise the invariant $f(\mathbf{s})$ is called relative [35, 44].

Notice that the problem of any classification of some set of objects (for example, the set of systems (2.1), (2.2)) implies a decomposition of this set on classes of identical (in a certain sense) elements. One of the most widespread methods of such decomposition is a description of system (2.1), (2.2) with the help of functions not depending on coordinate presentation of system (2.1), (2.2). This description is usually called invariant. Sometimes the invariant description of system (2.1), (2.2) is called an invariant (or algebraic) analysis.

The problem of invariant analysis of system (2.1),(2.2) with respect to action (2.3) was most in detail studied for the case $T = E_m$ and $W = E_p$, where E_m and E_p are identity matrices of degrees m and p. In this case the action (2.3) is called an action of similarity. In one or another form classification questions of linear control systems with respect to the action of similarity, their invariants and canonical forms were studied by many authors (see, for example, C. I. Byrnes and N. E. Hurt [9], M. Hazewinkel [18, 20], M. Hazewinkel and C. Martin [19], R. E. Kalman [28], A. Tannenbaum [45, 46]). We also note that in the book of D. Mumford and J. Fogarty [35] the tasks of invariant theory, directly relating to the classification questions of linear control systems, were considered. The topology problems of classification of linear control systems with respect to the action of similarity in detail were studied in articles of D. F. Delchamps [12], P. A. Fuhrmann and U. Helmke [14], and U. Helmke [21] – [23].

Note that a main attention of specialists on the mathematical system theory attract problems of invariant analysis (using geometric invariant theory) connected with the action of similarity. These problems are following: the finding of good canonical forms for linear systems, a computation of their invariants, a description of regular and stable systems, and also a construction of moduli spaces as quotients of algebraic varieties under algebraic group actions. It should be said that the indicated problems were considered with different positions by A. Tannenbaum [46, 47], S. Friedland [13], V. G. Lomadze [32], J. Rosenthal [41], W. Manthey and U. Helmke [33], M. Bader [1]. In our opinion the most complete solutions of indicated problems was got in [1]. In this work with the help of geometric invariant theory and methods of theory quivers compactifications of moduli spaces of linear dynamical systems were derived.

Except for the classification questions of linear control systems, invariant theory is widely used in the problem of output feed-back design both for linear and bilinear control systems (see, for examples, papers [2] – [6], [38], [39], [41, 42], and [50]). Here it should be said that in article [5] it was succeeded to get the constructive solution of output feed-back design problem for system (2.1), (2.2) in the case $mp > n$.

Consider two systems: $\mathbf{s}_1 = (C_1, A_1, B_1) \in \mathbb{S}_{open} \subset \mathbb{S}$ and $\mathbf{s}_2 = (C_2, A_2, B_2) \in \mathbb{S}_{open} \subset \mathbb{S}$, where $\mathbb{S}_{open}$ is an open subset in $\mathbb{S}$.

Equivalence problem. *It is necessary to find the set* $\mathbb{S}_{open} \subset \mathbb{S}$ *and to build the finite set of invariants* $\mathbf{f}_1(\mathbf{s}), ..., \mathbf{f}_k(\mathbf{s})$ *(absolute and relative) of group* $\mathbb{G}L$ *such that* $\forall \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{S}_{open}$ *from the conditions* $\mathbf{f}_1(\mathbf{s}_1) = \mathbf{f}_1(\mathbf{s}_2), ..., \mathbf{f}_k(\mathbf{s}_1) = \mathbf{f}_k(\mathbf{s}_2)$ *it follows* *that there exists the element* $\mathbf{g} = (S, T, W) \in \mathbb{G}L$ *such that*

$$A_2 = S^{-1}A_1S, \; B_2 = S^{-1}B_1T, \; C_2 = W^{-1}C_1S. \tag{2.4}$$

It should be said that in the present time there is a vast literature devoted to invariant theory and its applications to different tasks of mathematics, mechanics, physics, and control theory. However almost in all known manuscripts invariant theory is considered as part of algebraic geometry and theory of representations of groups. In addition, these treatises are intended for professional mathematicians and ineligible for specialists in applied system theory. It superfluously burdens an application of invariant theory to linear control systems, which this article is devoted. In this connection, basic results of the present work will be presented in terms of ordinary linear spaces, matrices and determinants. One of aims of the article is bringing of the solution of equivalence problem for linear control systems to such level, as it is done at description of invariant properties of a characteristic polynomial of square matrix.

In the present time the equivalence problem is not yet solved completely. In this connection we will solve this problem in a few stages.

On the first from these stages we study the simplified variant of equivalence problem, for which equation (2.2) in system (2.1), (2.2) is absent. In this case we suppose that the group $\mathbb{G}L = \mathbb{G}L(n, \mathbb{R}) \times \mathbb{G}L(m, \mathbb{R})$ and the space $\mathbb{S} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$.

Now we change the base field $\mathbb{R}$ and the group $\mathbb{G}L$ by the base field $\mathbb{C}$ and the special linear group $\mathbb{S}L = \mathbb{S}L(n, \mathbb{C}) \times \mathbb{S}L(m, \mathbb{C})$ saving its action (2.3) on the complex space $\mathbb{S} = \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$.

In this case all relative invariants of group $\mathbb{G}L$ become absolute invariants of group $\mathbb{S}L$. This circumstance allows us to use the Hilbert-Mumford theory [8, 35] for description of all invariants of group $\mathbb{S}L$ with respect to action (2.3). Therefore in further, all absolute and relative invariants we will call simply invariants.

Denote by $\mathbb{C}[\mathbb{S}]^{\mathbb{S}L}$ a ring of all invariants of group $\mathbb{S}L$ with respect to action (2.3) [35, 44]. A search problem of invariants can be essentially simplified if we will take advantage of the following concept [35].

Definition 2.2. (See [35]). The element $\mathbf{w} \in \mathbb{S}$ is called a null-form if for an arbitrary non-constant invariant $I(\cdot) \in \mathbb{C}[\mathbb{S}]^{\mathbb{S}L}$ $I(\mathbf{w}) = 0$.

Let $\mathbf{w}$ be an element of $\mathbb{S}$ and let $\mathbb{H}$ be an arbitrary subgroup of $\mathbb{S}L$. Denote by $\mathbb{O}^{\mathbb{H}}(\mathbf{w}) \subset \mathbb{S}$ an orbit of the point $\mathbf{w}$ with respect to action (2.3) of group $\mathbb{H}$. Let $\overline{\mathbb{O}^{\mathbb{H}}(\mathbf{w})} \subset \mathbb{S}$ be the closure of $\mathbb{O}^{\mathbb{H}}(\mathbf{w})$ in $\mathbb{S}$.

The following theorem is a basic instrument for search of null-forms.

Theorem 2.1. *(See [35]). The element* $\mathbf{w} \subset \mathbb{S}$ *is the null-form if and only if there exists a multiplicative one-parameter subgroup* $\mathbb{H} \subset \mathbb{S}L$ *such that the point* $\mathbf{0} = (0^{n\times n} \times 0^{n\times m}) \in \overline{\mathbb{O}^{\mathbb{H}}(\mathbf{w})}$.

2.1. Null-formes of space $\mathbb{S}$ for system (2.1)

Further, for system (2.1) we will use the designation (A, B), where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$. We will also call the system (A, B) by a system of type (n, m), where numbers n and m are dimensions of the state space and input space. In addition, we will designate the state space and input space by $\mathbb{X} \equiv \mathbb{C}^n$ and $\mathbb{U} \equiv \mathbb{C}^m$.

Without loss of generality, it is possible to consider that $\texttt{rank}B = m$. In addition, we will also assume that $n > m$, since in control theory this case is most important.

The following concept in linear control theory is fundamental.

Definition 2.3. (See [28]). The system (A, B) is called complete controllable if

$$\mathbb{X} = span\{B(\mathbb{U}), AB(\mathbb{U}), ..., A^{n-1}B(\mathbb{U})\}.$$

Let $\mathbb{X}_1 \subset \mathbb{X}$ be an invariant subspace in $\mathbb{X}$ with respect to the following action of operator A: $A\mathbb{X}_1 \subset \mathbb{X}_1$ [15].

We put $\mathbb{U}_1 = B^{-1}(\mathbb{X}_1 \cap B(\mathbb{U})) \subset \mathbb{U}$, where $B^{-1}(\mathbb{X})$ is a complete prototype of operator $B : \mathbb{U} \to \mathbb{X}$ [15].

Denote by $G|_{\mathbb{L}}$ the restriction of operator $G : \mathbb{C}^k \to \mathbb{C}^r$ to subspace $\mathbb{L} \subset \mathbb{C}^k$ [15]. (Here k and r are natural numbers.)

Definition 2.4. The pair of operators $(A|_{\mathbb{X}_1}, B|_{\mathbb{U}_1})$, where $A|_{\mathbb{X}_1}$ and $B|_{\mathbb{U}_1}$ are restrictions of operators A and B to subspaces $\mathbb{X}_1$ and $\mathbb{U}_1$ of dimensionalities n_1 and m_1, is called a subsystem of type (n_1, m_1) of system (A, B).

Definition 2.5. The pair of operators $(A|_{\mathbb{X}/\mathbb{X}_1}, B|_{\mathbb{U}/\mathbb{U}_1})$, where $A|_{\mathbb{X}/\mathbb{X}_1}$ and $B|_{\mathbb{U}/\mathbb{U}_1}$ are restrictions of operators A and B to factor-spaces $\mathbb{X}/\mathbb{X}_1$ and $\mathbb{U}/\mathbb{U}_1$ of dimensionalities $n - n_1$ and $m - m_1$, is called a factor-system of type $(n - n_1, m - m_1)$ of system (A, B) on the subsystem $(A|_{\mathbb{X}_1}, B|_{\mathbb{U}_1})$.

Theorem 2.2. *Let (A, B) be a system of type (n, m), $n > m$. Then the system (A, B) is a null-form if and only if the operator A is nilpotent and (A, B) contains the subsystem $(A|_{\mathbb{X}_1}, B|_{\mathbb{U}_1})$ of type (n_1, m_1), $n_1 \geq m_1$, such that*

$$\frac{n_1}{m_1} < \frac{n}{m}. \tag{2.5}$$

Proof. (a1) It is known [35,44] that in the suitable bases of spaces $\mathbb{C}^n$ and $\mathbb{C}^m$ the one-parametric group $\mathbb{H}$ can be represented by the group of diagonal matrices:

$$\mathbb{H} = \mathbb{H}_1 \times \mathbb{H}_2 = \begin{pmatrix} t^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{-\alpha_n} \end{pmatrix} \times \begin{pmatrix} t^{-\beta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{-\beta_m} \end{pmatrix},$$

where t is a real parameter, and real numbers $\alpha_1, \alpha_2, ..., \alpha_n,\ \beta_1, \beta_2, ..., \beta_m$ are satisfied to the following restrictions:

$$\alpha_1 + \alpha_2 + ... + \alpha_n = 0,\ \beta_1 + \beta_2 + ... + \beta_m = 0. \tag{2.6}$$

Note that if the group $\mathbb{H}$ acts on the system (A, B) by rule (2.3), then the elements of matrices A and B will be transformed on formulas:

$$a_{ik} \to a_{ik} t^{\alpha_i - \alpha_k},\ b_{ij} \to b_{ij} t^{\alpha_i - \beta_j};\ i, k = 1, ..., n; j = 1, ..., m.$$

The trivial system $(0, 0) \subset \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ lies in the closure of orbit $O^{\mathbb{H}}(A, B)$ of system (A, B) if and only if there exist the following limits:

$$\lim_{t\to\infty} \mathbb{H}_1^{-1} A \mathbb{H}_1 = 0,\ \lim_{t\to\infty} \mathbb{H}_1^{-1} B \mathbb{H}_2 = 0.$$

(a2) Both last limits take place if and only if there is a nontrivial joint solution of equalities (2.6) and the following system of inequalities and equalities:

$$b_{ij} = 0 \text{ or } \alpha_i < \beta_j,\ i = 1, ..., n;\ j = 1, ..., m; \tag{2.7}$$

$$a_{ii} = 0,\ 1 \le i \le n; \tag{2.8}$$

$$a_{ik} = 0 \text{ or } \alpha_i < \alpha_k,\ 1 \le k < i \le n. \tag{2.9}$$

First, we suppose that $b_{im} = 0$ for $i = 1, ..., n$. Then inequalities (2.7) replace by inequalities:

$$\alpha_i < \beta_j,\ i = 1, ..., n;\ j = 1, ..., m - 1. \tag{2.10}$$

Further, inequalities (2.7) are shown that numbers α_i (or β_j) can not have one sign. Therefore, we will assume that, for example, $\alpha_n > 0$. Now we choose numbers $\alpha_1, ..., \alpha_n$, which satisfy to conditions (2.7) and $\alpha_n > 0$.

Assume that the number β_i is given so that inequalities (2.10) will be fulfilled. Then the number β_m can be computed on formula (2.6): $\beta_m = -\beta_1 - ... - \beta_{m-1}$.

From here it follows that the system of inequalities (2.10) is solvable. Consequently, systems (2.7) – (2.8) are also solvable. From here it follows that if $n < m$, then taking into account conditions (2.9), which determines the nilpotent matrix A, all systems (A, B) of type (n, m) are null-forms. (By transformations from the group $\mathbb{S}L$ in the matrix B, it is possible to derive one zero column.)

If $n = m$ and $\det B \neq 0$, then the system (A, B) is not a null-form; if $\det B = 0$, then it can be reduced to the case $n < m$. Consequently, at $n > m$ the search of null-forms can be taken to the search of systems (A, B), for which $\texttt{rank} B = m$.

Further, changing bases in the spaces $\mathbb{X}$ and $\mathbb{U}$, it is possible to obtain that at $n_1 \le i \le n$ and $1 \le j \le m_1$ in the matrix B the elements $b_{ij} = 0$. In this case inequalities (2.7) can be transformed in inequalities:

$$\alpha_i < \beta_j,\ i = 1, ..., n_1;\ j = 1, ..., m; \tag{2.11}$$

$$\alpha_i < \beta_j,\ i = n_1 + 1, ..., n;\ j = m_1 + 1, ..., m. \tag{2.12}$$

Summing inequalities (2.11) n_1 times and taking into account the first from equalities (2.6), we get

$$-(\alpha_{n_1+1} + ... + \alpha_n) < n_1 \beta_j,\ j = 1, ..., m. \tag{2.13}$$

It is like easily to get from those inequalities (2.11) that

$$m\alpha_i < \beta_1 + ... + \beta_m = 0, \ i = 1, ..., n_1. \tag{2.14}$$

Now summing inequalities (2.13) up to $j = m_1$ and taking into account the second from equalities (2.6), we have

$$-m_1(\alpha_{n_1+1} + ... + \alpha_n) < -n_1(\beta_{m_1+1} + ... + \beta_m).$$

From here we get relation, which is equivalent to the following inequality:

$$(\alpha_{n_1+1} + ... + \alpha_n) > \frac{n_1}{m_1}(\beta_{m_1+1} + ... + \beta_m). \tag{2.15}$$

Having fulfilled similar computations for system (2.12), we get

$$(\alpha_{n_1+1} + ... + \alpha_n) > \frac{n - n_1}{m - m_1}(\beta_{m_1+1} + ... + \beta_m). \tag{2.16}$$

A comparison of inequalities (2.15) and (2.16) results in the double inequality

$$\frac{n - n_1}{m - m_1}(\beta_{m_1+1} + ... + \beta_m) > (\alpha_{n_1+1} + ... + \alpha_n) > \frac{n_1}{m_1}(\beta_{m_1+1} + ... + \beta_m). \tag{2.17}$$

(a3) From (2.14) and first from equalities (2.6), we have $\alpha_{n_1+1} + ... + \alpha_n > 0$.

Assume that $\beta_{m_1+1} + ... + \beta_m > 0$. Then the solvability of inequality (2.17) is possible only in the case $(n - n_1)/(m - m_1) > n_1/m_1$ (or $n/m > n_1/m_1$). If the case $\beta_{m_1+1} + ... + \beta_m < 0$ takes place, then the restriction $(n - n_1)/(m - m_1) < 0$ must be valid. This inequality results in inequalities: either $n > n_1$ and $m < m_1$ or $n < n_1$ and $m > m_1$ that it is impossible by virtue of the conditions $n > n_1$ and $m > m_1$.

Further, from conditions (2.9), which determine the form of the matrix A, it follows that n_1 is the dimension of the invariant with respect to operator A subspace $\mathbb{X}_1$, which contains subspace $B(\mathbb{U}_1) \subset \mathbb{X}_1$ spanned on m_1 first columns of the matrix B.

Suppose opposite: if in the space $B(\mathbb{U})$ there doesn't exist of the subspace $B(\mathbb{U}_1)$ such that

$$B(\mathbb{U}_1) \subset \mathbb{X}_1 = span\{B(\mathbb{U}_1), AB(\mathbb{U}_1), ..., A^{n-1}B(\mathbb{U}_1)\}$$

and $(\dim \mathbb{X}_1)/(\dim \mathbb{U}_1) = n_1/m_1 < n/m$, then system (A, B) is the null-form. Really, if $n_1/m_1 \geq n/m$, then $(n - n_1)/(m - m_1) \leq n_1/m_1$ and under the condition $\alpha_{n_1+1} + ... + \alpha_n > 0$ inequality (2.17) is incorrect. Consequently, the system of restrictions (2.7) – (2.9) is incompatible. The proof is finished. □

Notice that the nilpotency of matrix A [15] is necessary for an equality to zero of invariants of system (A, B) depending only on A. The equality to zero of invariants of system (A, B) depending on B is got without the use of concept of nilpotency. Thus, the following statement is obvious.

Corollary. *Let A be an arbitrary matrix of order n. Then under condition (2.5) of Theorem 2.2, all invariants of the system (A, B) depending on B are equal to zero.*

3. Stabilizer of system (A, B)

Let $\psi(\lambda)$ be a minimal polynomial of matrix $A \in \mathbb{C}^{n\times n}$ [15]. (It means that polynomial $\psi(\lambda)$ is the nonzero polynomial of the least degree $l \leq n$ such that $\psi(A) = 0$.)

Assume that a nonzero vector from $B(\mathbb{U})$ has a minimal polynomial $f_1(\lambda)$. Suppose also that this polynomial is the polynomial of the least degree among degrees of minimal polynomials of all nonzero vectors from $B(\mathbb{U})$.

Denote by $\mathbb{L}_1$ the set all nonzero vectors from $B(\mathbb{U})$, which have the same minimal polynomial $f_1(\lambda)$. Add the zero vector to the set $\mathbb{L}_1$. Denote the newly obtained set with the same symbol $\mathbb{L}_1$.

Lemma 3.1. *$\mathbb{L}_1$ is the linear subspace in $B(\mathbb{U})$.*

Proof. Let $\mathbb{U}_1 = B^{-1}(\mathbb{L}_1)$ be a complete prototype of space $\mathbb{L}_1$ in $\mathbb{U}$ with respect to the action of operator B. Then it is clear that subspace

$$\mathbb{X}_1 = span\{B(\mathbb{U}_1), AB(\mathbb{U}_1), ..., A^{n-1}B(\mathbb{U}_1)\}$$

is invariant with respect to the action of operator A and it minimal polynomial coincides with $f_1(\lambda)$. $\square$

Let $f_1(\lambda) \neq \psi(\lambda)$. Consider the factor-system $(A|_{\mathbb{X}/\mathbb{X}_1}, B|_{\mathbb{U}/\mathbb{U}_1})$. Then it is possible to find in the factor-space $(B(\mathbb{U}) + \mathbb{X}_1)/\mathbb{X}_1$ the subspace $\mathbb{L}_2/\mathbb{X}_1$, all nonzero vectors of which have the same minimal polynomial $f_2(\lambda)(\bmod\ \mathbb{X}_1)$ [15]. In addition, the degree of this polynomial will be minimum among degrees of all minimal polynomials, which can have nonzero vectors from $(B(\mathbb{U}) + \mathbb{X}_1)/\mathbb{X}_1$.

Now we build the invariant with respect to action of operator A subspace

$$\mathbb{X}_2 = span\{B(\mathbb{U}_2), AB(\mathbb{U}_2), ..., A^{n-1}B(\mathbb{U}_2)\} \subset \mathbb{X}.$$

Here $\mathbb{U}_2 = B^{-1}(\mathbb{L}_2)$ is the complete prototype of space $\mathbb{L}_2$ in $\mathbb{U}$ with respect to the action of operator B. If $\mathbb{X}_2 \neq \mathbb{X}$, then we consider the factor-system $(A|_{\mathbb{X}/\mathbb{X}_2}, B|_{\mathbb{U}/\mathbb{U}_2})$, where $\mathbb{U}_2 = B^{-1}(\mathbb{L}_2)$ is the complete prototype of subspace $\mathbb{L}_2$ in $\mathbb{U}$ with respect to the action of operator B, and so on. By virtue of finite dimensionality of the spaces $\mathbb{X}$ and $\mathbb{U}$ the indicated procedure will be finished on some stage r:

$$\mathbb{U}_r = \mathbb{U}, \quad \mathbb{X}_r = \mathbb{X} = span\{B(\mathbb{U}_r), AB(\mathbb{U}_r), ..., A^{n-1}(\mathbb{U}_r)\}.$$

Thus, we have the row of embedded in each other subsystems

$$(0, 0) \subset (A|_{\mathbb{X}_1}, B|_{\mathbb{U}_1}) \subset (A|_{\mathbb{X}_2}, B|_{\mathbb{U}_2}) \subset ... \subset (A|_{\mathbb{X}_r}, B|_{\mathbb{U}_r}) = (A, B) \tag{3.1}$$

such that any factor-system of this row possesses the following property: the state space $\mathbb{X}_i/\mathbb{X}_{i-1}$ of system $(A|_{\mathbb{X}_i/\mathbb{X}_{i-1}}, B|_{\mathbb{U}_i/\mathbb{U}_{i-1}})$ is generated by the factor-space $\mathbb{L}_i/\mathbb{X}_{i-1} \subset (B(\mathbb{U})+\mathbb{X}_i)/\mathbb{X}_{i-1}$, all nonzero vectors of which have the same minimal polynomial $f_i(\lambda)(\text{mod } \mathbb{X}_{i-1})$, $i = 1, ..., r$.

Definition 3.1. (See [20]). The maximal subgroup $\mathbb{S}tab_{\mathbb{S}L}(A, B)$ of group $\mathbb{S}L$ is given by the conditions

$$\mathbb{S}tab_{\mathbb{S}L}(A, B) = \{(S, T) \in \mathbb{S}L \mid AS = SA,\ BT = SB\}$$

is called a stabilizer of system (A, B).

Lemma 3.2. *Let $(S, T) \in \mathbb{S}tab_{\mathbb{S}L}(A, B)$ be an arbitrary element and $(A|_{\mathbb{X}_i}, B|_{\mathbb{U}_i})$ be an arbitrary system of row (3.1). Then the following inclusion takes place:*

$$(S, T) \cdot (A|_{\mathbb{X}_i}, B|_{\mathbb{U}_i}) \in (A|_{\mathbb{X}_i}, B|_{\mathbb{U}_i}),\ i = 1, ..., r. \tag{3.2}$$

Proof. We prove justice (3.1) for $i = 1$. Lemma 3.1 asserts that the subspace $\mathbb{L}_1$ is uniquely. Then from the definition of stabilizer we get that $SB(\mathbb{U}_1) = \mathbb{U}_1 \subset BT(\mathbb{U}) \subset B(\mathbb{U})$ and, by virtue of uniqueness of $B(\mathbb{U}_1)$ in $B(\mathbb{U})$, we have $SB(\mathbb{U}_1) \subset B(\mathbb{U}_1)$. Further, from the uniqueness $B(\mathbb{U}_1)$ and that $\mathbb{U}_1 = B^{-1}(\mathbb{L}_1)$ is the complete prototype $\mathbb{L}_1$ in $\mathbb{U}$, it follows that the subspace $\mathbb{U}_1$ is a unique in $\mathbb{U}$. It means that $T(\mathbb{U}_1) \subset \mathbb{U}_1$. In addition, we have

$$S(\mathbb{X}_1) = S\sum_{i=1}^{n-1} A^i B(\mathbb{U}_1) = \sum_{i=1}^{n-1} A^i SB(\mathbb{U}_1) = \sum_{i=1}^{n-1} A^i BT(\mathbb{U}_1) \subset \sum_{i=1}^{n-1} A^i B(\mathbb{U}_1) \subset \mathbb{X}_1.$$

The proof for case $i = 1$ is finished.

Now we lead an induction on i. Assume that for some $i < r$

$$(S, T) \cdot (A|_{\mathbb{X}_i}, B|_{\mathbb{U}_i}) \in (A|_{\mathbb{X}_i}, B|_{\mathbb{U}_i}).$$

Consider the factor-system $(A|_{\mathbb{X}_i/\mathbb{X}_{i-1}}, B|_{\mathbb{U}_i/\mathbb{U}_{i-1}})$. Then the proof of inclusion (3.2) at $i = 1$ word for a word is carried on the proof of inclusion

$$(S, T) \cdot (A|_{\mathbb{X}_i/\mathbb{X}_{i-1}}, B|_{\mathbb{U}_i/\mathbb{U}_{i-1}}) \in (A|_{\mathbb{X}_i/\mathbb{X}_{i-1}}, B|_{\mathbb{U}_i/\mathbb{U}_{i-1}}).$$

Taking into account the supposition of induction, we get the inclusion $(S, T) \cdot (A|_{\mathbb{X}_{i+1}}, B|_{\mathbb{U}_{i+1}}) \in (A|_{\mathbb{X}_{i+1}}, B|_{\mathbb{U}_{i+1}})$. It completes the proof of Lemma 3.2. □

Lemma 3.3. *Let (A, B) be a complete controllable system of type $(n, m), n \geq m, \mathtt{rank} B = m$, and all nonzero vectors of space $B(\mathbb{U})$ have the same minimal polynomial $f(\lambda)$. Then $\mathbb{S}tab_{\mathbb{S}L}(A, B) \cong \mathbb{S}L(m, \mathbb{C})$.*

Proof. We prove that system (A, B) is the direct sum of m copies the irreducible subsystems. (We remind that the system (A, B) is called irreducible if it contains only trivial subsystems $(0, 0)$ and (A, B) [44].)

At $m = 1$ the assertion of Lemma 3.3 is obvious. Assume that it is correctly for all $k \leq m-1$. Now we have to prove that (A, B) is the direct sum of equivalent irreducible subsystems for $k = m$.

Let $l = \deg f(\lambda)$. Choose in $B(\mathbb{U})$ an arbitrary base $b_1, ..., b_m$. Denote by the symbol $\mathbb{X}_i = \{b_i, Ab_i ..., A^{l-1}b_i\} \subset \mathbb{X}$ a cyclic with respect to action of operator A subspace in $\mathbb{X}$ [15]. (It is clear that $\dim \mathbb{X}_i = l,\ i = 1, ..., m$.)

Since the system (A, B) is the complete controllable, then we have $\mathbb{X} = \mathbb{X}_1 + ... + \mathbb{X}_m$. On the supposition of induction the last sum can be rewritten as $\mathbb{X} = \mathbb{X}_1 \oplus ... \oplus \mathbb{X}_{m-1} + \mathbb{X}_m \equiv \tilde{\mathbb{X}} + \mathbb{X}_m$. We will prove that this sum is direct.

Assume that the space $\mathbb{P} \equiv \tilde{\mathbb{X}} \cap \mathbb{X}_m \neq \emptyset$. It is clear that the space $\mathbb{P}$ is a proper cyclic subspace in $\mathbb{X}_m$ and $\tilde{\mathbb{X}}$.

Let $h(\lambda)$ be the minimal polynomial of space $\mathbb{P}$ (note that $\deg h(\lambda) < \deg f(\lambda)$). In this case there exists the vector $\alpha_1 b_1 + ... + \alpha_{m-1} b_{m-1} \in \tilde{\mathbb{X}}$ and the polynomial $g(\lambda) = f(\lambda)/h(\lambda)$ such that $g(A)(\alpha_1 b_1 + ... + \alpha_{m-1} b_{m-1}) \in \mathbb{P}$, and $g(A) b_m \in \mathbb{P}$.

Here two following cases are possible.

(i) $g(A)(\alpha_1 b_1 + ... + \alpha_{m-1} b_{m-1}) = g(A)(\alpha_m b_m)$. Then the polynomial $g(\lambda)$ is minimal for the nonzero vector $\alpha_1 b_1 + ... + \alpha_{m-1} b_{m-1} - \alpha_m b_m$. Since $\deg h(\lambda) < \deg f(\lambda)$, then this situation is impossible.

(ii) Let the space $\mathbb{Q} = \{g(A) b_1, ..., g(A) b_m\} \in \mathbb{P}$ be spanned on the vectors $g(A) b_1, ..., g(A) b_m$; it will be at least 2-dimensional. It is known that in any cyclic space of dimension k there exist cyclic spaces of all dimensions less than k [15].

Therefore, if $\mathbb{P}_1 \subset \mathbb{P}$ is a cyclic space of dimension $\deg_{\mathbb{C}} h(\lambda) - 1$, then $\mathbb{P}_1 \cap \mathbb{Q} = g(A) b \neq \emptyset$, where $b \in B(\mathbb{U})$ is a nonzero vector. From here it follows that there exists the polynomial $h_1(\lambda)$ such that $h_1(A) g(A) b = 0$. (Here $\deg_{\mathbb{C}} h_1(\lambda) g(\lambda) < \deg f(\lambda)$.) Again we get that the situation (ii) is impossible.

Thus, we have to have $P = \emptyset$ and $\mathbb{X} = \mathbb{X}_1 \oplus ... \oplus \mathbb{X}_m$. In addition, $B(\mathbb{U}) = \mathbb{X}_1 \cap B(\mathbb{U}) \oplus ... \oplus \mathbb{X}_m \cap B(\mathbb{U}) = b_1 \oplus ... \oplus b_m$. Consequently, there exist bases of the spaces $\mathbb{X}$ and $\mathbb{U}$, in which the matrices A and B can be represent in the following form:

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_1 \end{pmatrix}, \tag{3.3}$$

where $A_1 = A|_{\mathbb{X}_i},\ B_1 = b_i; i = 1, ..., m$. Now the statement of Lemma 3.3 follows from representation (3.3). $\square$

Denote by $\mathbb{X} \setminus \mathbb{X}_1$ a complement of $\mathbb{X}_1$ with respect to whole space $\mathbb{X}$. (Note that $\dim_{\mathbb{C}} \mathbb{X} \setminus \mathbb{X}_1 = \dim_{\mathbb{C}} \mathbb{X}/\mathbb{X}_1$, where $\dim_{\mathbb{C}} \mathbb{X}/\mathbb{X}_1$ is a dimension of factor-space $\mathbb{X}/\mathbb{X}_1$.)

We choose the bases of spaces $\mathbb{U}$ and $\mathbb{X}$ in accordance with Lemmas 3.1, 3.2, and 3.3. More precisely, the base $B(\mathbb{U})$ is formed by the bases of spaces

$$B(\mathbb{U}) \cap \mathbb{X}_1, B(\mathbb{U}) \cap (\mathbb{X}_2 \setminus \mathbb{X}_1), ..., B(\mathbb{U}) \cap (\mathbb{X}_m \setminus \mathbb{X}_{m-1}), \tag{3.4}$$

and the base $\mathbb{X}$ is formed by the bases of spaces

$$\mathbb{X}_1, \mathbb{X}_2 \setminus \mathbb{X}_1, ..., \mathbb{X}_m \setminus \mathbb{X}_{m-1}, \tag{3.5}$$

where each of bases $\mathbb{X}_{i+1} \setminus \mathbb{X}_i, i = 0, 1, ..., m-1 (\mathbb{X}_0 = 0)$ it is an association of bases of nonintersecting isomorphic cyclic spaces. Then matrices of operators A and B in bases (3.4) and (3.5) can be represented in the following forms:

$$A = \begin{pmatrix} A_{11} & & A_{1r} \\ & \ddots & \\ 0 & & A_{rr} \end{pmatrix}, B = \begin{pmatrix} B_{11} & & 0 \\ & \ddots & \\ 0 & & B_{rr} \end{pmatrix}, \tag{3.6}$$

where

$$A_{ii} = \begin{pmatrix} A_i & & 0 \\ & \ddots & \\ 0 & & A_i \end{pmatrix}, A_{ij} = \begin{pmatrix} A_{ij}^{(11)} & \dots & A_{ij}^{(1,m_j)} \\ \vdots & \dots & \vdots \\ A_{ij}^{(m_1,1)} & \dots & A_{ij}^{(m_i,m_j)} \end{pmatrix},$$

$$A_i = \begin{pmatrix} 0 & & 0 & a_{l_i}^{(i)} \\ 1 & & 0 & \vdots \\ & \ddots & & \vdots \\ 0 & & 1 & a_1^{(i)} \end{pmatrix}, A_{ij}^{(pq)} = \begin{pmatrix} 0 & \dots & 0 & * \\ \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & * \end{pmatrix},$$

$$B_{ii} = \begin{pmatrix} B_i & & 0 \\ & \ddots & \\ 0 & & B_i \end{pmatrix}, B_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here m_i is a multiple of including of block A_i in A_{ii} (B_i in B_{ii}); the matrix A_i has sizes $l_i \times l_i$, and matrix B_i has sizes $l_i \times 1$; $p = 1, ..., m_i; q = 1, ..., m_j$; $i, j = 1, ..., r; i < j$. (In addition, in matrices $A_{ij}^{(pq)}$ only the last column is not equal to zero.)

Let (3.6) be the matrix represent of system (A, B). Denote by $\mathbb{N} = \mathbb{N}_{\mathbb{X}} \times \mathbb{N}_{\mathbb{U}}$ a subset of group $\mathbb{S}L$, which determined as follows:

$$\mathbb{N}_{\mathbb{X}} \subset \mathbb{S}L(n, \mathbb{C}),\ \mathbb{N}_{\mathbb{U}} \subset \mathbb{S}L(m, \mathbb{C});$$

$$\mathbb{N}_{\mathbb{U}} = \begin{pmatrix} \mathbb{S}L(m_1, \mathbb{C}) & & \Gamma_{ij} \\ & \ddots & \\ 0 & & \mathbb{S}L(m_r, \mathbb{C}) \end{pmatrix}, \Gamma_{ij} = \begin{pmatrix} \gamma_{ij}^{(11)} & \dots & \gamma_{ij}^{(1,m_j)} \\ \vdots & \dots & \vdots \\ \gamma_{ij}^{(m_1,1)} & \dots & \gamma_{ij}^{(m_i,m_j)} \end{pmatrix};$$

$$\mathbb{N}_{\mathbb{X}} = \begin{pmatrix} \mathbb{S}L(m_1, \mathbb{C}) \otimes E_{l_1} & & L_{ij} \\ & \ddots & \\ 0 & & \mathbb{S}L(m_r, \mathbb{C}) \otimes E_{l_r} \end{pmatrix}.$$

Here $m_1 + ... + m_r = m$; $L_{ij} = \gamma_{ij}^{(pq)} \cdot (B_i, ..., A_i^{l_j - 1} B_i)$; $\gamma_{ij}^{(pq)}$ are arbitrary parameters; E_{l_i} are identity matrices of orders l_i; $m_1 l_1 + ... + m_r l_r = n$; $p = 1, ..., m_i; q = 1, ..., m_j$; $i < j; , j = 1, ..., r$. In this case it is easy to show that the following assertion takes place.

Lemma 3.4. *There exist bases of the spaces* $\mathbb{X}$ *and* $\mathbb{U}$*, in which* $\mathbb{S}tab_{\mathbb{S}L}(A, B) \subset \mathbb{N}$.

There are a lot of different variants of canonical forms for pair matrices (A, B) (see, for example [20], [33], [45] – [47], [50]). In order to build the stabilizer of system (A, B), we represented own variant (3.6) of such canonical form.

4. Actions of group $\mathbb{S}L$ on space $\mathbb{S}$

Denote by $\mathbb{S}/\mathbb{S}L$ a set of orbits all points from the space $\mathbb{S} = \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ with respect to action (2.3) of group $\mathbb{S}L$.

It is known that for a successful solution of invariant description of system (2.1) it is necessary to supply the space $\mathbb{S}$ (or some suitable subset $\mathbb{W} \subset \mathbb{S}$) by a projective variety structure [40], [45] – [47]. The points of this set are called stable. An exact definition of the set of stable points is such.

Definition 4.1. The system $(A, B) \in \mathbb{S}$ is called stable with respect to action (2.3) of group $\mathbb{S}L$, if there exists an open projective $\mathbb{S}L$-invariant set $\mathbb{W} \subset \mathbb{S}$ such that $(A, B) \in \mathbb{W}$ and orbits of all systems from $\mathbb{W}$ are closed in $\mathbb{W}$. A set $\mathbb{F}_s \subset \mathbb{S}$ consisting of all stable points is called stable.

Definition 4.2. The point (A, B) is called regular if the dimension of orbit $\mathbb{O}^{\mathbb{S}L}(A, B)$ of this point is maximal in $\mathbb{S}$.

Further, orbits of all points from $\mathbb{F}_s$ have the equal (maximal) dimension. Consequently, the stabilizers of these points must have the minimal dimension. Since for regular points is just the equality $\dim_{\mathbb{C}} \mathbb{S}tab_{\mathbb{S}L}(A, B) = 0$, then on the role of stable systems can pretended regular systems only [32, 35, 44]. (We note that the dimension of space of orbits $\mathbb{O}^{\mathbb{G}}(\mathbb{S})$ is given by the formula

$$\dim_{\mathbb{C}} \mathbb{O}^{\mathbb{S}L}(\mathbb{S}) = n^2 + nm - \dim_{\mathbb{C}} \mathbb{S}L(n, \mathbb{C}) - \dim_{\mathbb{C}} \mathbb{S}L(m, \mathbb{C}) = m(n-m) + 2.) \quad (4.1)$$

Theorem 4.1. *Let* (A, B) *be a system of type* (n, m), $n > m$. *Suppose that at least one of the following conditions is fulfilled:*

(i) $m = 1$ *and* (A, B) *does not contain nontrivial subsystems;*

(ii) $m > 1$ *and for an arbitrary nontrivial subsystem* $(A|_{\mathbb{X}_i}, B|_{\mathbb{U}_i}) \in (A, B)$ *of type* (n_i, m_i), $n_i > m_i$, *where* $i \in \mathbb{N}$ *and* $\mathbb{N}$ *is a set of indexes, the inequality*

$$\frac{n_i}{m_i} > \frac{n}{m} \quad (4.2)$$

takes place. Then the system (A, B) *is regular.*

Corollary. *Assume also that the numbers n and m are coprime. Then under the conditions of Theorem 4.1, the system* $(A, B) \in \mathbb{F}_s$.

Proof. The proof of this theorem is based on Lemmas 3.1 – 3.4. In essence, it is a modification of the proof proposed in [3]. □

The following example shows an importance of concept of coprime numbers. Consider the set $\mathbb{S}$ of systems of type $(4, 2)$. Let (A, B) be an arbitrary system from this set. Denote by $f(A, B) = \det(B, AB)$ an invariant polynomial of system (A, B). Really, if $(S, T) \in \mathbb{S}L$, then we have

$$\begin{aligned} f(S^{-1}AS, S^{-1}BT) &= \det(S^{-1}BT, S^{-1}ABT) \\ &= ((\det T)^2/\det S)\det(B, AB) = f(A, B). \end{aligned}$$

Consider the following system of type $(4, 2)$:

$$A = \begin{pmatrix} 0 & d_2 & 0 & c_1 \\ 1 & d_1 & 0 & c_2 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & 1 & d_1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.3}$$

where $c_1, c_2, d_1, d_2 \in \mathbb{C}$, and $c_1 \neq 0, c_2 \neq 0$. In this case, we get

$$\mathbb{S}tab_{\mathbb{S}L}(A, B) = \{(S, T)\} = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \alpha \neq 0.$$

Thus, we have $f(A, B) = 1 \neq 0$ and $\dim_{\mathbb{C}} \mathbb{S}tab_{\mathbb{S}L}(A, B) = 1$. Therefore, system (4.3) is not stable.

5. Invariant description of null-forms for system (A, B)

In Section 2 the existence conditions of null-forms were got in terms of some special subspaces in $\mathbb{X}$ and $\mathbb{U}$. However, for a description of ring of invariants $\mathbb{C}[\mathbb{S}]^{\mathbb{S}L}$, it is necessary to get these conditions in invariant terms of matrix pair (A, B) with respect to action (2.3) of group $\mathbb{S}L$.

Consider the matrix

$$R(A, B) = (B, AB, ..., A^{n-1}B) \in \mathbb{C}^{n \times nm}. \tag{5.1}$$

Then action (2.3) of group $\mathbb{S}L$ on space $\mathbb{C}^{n \times (n+m)}$ induces an action of the same group on space $\mathbb{C}^{n \times nm}$ by the following formula:

$$R(S^{-1}AS, S^{-1}BT) = S^{-1} \cdot R(A, B) \cdot \begin{pmatrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{pmatrix} = S^{-1} \cdot R(A, B) \cdot \bigoplus_{i=1}^{n} T. \tag{5.2}$$

Many questions of search of invariants for linear control systems may be related to problem of decomposability of polyvectors, which are constructed from vectors of linear space $\mathbb{C}^n$. The investigation relationships between polyvectors of $\mathbb{C}^n$, alternating multilinear forms on $\mathbb{C}^n$, hyperplanes of projective Grassmannians and regular spreads of projective spaces, it were represented in [10]. We use some constructions of this article in our own researches.

Denote by $\mathcal{M}$ a linear space spanned on minors of order n of matrix (5.1). Then in virtue of (5.2) $\mathbb{S}L(\mathcal{M}) \subset \mathcal{M}$. On the space $\mathcal{M}$ group $\mathbb{S}L_n$ acts by the multiplication on scalar $\det S = 1$, and group $\mathbb{S}L_m$ acts in accordance with representation $\bigwedge^n(\bigoplus\limits_{i=1}^{n} T)$. It is known [35, 37] that for search of all homogeneous polynomial invariants substantially depending on B, it is necessary to find a decomposition of the indicated representation on irreducible components. In obedience to the known result of representation theory of groups an arbitrary irreducible representation of group $\mathbb{S}L_m(\mathbb{C})$ is a tensor product of polyvector representations [10,19]. Then the decomposition on irreducible components have the form:

$$\bigwedge^n\left(\bigoplus_{i=1}^{n} T\right) = \bigoplus_{\omega=(n_1,\dots,n_d)} r_\omega \wedge^{n_1} T \otimes \wedge^{n_2} T \otimes \dots \otimes \wedge^{n_d} T,$$

where the summation is taken over all multiindexes $\omega = (n_1, \dots, n_d)$ such that $n_1 + \dots + n_d = n$ and $m \geq n_1 \geq \dots \geq n_d \geq 1$; r_ω is a multiple of appropriate irreducible representation in representation $\bigwedge^n(\bigoplus\limits_{i=1}^{n} T)$.

Below, we construct some examples of invariants for systems of type (n, m).

5.1. Invariants of systems of type $(4, 2)$

In future by a character $I_j(A, B); j \in \mathbb{J}$, we will designate an invariant of system (A, B), where $\mathbb{J}$ is a set of indexes.

1) $\omega = (n_1, n_2) = (2, 2)$; $r_\omega = 1$; $T = \mathbb{S}L(2, \mathbb{C})$, and we consider the representation $T \to (\wedge^2 T) \otimes (\wedge^2 T)$. In this case, $I_j(A, B) = \det(A^{i_1} B, A^{i_2} B)$, where i_1, i_2 are positive integers.

2) $\omega = (n_1, n_2, n_3) = (2, 1, 1)$; $r_\omega = 1$; $T = \mathbb{S}L(2, \mathbb{C})$, and we consider the representation $T \to (\wedge^2 T) \otimes T$. In this case

$$I_j(A, B) = \det \begin{pmatrix} \det(B, A^{i_1} b_1, A^{i_2} b_1) & \det(B, A^{i_1} b_1, A^{i_2} b_2) \\ \det(B, A^{i_1} b_2, A^{i_2} b_1) & \det(B, A^{i_1} b_2, A^{i_2} b_2) \end{pmatrix},$$

where (b_1, b_2) are columns of matrix B, and i_1, i_2 are positive integers.

5.2. Invariants of systems of type $(5, 2)$

1) $\omega = (n_1, n_2, n_3) = (2, 2, 1)$; $r_\omega = 1$; $T = \mathbb{S}L(2, \mathbb{C})$, and we consider the representation $T \to (\wedge^2 T) \otimes (\wedge^2 T) \otimes T$. In this case

$$I_j(A, B) = \det \begin{pmatrix} \det(B, A^{i_1} B, A^{i_2} b_1) & \det(B, A^{i_1} B, A^{i_2} b_2) \\ \det(B, A^{i_1} B, A^{i_3} b_1) & \det(B, A^{i_1} B, A^{i_3} b_2) \end{pmatrix},$$

where (b_1, b_2) are columns of matrix B, and i_1, i_2, i_3 are positive integers.

5.3. Invariants of systems of type $(5, 3)$

1) $\omega = (n_1, n_2) = (3, 2)$; $r_\omega = 1$; $T = \mathbb{S}L(3, \mathbb{C})$, and we consider the representation $T \to (\wedge^3 T) \otimes (\wedge^2 T)$. In this case

$$I_j(A, B)$$

$$= \det \begin{pmatrix} \det(B, A^{i_1}b_1, A^{i_1}b_2) & \det(B, A^{i_1}b_1, A^{i_1}b_3) & \det(B, A^{i_1}b_2, A^{i_1}b_3) \\ \det(B, A^{i_2}b_1, A^{i_2}b_2) & \det(B, A^{i_2}b_1, A^{i_2}b_3) & \det(B, A^{i_2}b_2, A^{i_2}b_3) \\ \det(B, A^{i_3}b_1, A^{i_3}b_2) & \det(B, A^{i_3}b_1, A^{i_3}b_3) & \det(B, A^{i_3}b_2, A^{i_3}b_3) \end{pmatrix},$$

where (b_1, b_2, b_3) are columns of matrix B, and i_1, i_2, i_3 are positive integers.

2) $\omega = (n_1, n_2, n_3) = (3, 1, 1)$; $r_\omega = 1$; $T = \mathbb{S}L(3, \mathbb{C})$, and we consider the representation $T \to (\wedge^3 T) \otimes T \otimes T$. In this case

$$I_j(A, B)$$

$$= \det \begin{pmatrix} \det(B, A^{i_1}b_1, A^{i_2}b_1) & \det(B, A^{i_1}b_1, A^{i_2}b_2) & \det(B, A^{i_1}b_1, A^{i_2}b_3) \\ \det(B, A^{i_1}b_2, A^{i_2}b_1) & \det(B, A^{i_1}b_2, A^{i_2}b_2) & \det(B, A^{i_1}b_2, A^{i_2}b_3) \\ \det(B, A^{i_1}b_3, A^{i_3}b_1) & \det(B, A^{i_1}b_3, A^{i_3}b_2) & \det(B, A^{i_1}b_3, A^{i_3}b_3) \end{pmatrix},$$

where (b_1, b_2, b_3) are columns of matrix B, and i_1, i_2, i_3 are positive integers.

Now we can begin to construct null-forms for systems of type (n, m). Denote by $a_i(A), i = 1, ..., n$, the coefficients of characteristic polynomial of matrix A. In addition, by the symbol $\mathbb{W}^\circ(n, m)$ we will denote a variety of all null-forms of space $\mathbb{S}$ with respect to action (2.3) of group $\mathbb{S}L$.

5.4. Null-forms of systems of type $(m + 1, m)$, $m > 1$

Construct the following matrix:

$$R(A, B) = \begin{pmatrix} \det(B, Ab_1), & ... & , \det(B, Ab_m) \\ \vdots & ... & \vdots \\ \det(B, A^m b_1), & ... & , \det(B, A^m b_m) \end{pmatrix} \in \mathbb{C}^{m \times m}. \tag{5.3}$$

Consider the invariant $I_1(A, B) = \det R(A, B)$ of matrix (5.3).

Theorem 5.1. *Let $n = m + 1,\ m \geq 1$. Then*

$$\mathbb{W}^\circ(m+1, m) = \{(A, B) \in \mathbb{C}^{n \times (n+m)} \mid a_1(A) = ... = a_n(A) = 0 \text{ and } I_1(A, B) = 0\}.$$

Proof. Assume that equality $I_1(A, B) = 0$ takes place. Then form (5.3) of matrix $R(A, B)$ allows to assert that columns of this matrix are linearly dependent. In other words, there exist numbers $\alpha_1, ..., \alpha_m \in \mathbb{C}$ not all equal to zero such that

$$\begin{pmatrix} \det(B, A(\alpha_1 b_1 + ... + \alpha_m b_m)) \\ \vdots \\ \det(B, A^m(\alpha_1 b_1 + ... + \alpha_m b_m)) \end{pmatrix} = 0. \tag{5.4}$$

It is possible to consider that $\mathtt{rank}B = m$. (Otherwise the system (A, B) is the null-form.) Then from (5.4) it follows that $A^i(\alpha_1 b_1 + ... + \alpha_m b_m) \in B(\mathbb{U}), i = 1, ..., n-1$, and, consequently, the space $B(\mathbb{U})$ contains the invariant subspace $\mathbb{V}$ spanned on the vectors

$$(\alpha_1 b_1 + ... + \alpha_m b_m), A(\alpha_1 b_1 + ... + \alpha_m b_m), ..., A^{n-1}(\alpha_1 b_1 + ... + \alpha_m b_m).$$

It is known that an arbitrary invariant space of matrix A contains an eigenvector of this matrix [15]. Thus, there is a nonzero vector $b \in \mathbb{V} \subset B(\mathbb{U})$ such that $Ab = \lambda b$, where λ is an eigenvalue of matrix A. It means that in the system (A, B) there exists the subsystem of type $(1, 1)$. In addition, if $a_1(A) = ... = a_n(A) = 0$, then in accord to Theorem 2.2 the system (A, B) is the null-form. □

5.5. Null-forms of systems of type $(2m, m)$, $m > 1$

Denote by $\alpha_1, ..., \alpha_n$ eigenvalues of matrix A and let

$$\mathrm{disc}(A) = (\alpha_2 - \alpha_1)^2(\alpha_3 - \alpha_1)^2 \cdot ... \cdot (\alpha_n - \alpha_1)^2 \cdot ... \cdot (\alpha_n - \alpha_{n-1})^2$$

be a discriminant of this matrix.

Let

$$Bin(\eta, \xi) = \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \frac{\eta!}{\xi!(\eta - \xi)!} \equiv \frac{\eta \cdot (\eta - 1) \cdot ... \cdot (\eta - \xi + 1)}{1 \cdot ... \cdot \xi}$$

be a binomial coefficient. Here η, ξ are positive integers, $0 \le \xi \le \eta$; $0! = 1! = 1$.

Introduce the invariants

$$I_j(A, B) = \det(B, A^{j-1}B), \qquad (5.5)$$

where $j = 1, ..., Bin(2m, m)/2$. (Note that $I_1(A, B) \equiv 0$.)

Theorem 5.2. *Let $n = 2m$, $m \ge 1$. Then*

$$\mathbb{W}^\circ(2m, m) = \{(A, B) \in \mathbb{C}^{n \times (n+m)} \mid a_1(A) = ... = a_n(A) = 0 \text{ and } I_j(A, B) = 0\},$$

where $I_j(A, B)$ are invariants (5.5).

Proof. Denote by $\mathbb{M}$ an open set in $\mathbb{C}^{n \times (n+m)}$ is given by the condition:

$$\mathbb{M} = \{(A, B) \in \mathbb{C}^{n \times (n+m)} \mid \mathrm{disc}(A) \ne 0\}.$$

Let $(A, B) \in \mathbb{M}$ be an arbitrary system. Then it is possible to consider that in a suitable base of space $\mathbb{X}$ the matrix $A = \mathtt{diag}\,(()\,\alpha_1, ..., \alpha_{2m})$.

Denote by $\Delta_{j_1 ... j_m}$ a minor located in the rows $1 \le j_1 < ... < j_m \le 2m$ of matrix B.

Let $\gamma_1 = \alpha_1 \cdot ... \cdot \alpha_m$, $\delta_1 = \alpha_{m+1} \cdot ... \cdot \alpha_{2m}$, ..., $\gamma_k = \alpha_{i_i} \cdot ... \cdot \alpha_{i_m}$, $\delta_k = \alpha_{j_1} \cdot ... \cdot \alpha_{j_m}$, where $i_1, ..., i_m, j_1, ..., j_m \in \{1, 2, ..., 2m\}$; $1 \le i_1 < ... < i_m \le 2m$, $1 \le j_1 < ... < j_m \le 2m$, and $(i_1, ..., i_m) \cap (j_1, ..., j_m) = \emptyset$; $k = Bin(2m, m)/2$.

Construct the matrix

$$Q(\alpha_1,...,\alpha_{2m}) = \begin{pmatrix} 2 & \cdots & 2 & \cdots & 2 \\ \gamma_1+\delta_1 & \cdots & \gamma_{i_1}+\delta_{j_1} & \cdots & \gamma_k+\delta_k \\ \gamma_1^2+\delta_1^2 & \cdots & \gamma_{i_1}^2+\delta_{j_1}^2 & \cdots & \gamma_k^2+\delta_k^2 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_1^{k-1}+\delta_1^{k-1} & \cdots & \gamma_{i_1}^{k-1}+\delta_{j_1}^{k-1} & \cdots & \gamma_k^{k-1}+\delta_k^{k-1} \end{pmatrix} \in \mathbb{C}^{k\times k}.$$

Taking into account the matrix $Q(\alpha_1,...,\alpha_{2m})$, the conditions $I_j(A,B)=0$ may be rewritten as

$$\begin{pmatrix} I_1 \\ \vdots \\ I_k \end{pmatrix} = Q(\alpha_1,...,\alpha_{2m})\cdot \begin{pmatrix} \Delta_{1...m}\cdot\Delta_{m+1...2m} \\ \vdots \\ \Delta_{i_1...i_m}\cdot\Delta_{j_1...j_m} \end{pmatrix}. \tag{5.6}$$

Assume that $\det Q(\alpha_1,...,\alpha_{2m}) \neq 0$. It is clear that if $I_1 = ... = I_k = 0$, then for all indicated above possible collections of indexes $i_1...i_m, j_1...j_m$ such that $\{i_1...i_m\}\cap\{j_1...j_m\} = 0$, system (5.6) has the trivial solutions only: $\Delta_{i_1...i_m}\cdot\Delta_{j_1...j_m} = 0$.

Further, invariant (5.5) can be written as:

$$\det(B, A^{j-1}B) = \sum_{i=1}^{k}(-1)^i f_{ij}(\alpha_1,...,\alpha_{2m})\Delta_{i_1...i_k}\cdot\Delta_{j_1...j_k}, \tag{5.7}$$

where $\{i_1,...,i_m,j_1...j_m\} = \{1,...,2m\}$, $1\le i_1<...<i_m\le 2m$; $1\le j_1<...<j_m\le 2m$.

From (5.7) it follows that equalities $\Delta_{i_1...i_m}\cdot\Delta_{j_1...j_m}=0$ take place if and only if the matrix (B,B) has the following form:

$$(B,B) = \left(\begin{array}{cccccc||cccccc} * & \cdots & * & * & \cdots & * & * & \cdots & * & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ * & \cdots & * & * & \cdots & * & * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 & * & \cdots & * \\ \vdots & N & \vdots & \vdots & \cdots & \vdots & \vdots & N & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 & * & \cdots & * \end{array}\right) \in \mathbb{C}^{2m\times 2m},$$

where $N\in\mathbb{C}^{(2m-2l+1)\times l}$ is the zero submatrix of matrix B and the symbol $*$ designates a nonzero element of matrix B; $l = m-1, m-2, ..., 1$.

It means that the space $B(\mathbb{U})$ intersects with the invariant subspace $\mathbb{V}\subset\mathbb{C}^{2m}(A\mathbb{V}\subset\mathbb{V})$ such that $\dim_{\mathbb{C}}\mathbb{V}\le 2l-1$ and $\dim_{\mathbb{C}} B(\mathbb{U})\cap\mathbb{V} = l$, where $l = m-1,...,1$. Consequently, the system (A,B) of type $(2m,m)$ contains a subsystem of type $(2l-1,l)$; $l\in\{m-1,...,1\}$. Thus, we have $(2l-1)/l < 2m/m = 2$ for $l>0$.

Denote by $\mathbb{K}$ an open set in $\mathbb{C}^{n\times(n+m)}$ is given by the following condition:

$$\mathbb{K} = \{(A,B) \in \mathbb{C}^{n\times(n+m)} \mid \det Q(\alpha_1,...,\alpha_{2m}) \cdot \mathrm{disc}(A) \neq 0\}.$$

It is known [44] that if the regular function $I_j(A,B) = 0$ on the open set $\mathbb{K} \subset \mathbb{C}^{n\times(n+m)}$, then this function is equal to zero everywhere on $\mathbb{C}^{n\times(n+m)}$. Now we add conditions $I_j(A,B) = 0, j = 2,...,k$, by conditions $a_i(A) = 0, i = 1,...,2m$. Then according to Theorem 2.2, we obtain that (A,B) is the null-form. □

5.6. Null-forms of systems of type (pm,m), $p > 2$, $m > 1$

In this subsection the results of previous subsection will be generalized.
Introduce the invariants

$$I_j(A,B) = \det(B, AB, ..., A^{p-2}B, A^jB), \tag{5.8}$$

where $j = p-1, ..., Bin(pm,m)$.

Theorem 5.3. *Let $n = pm$, $p > 2$, $m \geq 1$. Then*

$$\mathbb{W}^\circ(pm,m) = \{(A,B) \in \mathbb{C}^{n\times(n+m)} \mid a_1(A) = ... = a_n(A) = 0 \text{ and } I_j(A,B) = 0\},$$

where $I_j(A,B)$ are invariants (5.8).

Proof. A proof of this theorem almost word for a word repeats the proof of Theorem 5.2. It is necessary only to specify some details.

Let $\det Q(\alpha_1,...,\alpha_{pm}) \neq 0$. Then equations (5.6), (5.7), and the matrix (B,B) are replaced accordingly by equations

$$\begin{pmatrix} I_1 \\ \vdots \\ I_r \end{pmatrix} = Q(\alpha_1,...,\alpha_{pm}) \cdot \begin{pmatrix} \Delta_{1...m} \cdot ... \cdot \Delta_{(p-1)m+1...pm} \\ \vdots \\ \Delta_{i_1...i_m} \cdot ... \cdot \Delta_{j_1...j_m} \end{pmatrix}, \tag{5.9}$$

$$\det(B, AB, ..., A^{p-1}B, A^{j-1}B) = \sum_{i=1}^{r} (-1)^i f_{ij}(\alpha_1,...,\alpha_{pm}) \Delta_{i_1...i_1} \cdot ... \cdot \Delta_{j_1...j_k}, \tag{5.10}$$

where $r = Bin(pm,m)$; $\{i_1,...,i_m,...,j_1,...,j_m\} = \{1,...,pm\}$; $1 \leq i_1 < ... < i_m \leq pm$;...;$1 \leq j_1 < ... < j_m \leq pm$, and the matrix

$$\underbrace{(B,...,B)}_{pm} = \left(\begin{array}{cccccc|c|cccccc} * & \cdots & * & * & \cdots & * & & * & \cdots & * & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ * & \cdots & * & * & \cdots & * & & * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * & \cdots & 0 & \cdots & 0 & * & \cdots & * \\ \vdots & N & \vdots & \vdots & \cdots & \vdots & & \vdots & N & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * & & 0 & \cdots & 0 & * & \cdots & * \end{array}\right) \in \mathbb{C}^{pm\times pm}.$$

In order that $I_1 = ... = I_r = 0$, it is necessary that $\Delta_{i_1...i_m} \cdot ... \cdot \Delta_{j_1...j_m} = 0$. Then from (5.9), (5.10), and the representation of matrix $(B, ..., B)$ it follows that the space $B(\mathbb{U})$ intersects with the invariant subspace $\mathbb{V} \subset \mathbb{C}^{pm}(A\mathbb{V} \subset \mathbb{V})$ such that $\dim_{\mathbb{C}} \mathbb{V} \leq pl - 1$ and $\dim_{\mathbb{C}} B(\mathbb{U}) \cap \mathbb{V} = l$, where $l = m - 1, ..., 1$. Consequently, the system (A, B) of type (pm, m) contains a subsystem of type $(pl - 1, l)$; $l \in \{m - 1, ..., 1\}$. Thus, we have $(pl - 1)/l < pm/m = p$ for $l > 0$. □

5.7. Null-forms of systems of type $(pm + 1, m)$, $p \geq 2$, $m > 1$

Let $B = (b_1, ..., b_m)$. Introduce the matrix $R_1(A, B) \in \mathbb{C}^{(pm-m+1)\times m}$, where

$$R_1(A,B)$$

$$= \begin{pmatrix} \det(B, ..., A^{p-1}B, A^p b_1) & \cdots & \det(B, ..., A^{p-1}B, A^p b_m) \\ \vdots & \cdots & \vdots \\ \det(B, ..., A^{p-1}B, A^{pm} b_1) & \cdots & \det(B, ..., A^{p-1}B, A^{pm} b_m) \end{pmatrix}. \quad (5.11)$$

Introduce the invariants $I_{1j}(A, B)$, which are all minors of degree m of matrix $R_1(A, B)$; $j = 1, ..., Bin(pm, m)$.

Theorem 5.4. *Let $n = pm + 1$, $p \geq 2$, $m > 1$. Then*

$$\mathbb{W}^\circ(pm + 1, m) = \{(A, B) \in \mathbb{C}^{n\times(n+m)}$$

$$| \ a_1(A) = ... = a_n(A) = 0; I_{1j}(A, B) = 0, j = 1, ..., Bin(pm, m)\}.$$

Proof. Assume that equalities $I_{1j}(A, B) = 0$, $j = 1, ..., Bin(pm, m)$, take place. Then from (5.11) it follows that columns of matrix $R_1(A, B)$ are linearly dependent. Consequently, there exist numbers $\alpha_1, ..., \alpha_m \in \mathbb{C}$ not all equal to zero such that

$$\begin{pmatrix} \det(B, AB, ..., A^{p-1}B, A^p(\alpha_1 b_1 + ... + \alpha_m b_m)) \\ \vdots \\ \det(B, AB, ..., A^{p-1}B, A^{pm}(\alpha_1 b_1 + ... + \alpha_m b_m)) \end{pmatrix} = 0. \quad (5.12)$$

Equalities (5.12) can be rewritten as $\wedge^{pm+1}(B, AB, ..., A^{p-1}B, A^j b) = 0$, where $b = \alpha_1 b_1 + ... + \alpha_m b_m = (\Delta_1, ..., \Delta_{pm+1})^T$, $\Delta_1, ..., \Delta_{pm+1}$ are coordinates of vector b, $j = p, ..., pm$.

Thus, we have

$$I_{1j}(A, B) = \sum_{i=1}^{k} (-1)^i f_{ij}(\alpha_1, ..., \alpha_{pm+1}) \Delta_{i_1...i_m} \cdot ... \cdot \Delta_{j_1...j_m} \cdot ... \cdot \Delta_{l_1...l_m} \cdot \Delta_i, \quad (5.13)$$

where $\{i_1, ..., i_2, ..., j_1, ..., j_m, ..., l_1, ..., l_m, i\} = \{1, ..., pm + 1\}$; $k = Bin(pm, m)$.

From (5.13) it follows that equalities $\Delta_{i_1...i_m} \cdot ... \cdot \Delta_{j_1...j_m} \cdot ... \cdot \Delta_{l_1...l_m} \cdot \Delta_i = 0$ take place if and only if the matrix $(B, ..., B, b)$ has the form

$$\underbrace{(B, ..., B, b)}_{pm+1}$$

$$= \left(\begin{array}{cccccc|c|cccccc||c} * & \cdots & * & * & \cdots & * & & * & \cdots & * & * & \cdots & * & * \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * & & * & \cdots & * & * & \cdots & * & * \\ 0 & \cdots & 0 & * & \cdots & * & \cdots & 0 & \cdots & 0 & * & \cdots & * & 0 \\ \vdots & N & \vdots & \vdots & \cdots & \vdots & & \vdots & N & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * & & 0 & \cdots & 0 & * & \cdots & * & 0 \end{array} \right) \in \mathbb{C}^{(pm+1)\times(pm+1)},$$

where $N \in \mathbb{C}^{(pm-pl+1)\times l}$ is the zero submatrix of matrix B and the symbol $*$ is a nonzero element of matrix B; $l = m-1, m-2, ..., 1$. In addition, the column b has $pm - pl + 1$ zero coordinates.

In order that $I_{11} = ... = I_{1r} = 0$, it is necessary that $\Delta_{i_1...i_m} \cdot ... \cdot \Delta_{j_1...j_m} \cdot ... \cdot \Delta_{l_1...l_m} \cdot \Delta_i = 0$. Then from the representation of matrix $(B, ..., B, b)$ it follows that the space $B(\mathbb{U})$ intersects with the invariant subspace $\mathbb{V} \subset \mathbb{C}^{pm+1}(A\mathbb{V} \subset \mathbb{V})$ such that $\dim_{\mathbb{C}} \mathbb{V} \leq pl$ and $\dim_{\mathbb{C}} B(\mathbb{U}) \cap \mathbb{V} = l$, where $l = m-1, ..., 1$. Consequently, the system (A, B) of type $(pm+1, m)$ contains a subsystem of type (pl, l); $l \in \{m-1, ..., 1\}$. Thus, we have $pl/l = p < (pm+1)/m$ for $l > 0$. Then according to Theorem 2.2, we obtain that (A, B) is the null-form. □

5.8. Null-forms of systems of type $(pm-1, m)$, $p \geq 2$, $m > 1$

Let $B = (b_1, ..., b_m)$. Introduce the matrix $R_{m-1}(A, B) \in \mathbb{C}^{(pm-m+1)\times m}$, where

$$R_{m-1}(A,B)$$

$$= \begin{pmatrix} \det(B, ..., A^p(b_1, ..., b_{m-1})) & \cdots & \det(B, ..., A^p(b_2, ..., b_m)) \\ \vdots & \cdots & \vdots \\ \det(B, ..., A^{pm}(b_1, ..., b_{m-1})) & \cdots & \det(B, ..., A^{pm}(b_2, ..., b_m)) \end{pmatrix}. \quad (5.14)$$

Introduce the invariants $I_{m-1,j}(A, B)$; $j = 1, ..., Bin(pm, m)$, which are all minors of degree m of matrix $R_{m-1}(A, B)$.

Theorem 5.5. *Let $n = pm - 1$, $p \geq 2$, $m > 1$. Then*

$$\mathbb{W}^\circ(pm-1, m) = \{(A, B) \in \mathbb{C}^{n\times(n+m)}$$

$$\mid a_1(A) = ... = a_n(A) = 0; I_{m-1,j}(A, B) = 0, j = 1, ..., Bin(pm, m)\}.$$

Proof. Assume that equalities $I_{m-1,j}(A, B) = 0$, $j = 1, ..., Bin(pm, m)$, take place. Then from (5.14) it follows that columns of matrix $R_{m-1}(A, B)$ are linearly dependent. Consequently, there exist numbers $\alpha_1, ..., \alpha_m \in \mathbb{C}$ not all equal to zero such that

$$\begin{pmatrix} \det(B, AB, ..., A^{p-1}B, \alpha_1 A^p(b_1, ..., b_{m-1}) + ... + \alpha_m A^p(b_{m-1}, ..., b_m)) \\ \vdots \\ \det(B, AB, ..., A^{p-1}B, \alpha_1 A^{pm}(b_1, ..., b_{m-1}) + ... + \alpha_m A^p(b_{m-1}, ..., b_m)) \end{pmatrix} = 0. \quad (5.15)$$

According to the known result of external algebra [10,19] any $(m-1)$-polyvector built from $m-1$ vectors of m-dimensional space $B(\mathbb{U})$ is simple. In considered case it means that there exist numbers $\alpha_1, ..., \alpha_m \in \mathbb{C}$ not all equal to zero such that

$$\wedge^{m-1}(\alpha_1(b_1, ..., b_{m-1}) + ... + \alpha_m(b_{m-1}, ..., b_m)) = \wedge^{m-1}(q_1, ..., q_{m-1}),$$

where $q_i \in B(\mathbb{U})$, $i = 1, ..., m-1$.

Without loss of generality, it is possible to consider that $b_1 = q_1, ..., b_{m-1} = q_{m-1}$. Then equalities (5.15) can be rewritten as

$$\wedge^{pm-1}(B, AB, ..., A^{p-1}B, A^j(b_1, ..., b_{m-1})) = 0,$$

where $j = p, ..., pm$.

Thus, we have

$$I_{m-1,j}(A, B) = \sum_{i=1}^{k}(-1)^i f_{ij}(\alpha_1, ..., \alpha_{pm-1})\Delta_{i_1...i_m} \cdot ...\Delta_{j_1...j_m} \cdot \Delta_{l_1...l_{m-1}}, \quad (5.16)$$

where $\{i_1, ..., i_m, ..., j_1, ..., j_m, l_1, ..., l_{m-1}\} = \{1, ..., pm-1\}$, $k = Bin(pm, m)$, and $\Delta_{l_1...l_{m-1}}$ are minors of matrix $(b_1, ..., b_{m-1})$ of order $m-1$.

From (5.16) it follows that equalities $\Delta_{i_1...i_m} \cdot ... \cdot \Delta_{j_1...j_m} \cdot \Delta_{l_1...l_{m-1}} = 0$ take place if and only if the matrix $(B, ..., B, b_1, ..., b_{m-1})$ has the form:

$$\underbrace{(B, ..., B, b_1, ..., b_{m-1})}_{pm-1}$$

$$= \left(\begin{array}{cccccc|c|cccccc||ccc}
* & \cdots & * & * & \cdots & * & & * & \cdots & * & * & \cdots & * & * & \cdots & * \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
* & \cdots & * & * & \cdots & * & & * & \cdots & * & * & \cdots & * & * & \cdots & * \\
0 & \cdots & 0 & * & \cdots & * & ... & 0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 \\
\vdots & N & \vdots & \vdots & \cdots & \vdots & & \vdots & N & \vdots & \vdots & \cdots & \vdots & \vdots & N & \vdots \\
0 & \cdots & 0 & * & \cdots & * & & 0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0
\end{array}\right) \in \mathbb{C}^{r \times r},$$

where $r = pm-1, N \in \mathbb{C}^{(pm-pl+1)\times l}$ is the zero submatrix of matrix B, and the symbol $*$ is a nonzero element of matrix B; $l = m-1, m-2, ..., 1$.

In order that $I_{m-1,1} = ... = I_{m-1,k} = 0$, it is necessary that $\Delta_{i_1...i_m} \cdot ... \cdot \Delta_{j_1...j_m} \cdot \Delta_{l_1...l_{m-1}} = 0$. Then from the representation of matrix $(B, ..., B, b_1, ..., b_{m-1})$ it follows that the space $B(\mathbb{U})$ intersects with the invariant subspace $\mathbb{V} \subset \mathbb{C}^{pm-1}$ $(A\mathbb{V} \subset \mathbb{V})$ such that $\dim_{\mathbb{C}} \mathbb{V} \leq pl$ and $\dim_{\mathbb{C}} B(\mathbb{U}) \cap \mathbb{V} = l$, where $l = m-1, ..., 1$. Consequently, the system (A, B) of type $(pm-1, m)$ contains a subsystem of type $(pl-1, l)$; $l \in \{m-1, ..., 1\}$. Thus, we have $(pl-1)/l < (pm-1)/m$ for $l > 0$. Then according to Theorem 2.1, we obtain that (A, B) is the null-form. $\square$

6. Stability criterions for system (A, B)

Now we explain an importance of stability concept. Denote by $\mathbb{C}[A,B]^{\mathbb{S}L}$ the ring of all invariants of space $\mathbb{C}^{n\times(n+m)}$ with respect to action (2.3) of group $\mathbb{S}L$. Assume that

$$a_1 = a_1(A), ..., a_n = a_n(A), I_1 = I_1(A,B), ..., I_r = I_r(A,B)$$

is a set of homogeneous polynomial generators of ring $\mathbb{C}[A,B]^{\mathbb{S}L}$. (Since the group $\mathbb{S}L$ is reductive, then the set of generators is finite [35].) Thus, it is possible to write $\mathbb{C}[A,B]^{\mathbb{S}L} = \mathbb{C}[a_1,...,a_n,I_1,...,I_r]$.

Consider two systems (A_1,B_1), $(A_2,B_2) \in \mathbb{C}^{n\times(n+m)}$, which are determined by the same set of invariants: $a_1,...,a_n,I_1,...,I_r$. (In other words we have to have: $a_1(A_1) = a_1(A_2), ..., a_n(A_1) = a_n(A_2), I_1(A_1,\ B_1) = I_1(A_2,B_2), ..., I_r(A_1,B_1) = I_r(A_2,B_2)$.) Are there matrices $S \in \mathbb{G}L(n,\mathbb{C})$ and $T \in \mathbb{G}L(m,\mathbb{C})$ such that $A_2 = S^{-1}A_1S$ and $B_2 = S^{-1}B_1T$? The example of system (4.3) shows that the answer on this question is negative.

In [35, 45] it is shown that the set of invariants $a_1(A),...,a_n(A), I_1(A,B),..., I_r(A,B)$ determines a unique $\mathbb{S}L$-orbit (or $\mathbb{G}L$-orbit) of system (A,B) if and only if $(A,B) \in \mathbb{F}_s$. This circumstance explains the importance of stability concept.

The following theorem specifies Theorem 4.1.

Assume that $n = (p-1)m + s, 1 < s < m$, where s is a natural number. Consider in the representation $\bigwedge^n(\bigoplus_{i=1}^{n} T)$ simplest components of kind $\wedge^m T^{\otimes(p-1)} \otimes \wedge^s T = \det(T)^{p-1} \otimes \wedge^s T$.

Introduce the invariant $f_s(A,B) = \det(\mathbf{z}_1,...,\mathbf{z}_l), l = Bin(m,s)$, where all vectors $\mathbf{z}_i$ belong to different submodules of kind $\wedge^m T^{\otimes(p-1)} \otimes \wedge^s T = \det(T)^{p-1} \otimes \wedge^s T$. For the systems (A,B) of type (n,m) indicated invariant is $f_s(A,B) = \det G(A,B)$, and elements of square matrix $G(A,B)$ of order $l = Bin(m,s)$ are:

$$g_{ij} = \det(B, AB, ..., A^{p-1}B, A^j(b_{i_1},...,b_{i_s})), j \geq p.$$

Theorem 6.1. *Let number n and m be coprime numbers and $f_s(A,B) = \det(\mathbf{z}_1, ..., \mathbf{z}_l)$ be an invariant substantially depending on B. We will assume that vectors $\mathbf{z}_1,...,\mathbf{z}_l$ belong to different components of representation $\bigwedge^n(\bigoplus_{i=1}^{n} T)$, which are isomorphic to $(\det T)^{p-1} \otimes \wedge^s T$. If $f_s(A,B) \neq 0$, then system (A,B) is stable.*

Proof. Since $f_s(A,B) \neq 0$, then the matrix $G(A,B) = (\mathbf{z}_1,...,\mathbf{z}_l)$ is not singular. If $(S,T) \in \mathbb{S}tab_{\mathbb{S}L}(A,B)$, then

$$G((S,T)\circ(A,B)) = (\det S^{-1})^q (\det T)^{p-1} G(A,B) \cdot \wedge^s T = G(A,B).$$

By virtue of invertibility of the matrix $G(A,B)$, we get that $\wedge^s T = \lambda E_s$, where E_s is the identity matrix of order $l = Bin(m,s)$. Note that the homomorphism $T \to \wedge^s T$ has the identity core $\{\zeta_s E_m\}$, where $\zeta_s = \sqrt[s]{1}$. Therefore, $\lambda = \zeta_s$,

$S = \zeta_s E_n$ and $\mathbb{S}tab_{\mathbb{S}L}(A,B)$ consists of the finite number s of pairs matrices $(\zeta_s E_n, \zeta_s E_m)$. Therefore, all points of the invariant set $\mathbb{M}_{f_s} \in \mathbb{C}^{n\times(n+m)}$ such that $f_s(A,B) \neq 0$ are regular and, consequently, its points are stable. $\square$

Theorem 6.2. *Let* $\mathrm{disc}(A) \neq 0$. *Suppose also that* $A = \mathtt{diag}\left(()\,\lambda_1, ..., \lambda_n\right)$ *is the diagonal matrix in some base of space* $\mathbb{C}^n$*, and in the same base of space* $\mathbb{C}^n$ *all minors of matrix* B *are distinct from zero. Then system* (A,B) *is stable.*

Proof. In the case $(n,m) = (4,2)$ this theorem was proved in [24].

(a) Let $\phi : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$ be a morphism of algebraic manifolds, which satisfies to the following condition:

$$\forall (S,T) \in \mathbb{G}L \text{ and } \forall (A,B) \in \mathbb{C}^{n\times(n+m)}\ \phi(S^{-1}AS, S^{-1}BT) = (S,T)\circ\phi(A,B). \tag{6.1}$$

Show that system (A,B) is stable if and only if the system $\phi(A,B)$ is stable.

Let the system $\phi(A,B)$ be stable. Then there is an invariant opened set $\mathbb{M}_f$ is defined by an invariant $f(A,B)$ such that $f(\phi(A,B)) \neq 0$ and action of group $\mathbb{G}L$ on $\mathbb{M}_f$ is closed. Then the invariant $\phi(f)$ determines the invariant open set $\mathbb{M}_{\phi(f)} \subset \mathbb{C}^{n\times(n+m)}$, on which $\mathbb{G}L$ acts with closed orbits. The inverse assertion can be got if instead of the isomorphism ϕ to consider the inverse morphism ϕ^{-1}.

(b) Now we consider an automorphism $\phi_\lambda : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times(n+m)}$, which is given by the rule: $\phi_\lambda(A,B) = (A + \lambda E, B)$, where E is the identity matrix. (It is easily to check that ϕ_λ satisfies to condition (6.1).)

It is obvious that $A + \lambda E = \mathtt{diag}\left(()\,\lambda_1 + \lambda, ..., \lambda_n + \lambda\right)$ and $\mathrm{disc}(A+\lambda E) \neq 0$. Then there is a number $\lambda \in \mathbb{C}$ such that $\det(A + \lambda E) \neq 0$. (It is enough to take $\lambda \neq -\lambda_i, i = 1, ..., n$.)

Let a system $(\wedge^m A, \wedge^m B)$ be an m-exterior degree of the system (A,B). Assume that for the system $(\wedge^m A, \wedge^m B)$ the following condition

$$\mathtt{rank}(\wedge^m B, \wedge^m(AB), ..., \wedge^m(A^{n-1}B)) < Bin(n,m)$$

is fulfilled. (The system $(\wedge^m A, \wedge^m B)$ is not complete controllable.)

It means that under the conditions of Theorem 6.2 there exist collections of indexes $(i_1, ..., i_m)$ and $(j_1, ..., j_m)$, where $1 \leq i_1 < ... < i_m \leq n$ and $1 \leq j_1 < ... < j_m \leq n$ such that

$$\{(i_1, ..., i_m)\} \cap \{(j_1, ..., j_m)\} = \varnothing \text{ and } \lambda_{i_1}\cdot ... \cdot \lambda_{i_m} = \lambda_{j_1}\cdot ... \cdot\lambda_{j_m}. \tag{6.2}$$

Then there exists a number $\lambda \in \mathbb{C}$ such that for the matrix $A + \lambda E$ condition (6.2) is not fulfilled.

Indeed, otherwise for all λ

$$(\lambda_{i_1} + \lambda)\cdot ... \cdot(\lambda_{i_m} + \lambda) = (\lambda_{j_1} + \lambda)\cdot ... \cdot(\lambda_{j_m} + \lambda). \tag{6.3}$$

Equality (6.3) means that

$$h_k(\lambda_{i_1}, ..., \lambda_{i_m}) = h_k(\lambda_{j_1}, ..., \lambda_{j_m})$$

for any elementary symmetric polynomials $h_k(\lambda_{i_1}, ..., \lambda_{i_m})$, $k = 1, ..., m$ [15]. However, it is impossible since $\lambda_i \neq \lambda_j$ at $i \neq j$.

Further, since the set of all numbers λ such that

$$\mathrm{disc}(\wedge^m(A + \lambda E)) \neq 0 \text{ and } \det A \neq 0 \tag{6.4}$$

is open in the Zariski topology and the Euclidean topology of space $\mathbb{C}$ [44], then their intersection is not empty. Consequently, there will be numbers $\lambda \in \mathbb{C}$, for which both conditions (6.4) are valid simultaneously. Then the system $\phi_\lambda(A, B)$ is stable and, in obedience to the item (a), the system (A, B) will be also stable. It completes proof of item (b) and all Theorem 6.2. □

Theorem 6.3. *System (A, B) of type (n, m) is stable with respect to action (2.3) of group $\mathbb{G}L$ if and only if it is stable with respect to action (2.3) of group $\mathbb{S}L$.*

Proof. We will consider the morphism of algebraic manifolds:

$$\phi : \mathbb{C}^{n\times(n+m)} \to \mathbb{C}^{n\times n} \times \mathbb{C}^N, N = Bin(n, m),$$

$$\phi(A, B) = (A, \wedge^m B).$$

Notice that the image of morphism ϕ is a close variety in space $\mathbb{C}^{n\times n} \times \mathbb{C}^N$ and orbits of group $\mathbb{S}L$ the morphism ϕ transfers in the orbits of group $\mathbb{G}L(n, \mathbb{C})$ in space $\mathbb{C}^{n\times(n+m)}$ at action

$$\mathbb{S}L(n, \mathbb{C}) \times \mathbb{C}^{n\times n} \times \mathbb{C}^N \to \mathbb{C}^{n\times n} \times \mathbb{C}^N,$$

$$(S, A, \wedge^m B) = (S^{-1}AS, \wedge^m(S^{-1}B)).$$

It completes the proof. □

6.1. Stability of systems of type $(4, 2)$

In this subsection we show difficulties, which can arise up at construction of the set of all stable systems in case if m is a divisor of n (see [24]).

First of all, we show that if system (A, B) is stable then the matrix A is cyclic [15]. For this purpose we present the various Jordan formes of noncyclic matrix of order 4:

$$a) \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_1 & & \\ & & \alpha_2 & \\ 0 & & & \alpha_3 \end{pmatrix}, \ b) \begin{pmatrix} \alpha_1 & 1 & & 0 \\ & \alpha_1 & & \\ & & \alpha_1 & \\ 0 & & & \alpha_2 \end{pmatrix}, \ c) \begin{pmatrix} \alpha_1 & 1 & & 0 \\ & \alpha_1 & & \\ & & \alpha_1 & 1 \\ 0 & & & \alpha_1 \end{pmatrix},$$

$$d) \begin{pmatrix} \alpha_1 & 1 & & 0 \\ & \alpha_1 & 1 & \\ & & \alpha_1 & \\ 0 & & & \alpha_1 \end{pmatrix}, \ e) \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_1 & & \\ & & \alpha_1 & \\ 0 & & & \alpha_2 \end{pmatrix}, \ f) \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_1 & & \\ & & \alpha_1 & \\ 0 & & & \alpha_1 \end{pmatrix}.$$

Thus, there are six such forms.

Consider, for example, the system (A, B) of type $(4, 2)$, for which matrix A looks like a) and matrix B is arbitrary. Apply to (A, B) the transformation $(S, T) \in \mathbb{G}L$, where

$$S = \begin{pmatrix} \alpha & \beta & & 0 \\ \gamma & \delta & & \\ & & \nu & \\ 0 & & & \mu \end{pmatrix}$$

and numbers $\alpha, \beta, \gamma, \delta, \mu$, and ν satisfy a unique condition: $\det S \neq 0$. Then system (A, B) can be transformed to the following system:

$$S^{-1}AS = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_1 & & \\ & & \alpha_2 & \\ 0 & & & \alpha_3 \end{pmatrix}, S^{-1}BT = \begin{pmatrix} 0 & * \\ 1 & * \\ 1 & * \\ 0 & * \end{pmatrix},$$

where by character $*$ arbitrary numbers are marked.

From here it follows that the system (A, B) contains a subsystem of type $(2, 1)$. Consequently, according to Theorem 4.1, the system (A, B) is nonstable. By applying the same arguments to one of matrices b) - f), we obtain a similar result: the conditions of Theorem 4.1 are not valid.

We reduce the matrix A of system (A, B) to triangular form [15]. Then analysis of all possible Jordan forms of A results in the conclusion: if the system is not stable then there exists a minor of the second order of matrix B, which equal to zero. Otherwise, if the system (A, B) is stable, then there is even one nonzero minor of the second order of matrix B.

Denote by $I_0 = \det(B, B) \equiv 0, I_1 = \det(B, AB)$, $I_2 = \det(B, A^2B)$, $a_1, ..., a_4$ invariants of system (A, B).

Theorem 6.4. *Let (A, B) be a system of type $(4, 2)$. Then the set $\mathbb{F}_s$ of all $\mathbb{S}L$-stable systems with respect to action (2.3) is determined by the condition:*

$$\mathbb{F}_s = \{(A, B) \in \mathbb{C}^{4\times6} \mid f_s(A, B) \neq 0\},$$

where the polynomial invariant

$$f_s(A, B) = (a_1a_2a_3 - a_3^2 - a_1^2a_4)I_1^3 - (a_1a_3 + a_2^2 - 4a_4)I_1^2I_2 + 2a_2I_1I_2^2 - I_2^3$$

has degree 24 with respect to elements a_{ij} and b_{ik} of matrices A and B.

Proof. Assume that for the matrix A, we have $\text{disc}(A) \neq 0$. In this case the system (A, B) of type $(4, 2)$ may be transformed to the following aspect:

$$A = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \alpha_3 & \\ 0 & & & \alpha_4 \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}.$$

Denote by $\wedge^2 B = (\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34})^T$ a bivector of matrix B; here $\Delta_{ij} = b_{i1}b_{j2} - b_{i2}b_{j1}$ are minors of matrix B; $i, j = 1, ..., 4; i < j$. Then the invariants I_0, I_1, and I_2 of system (A, B) are given by the formulas

$$\begin{pmatrix} 0 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ \alpha_1\alpha_2 + \alpha_3\alpha_4 & \alpha_1\alpha_3 + \alpha_2\alpha_4 & \alpha_1\alpha_4 + \alpha_2\alpha_3 \\ \alpha_1^2\alpha_2^2 + \alpha_3^2\alpha_4^2 & \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_4^2 & \alpha_1^2\alpha_4^2 + \alpha_2^2\alpha_3^2 \end{pmatrix} \cdot \begin{pmatrix} \Delta_{12}\Delta_{34} \\ -\Delta_{13}\Delta_{24} \\ \Delta_{14}\Delta_{23} \end{pmatrix}. \tag{6.5}$$

The elementary computations show that

$$\det \begin{pmatrix} 2 & 2 & 2 \\ \alpha_1\alpha_2 + \alpha_3\alpha_4 & \alpha_1\alpha_3 + \alpha_2\alpha_4 & \alpha_1\alpha_4 + \alpha_2\alpha_3 \\ \alpha_1^2\alpha_2^2 + \alpha_3^2\alpha_4^2 & \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_4^2 & \alpha_1^2\alpha_4^2 + \alpha_2^2\alpha_3^2 \end{pmatrix} = 2\ (\operatorname{disc}(A))^{1/2}.$$

Let $p_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $p_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$, and $p_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$. Since we consider that $\operatorname{disc}(A) \neq 0$, then from (6.5) we have

$$\begin{pmatrix} \Delta_{12}\Delta_{34} \\ -\Delta_{13}\Delta_{24} \\ \Delta_{14}\Delta_{23} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ \alpha_1\alpha_2 + \alpha_3\alpha_4 & \alpha_1\alpha_3 + \alpha_2\alpha_4 & \alpha_1\alpha_4 + \alpha_2\alpha_3 \\ \alpha_1^2\alpha_2^2 + \alpha_3^2\alpha_4^2 & \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_4^2 & \alpha_1^2\alpha_4^2 + \alpha_2^2\alpha_3^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ I_1 \\ I_2 \end{pmatrix}$$

$$= \frac{1}{(2\operatorname{disc}(A))^{1/2}} \begin{pmatrix} 0.5p_2p_3(p_2 - p_3) & (p_2^2 - p_3^2) & (p_2 - p_3) \\ 0.5p_1p_3(p_1 - p_3) & (p_1^2 - p_3^2) & (p_1 - p_3) \\ 0.5p_1p_2(p_1 - p_2) & (p_1^2 - p_2^2) & (p_1 - p_2) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ I_1 \\ I_2 \end{pmatrix}$$

$$= \frac{1}{(2\operatorname{disc}(A))^{1/2}} \begin{pmatrix} -(p_2 - p_3)(p_2 + p_3)I_1 + (p_2 - p_3)I_2) \\ -(p_1 - p_3)(p_1 + p_3)I_1 + (p_1 - p_3)I_2) \\ -(p_1 - p_2)(p_1 + p_2)I_1 + (p_1 - p_2)I_2) \end{pmatrix}.$$

From the last relations we get

$$\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{23}\Delta_{24}\Delta_{34}$$

$$= -\frac{1}{8\operatorname{disc}(A)}(-(p_2 + p_3)I_1 + I_2)(-(p_1 + p_3)I_1 + I_2)(-(p_1 + p_2)I_1 + I_2)$$

$$= -\frac{-(p_1 + p_2)(p_1 + p_3)(p_2 + p_3)I_1^3}{8\operatorname{disc}(A)}$$

$$-\frac{((p_1 + p_3)(p_2 + p_3) + (p_1 + p_3)(p_1 + p_2) + (p_1 + p_2)(p_2 + p_3))I_1^2I_2}{8\operatorname{disc}(A)}$$

$$+\frac{2(p_1 + p_2 + p_3)I_1I_2^2 - I_2^3}{8\operatorname{disc}(A)}$$

$$= -\frac{-(a_1a_2a_3 - a_3^2 - a_1^2a_4)I_1^3 + (a_1a_3 + a_2^2 - 4a_4)I_1^2I_2 - 2a_2I_1I_2^2 + I_2^3}{8\operatorname{disc}(A)}. \tag{6.6}$$

Above it was noted that the system (A, B) would be not stable if either even one minor $\Delta_{ij} = 0$ or $\mathrm{disc}(A) = 0$. Then from (6.6) it follows that (A, B) is $\mathbb{S}L$-stable if the invariant

$$f_s(A, B) = (a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) I_1^3 - (a_1 a_3 + a_2^2 - 4a_4) I_1^2 I_2 + 2a_2 I_1 I_2^2 - I_2^3 \neq 0.$$

Consider the cyclic matrix

$$\begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \alpha_3 & \\ 0 & & & \alpha_4 \end{pmatrix}. \tag{6.7}$$

Apply to this matrix the transformation

$$S = \begin{pmatrix} 1 & 1/(\alpha_1 - \alpha_2) & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix};$$

then, we have

$$S^{-1}AS = \begin{pmatrix} \alpha_1 & 1 & & 0 \\ & \alpha_2 & & \\ & & \alpha_3 & \\ 0 & & & \alpha_4 \end{pmatrix}.$$

Let $T = E$ be the identity matrix of order 2. Then

$$f_s(S^{-1}AS, S^{-1}BT) = (\det S)^{l_S} (\det T)^{l_T} f_s(A, B) = f_s(A, B).$$

Since we have

$$\lim_{(\alpha_1 - \alpha_2) \to 0} S^{-1}AS = \begin{pmatrix} \alpha_1 & 1 & & 0 \\ & \alpha_1 & & \\ & & \alpha_3 & \\ 0 & & & \alpha_4 \end{pmatrix},$$

then even if $(\alpha_1 - \alpha_2) \to 0$ the invariant polynomial $f_s(A, B)$ is saved.

Applying similar methods, it is possible to get from matrix (6.7) an arbitrary cyclic matrix A, for which $\mathrm{disc}(A) = 0$. (Note that if matrix (6.7) is noncyclic, then the last assertion is incorrect.) Consequently, the polynomial $f_s(A, B)$ saves and for cyclic matrices A such that $\mathrm{disc}(A) = 0$.

Finally, we note that for noncyclic matrices A system (A, B) type $(4, 2)$ is not stable. The proof is finished. □

6.2. Stability of systems of type $(m + 1, m)$, $m > 1$

(a) $m = 2$. Let $A = \texttt{diag}\,(()\,\alpha_1, \alpha_2, \alpha_3)$, $\wedge^2 B = (\Delta_{12}, \Delta_{13}, \Delta_{23})^T$, and $\mathrm{disc}(A) \neq 0$. We take advantage of Theorems 5.1 and 6.1.

Consider the invariant $I(A,B) = \det R(A,B)$, where $R(A,B)$ is matrix (5.3) at $n=3$ and $m=2$. Then we have

$$I(A,B)$$

$$= \det \begin{pmatrix} \alpha_1\Delta_{23}b_{11} - \alpha_2\Delta_{13}b_{21} + \alpha_3\Delta_{12}b_{31} & \alpha_1\Delta_{23}b_{12} - \alpha_2\Delta_{13}b_{22} + \alpha_3\Delta_{12}b_{32} \\ \alpha_1^2\Delta_{23}b_{11} - \alpha_2^2\Delta_{13}b_{21} + \alpha_3^2\Delta_{12}b_{31} & \alpha_1^2\Delta_{23}b_{12} - \alpha_2^2\Delta_{13}b_{22} + \alpha_3^2\Delta_{12}b_{32} \end{pmatrix}$$

$$= (\alpha_1\alpha_2(\alpha_1-\alpha_2) - \alpha_1\alpha_3(\alpha_1-\alpha_3) + \alpha_2\alpha_3(\alpha_2-\alpha_3))\Delta_{12}\Delta_{13}\Delta_{23}$$

$$= (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)\Delta_{12}\Delta_{13}\Delta_{23}.$$

It is clear that $I(A,B)$ is $\mathbb{G}L$-invariant, but it is not $\mathbb{S}L$-invariant. Therefore, we have to take the invariant

$$I^2(A,B) = \operatorname{disc}(A)\Delta_{12}^2\Delta_{13}^2\Delta_{23}^2.$$

Now we must show that if the matrix A is noncyclic then $I^2(A,B) = 0$. It can be checked by the methods of subsection 6.1.

Thus, the set of all $\mathbb{S}L$-stable systems (A,B) of type $(3,2)$ is given by the condition

$$\mathbb{F}_s = \{(A,B) \in \mathbb{C}^{3\times 5} \mid I^2(A,B) \neq 0\}.$$

(b) $m > 2$. A proof of this case word for a word repeats the previous proof. In this case we get

$$I^2(A,B) = \operatorname{disc}(A) \underbrace{\Delta_{1\ldots m}^2 \cdot \ldots \cdot \Delta_{2\ldots m+1}^2}_{m+1}.$$

Therefore, the set of all $\mathbb{S}L$-stable systems (A,B) of type $(m+1,m)$ is given by the condition

$$\mathbb{F}_s = \{(A,B) \in \mathbb{C}^{(m+1)\times(2m+1)} \mid I^2(A,B) \neq 0\}.$$

7. Description of ring of invariants for system (A,B)

7.1. Structure of invariants of group $\mathbb{S}L$ for system (A,B)

Denote by $a_1,\ldots,a_n$ the coefficients of the characteristic polynomial of matrix A. Let $f(A,B) \in \mathbb{C}[A,B]^{\mathbb{S}L}$ be an invariant of group $\mathbb{S}L = \mathbb{S}L(n,\mathbb{C}) \times \mathbb{S}L(m,\mathbb{C})$ with respect to action (2.3). Let also the vector $(\Delta_1,\ldots,\Delta_r)^T = \wedge^m(B)$ be the exterior degree of matrix B; $r = Bin(n,m)$.

Theorem 7.1. *Any polynomial $\mathbb{S}L$-invariant of group $\mathbb{S}L$ is a function of elements $a_1,\ldots,a_n$, $\Delta_1,\ldots,\Delta_r$.*

Proof. We represent the polynomial $f(A,B)$ in the following form:

$$f(A,B) = \sum_{i=1}^{r} g_i(A) \otimes v_i(B),$$

where $g_i(A)$ ($v_i(B)$) are polynomials depending only on the elements of matrix A (the elements of matrix B) and the polynomials $g_i(A)$ it is possible to choose linearly independent.

Further, $\forall T \in \mathbb{S}L(m,\mathbb{C})$, we have

$$\sum_{i=1}^{r} g_i(A) \otimes v_i(B) = f(A,B) = f(A,BT) = \sum_{i=1}^{r} g_i(A) \otimes v_i(BT).$$

Consequently, from here it follows that $\sum\limits_{i=1}^{r} g_i(A) \otimes [v_i(B) - v_i(BT)] = 0$.

By virtue of the linear independence of polynomials $g_i(A)$ over $\mathbb{C}$, we get that $\forall i$ and for all $T \in \mathbb{S}L(m,\mathbb{C})$ $v_i(B) = v_i(BT)$. In other words, the inclusion $v_i(B) \in \mathbb{C}[B]^{\mathbb{S}L(m,\mathbb{C})} = \mathbb{C}[\Delta_1, ..., \Delta_r]$ takes place. Consequently, $f(A,B) = h(A, \Delta_1, ..., \Delta_r)$ is a polynomial of elements of matrix A and coordinates of polyvector $\wedge^m B$. Thus, $\forall (S,T) \in \mathbb{S}L$ we have

$$(S,T) \circ f(A,B) = (S^{-1}AS, S^{-1}BT) = h(A, \wedge^m (S^{-1}BT)) = h(A, \Delta_1, ..., \Delta_r),$$

where $h(...)$ is a homogeneous polynomial of elements of matrix A and $\Delta_1, ..., \Delta_r$. Besides, if we take into account that any $\mathbb{S}L(n,\mathbb{C})$-invariant of matrix A is a polynomial of elements $a_1, ..., a_n$, then the invariant $f(A,B)$ is the function of $a_1, ..., a_n, \Delta_1, ..., \Delta_r$. $\square$

7.2. Ring of invariants for system (A,B) of type $(2p,2)$, $p > 1$

Theorem 7.2. *Let $I_1(A,B), ..., I_{2p-2}(A,B)$ be invariants (5.8). Then the following equality*

$$\mathbb{C}[A,B]^{\mathbb{S}L} = \mathbb{C}[a_1, ..., a_{2p}, I_1, ..., I_{2p-2}]$$

takes place. Moreover, the number $4p-2$ of generators of ring $\mathbb{C}[A,B]^{\mathbb{S}L}$ is minimal.

Proof. (a) $p = 2$. In this case the proof easily can be got from the proofs of Theorems 5.2 and 6.4.

(b) $p = 3$. Let $A = \mathtt{diag}\,(()\,\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$, $B = (b_1, b_2) \in \mathbb{C}^{5\times 2}$, $\wedge^2 B = (\Delta_{12}, \Delta_{13}, ..., \Delta_{56})^T$, and $\mathrm{disc}(A) \neq 0$.

Represent the equivalence $\det(B,B,B) \equiv 0$ in the following form:

$$\det(B,B,B) = \Delta_{12}\Delta_{34}\Delta_{56} - \Delta_{13}\Delta_{25}\Delta_{46} + \Delta_{14}\Delta_{26}\Delta_{45}$$

$$-\Delta_{15}\Delta_{24}\Delta_{36} + \Delta_{16}\Delta_{35}\Delta_{24} \equiv 0.$$

Construct invariants (5.8):

$$\begin{pmatrix} I_0 = \det(B, AB, AB) \\ I_1 = \det(B, AB, A^2B) \\ I_2 = \det(B, AB, A^3B) \\ I_3 = \det(B, AB, A^4B) \\ I_4 = \det(B, AB, A^5B) \end{pmatrix} = Q(\alpha_1, ..., \alpha_6) \cdot \begin{pmatrix} \Delta_{12}\Delta_{34}\Delta_{56} \\ -\Delta_{13}\Delta_{25}\Delta_{46} \\ \Delta_{14}\Delta_{26}\Delta_{45} \\ -\Delta_{15}\Delta_{24}\Delta_{36} \\ \Delta_{16}\Delta_{35}\Delta_{24} \end{pmatrix}, \tag{7.1}$$

where $I_0 \equiv 0$ and

$$Q(\alpha_1, ..., \alpha_6)$$

$$= \alpha_1 \cdot ... \cdot \alpha_6 \cdot \begin{pmatrix} \frac{1}{\alpha_3\alpha_4} + \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_5\alpha_6} & \cdots & \frac{1}{\alpha_2\alpha_4} + \frac{1}{\alpha_3\alpha_5} + \frac{1}{\alpha_1\alpha_6} \\ \vdots & \cdots & \vdots \\ \frac{\alpha_5^4\alpha_6^4}{\alpha_3\alpha_4} + \frac{\alpha_3^4\alpha_4^4}{\alpha_1\alpha_2} + \frac{\alpha_1^4\alpha_2^4}{\alpha_5\alpha_6} & \cdots & \frac{\alpha_3^4\alpha_5^4}{\alpha_2\alpha_4} + \frac{\alpha_1^4\alpha_6^4}{\alpha_3\alpha_5} + \frac{\alpha_2^4\alpha_4^4}{\alpha_1\alpha_6} \end{pmatrix} \in \mathbb{C}^{5\times 5}.$$

Assume that $\det Q(\alpha_1, ..., \alpha_6) \neq 0$. Let also $I_1 = ... = I_4 = 0$. Then from (7.1) it follows that

$$\Delta_{12}\Delta_{34}\Delta_{56} = \Delta_{13}\Delta_{25}\Delta_{46} = \Delta_{14}\Delta_{26}\Delta_{45} = \Delta_{15}\Delta_{24}\Delta_{36} = \Delta_{16}\Delta_{35}\Delta_{24} = 0. \tag{7.2}$$

Researches of system (7.2) result in one of systems of equations: either $\Delta_{12} = \Delta_{13} = \Delta_{14} = \Delta_{23} = \Delta_{24} = \Delta_{34} = 0$ or ... or $\Delta_{34} = \Delta_{35} = \Delta_{36} = \Delta_{45} = \Delta_{46} = \Delta_{56} = 0$. According to Theorem 5.3 it means that there exists a subsystem of type $(2, 1)$ of system (A, B). If it is assertion to complement by conditions $a_1 = ... = a_6 = 0$, then we get that the system (A, B) is the null-form.

Now we suppose that $g(A, B) \in \mathbb{C}[A, B]^{\mathbb{S}L}$ is the homogeneous polynomial invariant such that $g(A, B) = 0$ in all roots of polynomials $a_1, ..., a_6, I_1, ..., I_4$. Then by the known Hilbert theorem [35] there exists an integer number $d \geq 1$ such that $g^d(A, B) \in \mathbb{C}[a_1, ..., a_6, I_1, ..., I_4]$. Assume that for the polynomial invariant $g(A, B)$ such that $g(A, B) = 0$, we have $d > 1$. (In other words, $g(A, B) \notin \mathbb{C}[a_1, ..., a_6, I_1, ..., I_4]$.) It is clear that the invariants $I_1, ..., I_4$ are polynomials of degree 3 with respect to $\Delta_{12}, ..., \Delta_{56}$. In addition, degree 3 is the minimal degree with respect to $\Delta_{12}, ..., \Delta_{56}$ of all invariants depending on B in the ring $\mathbb{C}[A, B]^{\mathbb{S}L}$. Therefore, there exists the invariant $g(A, B)$ such that $d = 1$, and therefore, $\mathbb{C}[a_1, ..., a_6, I_1, ..., I_4] = \mathbb{C}[A, B]^{\mathbb{S}L}$.

(c) $p > 3$. Let $A = \mathtt{diag}\,(()\, \alpha_1, ..., \alpha_{2p})$, $\wedge^2 B = (\Delta_{12}, \Delta_{13}, ..., \Delta_{2p-1,2p})^T$, and $\mathrm{disc}(A) \neq 0$.

A proof of the case $p > 3$ repeats the proof of Theorem 7.2 for case $p = 3$. It is necessary only to do some generalizations. For $p > 3$ system (7.1) has such form:

$$\begin{pmatrix} I_0 \\ I_1 \\ \vdots \\ I_{2p-2} \end{pmatrix} = Q(\alpha_1, ..., \alpha_{2p}) \cdot \underbrace{\begin{pmatrix} \Delta_{12}\Delta_{34} \cdot ... \cdot \Delta_{2p-1,2p} \\ -\Delta_{13}\Delta_{25} \cdot ... \cdot \Delta_{2p-2,2p} \\ \vdots \\ (-1)^l \Delta_{i_1 i_2}\Delta_{j_1 j_2} \cdot ... \cdot \Delta_{k_1 k_2} \end{pmatrix}}_{p \text{ factors}}, \tag{7.3}$$

where $I_0 \equiv 0$, $Q(\alpha_1, ..., \alpha_{2p}) \in \mathbb{C}^{(2p-1)\times(2p-1)}$, $1 \le i_1 < i_2 \le 2p$, ..., $1 \le k_1 < k_2 \le 2p$, $l = 1, ..., 2p-1$, and permutations (i_1, i_2),...,(k_1, k_2) are satisfied to the condition $(i_1, i_2, ..., k_1, k_2) \in \{1, 2, ..., 2p-1, 2p\}$. There are all $(2p-1)!! = 1 \cdot 3 \cdot 5 \cdot ... \cdot (2p-1)$ permutations.

Further, let $I_1 = ... = I_{2p-2} = 0$. If $\det Q(\alpha_1, ..., \alpha_{2p}) \neq 0$, then from (7.3) it follows that $\Delta_{12}\Delta_{34} \cdot ... \cdot \Delta_{2p-1,2p} = ... = \Delta_{i_1 i_2}\Delta_{j_1 j_2} \cdot ... \cdot \Delta_{k_1 k_2} = 0$. It means that there exists a submatrix $B_{p-1} \in \mathbb{C}^{(p-1)\times 2}$ of matrix B such that $\wedge^2 B_{p-1} = 0$. A further algorithm of proof is obvious.

Let us compute the dimension of space of orbits for system of type $(2p, 2)$. From formula (4.1) it follows that for the system of type $(2p, 2)$ we have $\dim_{\mathbb{C}} \mathbb{O}^{\mathbb{S}L}(\mathbb{S}) = 4p - 2$. It means that in Theorem 7.2 the number of generators of ring $\mathbb{C}[A, B]^{\mathbb{S}L}$ is minimal.□

7.3. Ring of invariants for system (A, B) of type $(2p+1, 2)$, $p \ge 1$

Theorem 7.3. *Let*

$$I_{ij}(A, B) = \det \begin{pmatrix} \det(B, AB, ..., A^{p-1}B, A^i b_1) & \det(B, AB, ..., A^{p-1}B, A^i b_2) \\ \det(B, AB, ..., A^{p-1}B, A^j b_1) & \det(B, AB, ..., A^{p-1}B, A^j b_2) \end{pmatrix},$$

$p \le i < j \le 2p+1$, *be invariants (5.11). Then the following equality*

$$\mathbb{C}[A, B]^{\mathbb{S}L} = \mathbb{C}[a_1, ..., a_{2p+1}, \underbrace{I_{p,p+1}, ..., I_{p,2p}}_{p}, \underbrace{I_{p+1,p+2}, ..., I_{p+1,2p}}_{p-1}]$$

takes place. Moreover, the number $4p$ of generators of ring $\mathbb{C}[A, B]^{\mathbb{S}L}$ is minimal.

Proof. (a) $p = 2$. Let $A = \mathtt{diag}\,(()\,\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $B = (b_1, b_2) \in \mathbb{C}^{5\times 2}$, $\wedge^2 B = (\Delta_{12}, ..., \Delta_{45})^T$, and $\mathrm{disc}(A) \neq 0$.

Represent the functions $\det(B, B, b_1) \equiv 0$ and $\det(B, B, b_2) \equiv 0$ in the following forms:

$$\begin{gathered} \det(B, B, b_1) \\ = 2b_{11}(\Delta_{23}\Delta_{45} - \Delta_{24}\Delta_{35} + \Delta_{25}\Delta_{34}) - 2b_{21}(\Delta_{13}\Delta_{45} - \Delta_{14}\Delta_{35} + \Delta_{15}\Delta_{34}) \\ +2b_{31}(\Delta_{12}\Delta_{45} - \Delta_{14}\Delta_{25} + \Delta_{24}\Delta_{15}) - 2b_{41}(\Delta_{12}\Delta_{35} - \Delta_{13}\Delta_{25} + \Delta_{23}\Delta_{15}) \\ +2b_{51}(\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{23}\Delta_{34}) \equiv 0 \end{gathered}$$

and

$$\begin{gathered} \det(B, B, b_2) \\ = 2b_{12}(\Delta_{23}\Delta_{45} - \Delta_{24}\Delta_{35} + \Delta_{25}\Delta_{34}) - 2b_{22}(\Delta_{13}\Delta_{45} - \Delta_{14}\Delta_{35} + \Delta_{15}\Delta_{34}) \\ +2b_{32}(\Delta_{12}\Delta_{45} - \Delta_{14}\Delta_{25} + \Delta_{24}\Delta_{15}) - 2b_{42}(\Delta_{12}\Delta_{35} - \Delta_{13}\Delta_{25} + \Delta_{23}\Delta_{15}) \\ +2b_{52}(\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{23}\Delta_{34}) \equiv 0. \end{gathered}$$

Taking into account the structure of functions $\det(B,B,b_1)$, $\det(B,B,b_2)$, we form the following zero (2×2)-matrix:

$$R_0 = 2\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

$$\times\begin{pmatrix} b_{11}(\Delta_{23}\Delta_{45}-\Delta_{24}\Delta_{35}+\Delta_{25}\Delta_{34}) & b_{12}(\Delta_{23}\Delta_{45}-\Delta_{24}\Delta_{35}+\Delta_{25}\Delta_{34}) \\ b_{21}(\Delta_{13}\Delta_{45}-\Delta_{14}\Delta_{35}+\Delta_{15}\Delta_{34}) & b_{22}(\Delta_{13}\Delta_{45}-\Delta_{14}\Delta_{35}+\Delta_{15}\Delta_{34}) \\ b_{31}(\Delta_{12}\Delta_{45}-\Delta_{14}\Delta_{25}+\Delta_{24}\Delta_{15}) & b_{32}(\Delta_{12}\Delta_{45}-\Delta_{14}\Delta_{25}+\Delta_{24}\Delta_{15}) \\ b_{41}(\Delta_{12}\Delta_{35}-\Delta_{13}\Delta_{25}+\Delta_{23}\Delta_{15}) & b_{42}(\Delta_{12}\Delta_{35}-\Delta_{13}\Delta_{25}+\Delta_{23}\Delta_{15}) \\ b_{51}(\Delta_{12}\Delta_{34}-\Delta_{13}\Delta_{24}+\Delta_{23}\Delta_{34}) & b_{52}(\Delta_{12}\Delta_{34}-\Delta_{13}\Delta_{24}+\Delta_{23}\Delta_{34}) \end{pmatrix}.$$

Now we use the structure of matrix R_0 for computation of invariants. Then we have:

$$I_{ij}(A,B) = \det\begin{pmatrix} \det(B,AB,A^ib_1) & \det(B,AB,A^ib_2) \\ \det(B,AB,A^jb_1) & \det(B,AB,A^jb_2) \end{pmatrix}$$

$$= \det\Big[\begin{pmatrix} \alpha_1^i & -\alpha_2^i & \alpha_3^i & -\alpha_4^i & \alpha_5^i \\ \alpha_1^j & -\alpha_2^j & \alpha_3^j & -\alpha_4^j & \alpha_5^j \end{pmatrix}\times$$

$$\mathtt{diag}\,(()\,(\alpha_2\alpha_3+\alpha_4\alpha_5)\Delta_{23}\Delta_{45}-(\alpha_2\alpha_4+\alpha_3\alpha_5)\Delta_{24}\Delta_{35}+(\alpha_2\alpha_5+\alpha_3\alpha_4)\Delta_{25}\Delta_{34},$$

$$(\alpha_1\alpha_3+\alpha_4\alpha_5)\Delta_{13}\Delta_{45}-(\alpha_1\alpha_4+\alpha_3\alpha_5)\Delta_{14}\Delta_{35}+(\alpha_1\alpha_5+\alpha_3\alpha_4)\Delta_{15}\Delta_{34},$$

$$(\alpha_1\alpha_2+\alpha_4\alpha_5)\Delta_{12}\Delta_{45}-(\alpha_1\alpha_4+\alpha_2\alpha_5)\Delta_{14}\Delta_{25}+(\alpha_2\alpha_4+\alpha_1\alpha_5)\Delta_{24}\Delta_{15},$$

$$(\alpha_1\alpha_2+\alpha_3\alpha_5)\Delta_{12}\Delta_{35}-(\alpha_1\alpha_3+\alpha_2\alpha_5)\Delta_{13}\Delta_{25}+(\alpha_2\alpha_3+\alpha_1\alpha_5)\Delta_{23}\Delta_{15},$$

$$(\alpha_1\alpha_2+\alpha_3\alpha_4)\Delta_{12}\Delta_{34}-(\alpha_1\alpha_3+\alpha_2\alpha_4)\Delta_{13}\Delta_{24}+(\alpha_2\alpha_3+\alpha_3\alpha_4)\Delta_{23}\Delta_{34})\cdot B\Big]. \quad (7.4)$$

We represent system (7.4) in such aspect:

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{23} \\ I_{24} \\ I_{34} \end{pmatrix} = \wedge^2\left(\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ \alpha_1 & -\alpha_2 & \alpha_3 & -\alpha_4 & \alpha_5 \\ \alpha_1^2 & -\alpha_2^2 & \alpha_3^2 & -\alpha_4^2 & \alpha_5^2 \\ \alpha_1^3 & -\alpha_2^3 & \alpha_3^3 & -\alpha_4^3 & \alpha_5^3 \\ \alpha_1^4 & -\alpha_2^4 & \alpha_3^4 & -\alpha_4^4 & \alpha_5^4 \end{pmatrix}\cdot D\cdot B\right), \quad (7.5)$$

where $D = \mathtt{diag}\,(()\,d_1,...,d_5)$ is the diagonal matrix from system (7.4).

In order that system (7.5) had trivial solution only, it is necessary that $I_{23} = I_{24} = I_{34} = 0$. Indeed, since $\mathrm{disc}(A)\neq 0$, then from (7.5) it follows that $\wedge^2(D\cdot B) = 0$. The last equality is possible if and only if

$$d_id_j\Delta_{ij} = 0;\ 1\le i<j\le 5. \quad (7.6)$$

The following variants are here possible.

(a1) $d_1 \neq 0, ..., d_5 \neq 0$. Then we have $\wedge^2 B = 0$. It means $\Delta_{ij} = 0;\ 1 \leq i < j \leq 5$.

(a2) $d_1 = 0, d_2 \neq 0, ..., d_5 \neq 0$. Then $\wedge^2 B_1 = 0$, where B_1 is the submatrix of matrix B without the first row.

(a3) $d_1 = d_2 = 0,\ d_3 \neq 0, ..., d_5 \neq 0$. Then $\wedge^2 B_{12} = 0$, where B_{12} is the submatrix of matrix B without the first and second rows.

In all other cases system (7.6) does not have solutions. From here it follows that in the space $\mathbb{X}$ there exists the space $\mathbb{X}_1$ such that $\dim \mathbb{X}_1 \cap B(\mathbb{U}) = 1$ and $\dim \mathbb{X}_1/1 < 5/2$. It means that the system (A, B) is the null-form, which is defined by the invariants $a_1, ..., a_5, I_{23}, I_{24}, I_{34}$(see Theorems 2.2 and 5.4).

(b) $p > 2$. In order that to generalize the proof of Theorem 7.3 in this case, it is necessary to do alterations in the structure of diagonal matrix D only. The diagonal elements of this matrix will be have the form

$$d_i = \sum_{j=1}^{2p-1} (-1)^i f_{ij}(\alpha_1, ..., \alpha_{2p+1}) \underbrace{\Delta_{i_1 i_2} \cdot ... \cdot \Delta_{j_1 j_2}}_{p},$$

where $\{i_1, i_2, ..., j_1, j_2\} = \{1, ..., 2p+1\};\ 1 \leq i_1 < i_2 \leq 2p+1, ...,\ 1 \leq j_1 < j_2 \leq 2p+1;\ i = 1, ..., 2p+1$.

In order that the conditions of Theorem 2.1 were satisfied, it is necessary the equality to zero of $Bin(p+1, 2) = p(p+1)/2$ invariants (5.11). However, among these invariants there exist $(p-1)(p-2)/2$ syzygies [37, 44].

Invariants (5.11) are coordinates of bivector

$$\wedge^2 \begin{pmatrix} \det(B, AB, ..., A^{p-1}B, A^p b_1) & \det(B, AB, ..., A^{p-1}B, A^p b_2) \\ \vdots & \vdots \\ \det(B, AB, ..., A^{p-1}B, A^{2p} b_1) & \det(B, AB, ..., A^{p-1}B, A^{2p} b_2) \end{pmatrix}.$$

Thus, among generators of the ring $\mathbb{C}[A, B]^{\mathbb{S}L}$ there must be $p(p+1)/2 - (p-1)(p-2)/2 = 2p-1$ algebraically independent invariants (5.11). These invariants are indicated in Theorem 7.3.

Now we compute the dimension of space of orbits for system of type $(2p+1, 2)$.

From formula (4.1) it follows that for the system of type $(2p+1, 2)$ we have $\dim_{\mathbb{C}} \mathbb{O}^{\mathbb{S}L}(\mathbb{S}) = 4p$. It means that in Theorem 7.3 the number of generators of ring $\mathbb{C}[A, B]^{\mathbb{S}L}$ is minimal. $\square$

7.4. Ring of invariants for system (A, B) of type (n, m), $n > m$

As regards of a description of rings of invariants for systems of arbitrary types, we can state the following reasons.

Denote by $\mathbb{W}^\circ \subset \mathbb{S}$ an algebraic variety of all null-forms of space $\mathbb{S}$ with respect to action (2.3) of group $\mathbb{S}L$. Let also $f_1(\cdot), ..., f_k(\cdot)$ be homogeneous invariant polynomials defining the variety $\mathbb{W}^\circ$. (The last means that if $\mathbb{W}_{\max}$ is the maximal subvariety in $\mathbb{S}$ such that $\forall y \in \mathbb{S}_{\max}\ \ f_1(y) = 0, ..., f_k(y) = 0$, then $\mathbb{W}_{\max} = \mathbb{W}^\circ$.)

We will denote by $\mathbb{C}[f_1,...,f_k]$ a ring of invariants generating subvariety $\mathbb{W}^{\circ}$. Then the following theorem holds true.

Theorem 7.4. *(See [35]).* $\mathbb{C}[\mathbb{S}]^{\mathbb{S}L} = \overline{\mathbb{C}[f_1,...,f_k]}$, *where the symbol* $\overline{\mathbb{C}[f_1,...,f_k]}$ *means the integer closure of the ring* $\mathbb{C}[f_1,...,f_k] \subset \mathbb{C}[\mathbb{S}]^{\mathbb{S}L}$ *in the ring* $\mathbb{C}[\mathbb{S}]^{\mathbb{S}L}$.

Now we return to the equivalence problem. From Theorem 7.4 it follows that although rings $\mathbb{C}[f_1,...,f_k]$ and $\mathbb{C}[\mathbb{S}]^{\mathbb{S}L}$ do not coincide, but their quotient fields coincide: $\mathbb{C}(f_1,...,f_k) = \mathbb{C}(\mathbb{S})^{\mathbb{S}L}$. It means that invariants $f_1(\cdot),...,f_k(\cdot)$ define a point $(A,B) \in \mathbb{S}$ to within a birational equivalence, and consequently, their knowledge completely gives the solution of equivalence problem for systems $(A_1,B_1) \in \mathbb{F}_s \subset \mathbb{S}$ and $(A_2,B_2) \in \mathbb{F}_s \subset \mathbb{S}$.

Since in general case the set $\mathbb{F}_s$ of stable systems is unknown, for the solution of the equivalence problem an open subset in $\mathbb{F}_s$, which is defined by the conditions of Theorem 6.2, it can be used.

For the decision of equivalence problem Theorem 7.4 can be used as follows. Assume that the homogeneous invariants $a_1(A),...,a_n(A), I_1(A,B),...,I_r(A,B)$ define the manifold $\mathbb{W}^{\circ} \subset \mathbb{S}$ of null-forms (A,B) of type (n,m). It means that polynomials $a_1(A),...,a_n(A), I_1(A,B), ..., I_r(A,B)$ are generators of the quotient field $\mathbb{C}(A,B)^{\mathbb{S}L}$ of ring $\mathbb{C}[A,B]^{\mathbb{S}L}$.

Let $\mathcal{I} = \mathbb{C}[a_1,...,a_n, I_1,...,I_r] \subset \mathbb{C}[A,B]$ be an ideal, which is generated by the polynomials $a_1,...,a_n, I_1,...,I_r$ in the ring of polynomials from elements $a_{11},...,a_{nn}, b_{11},...,b_{nm}$ of matrices A and B.

Let $g(A,B)$ be an polynomial in $\mathbb{C}[A,B]$. Assume that there exists an integer k such that from the condition $g^k(A,B) \in \mathcal{I}$ it follows that $g(A,B) \in \mathcal{I}$.

Definition 7.1. (See [35, 44]). The ideal $\mathcal{I} \in \mathbb{C}[A,B]$ is called a radical ideal if from the condition $g^k(A,B) \in \mathcal{I}$ it follows that $g(A,B) \in \mathcal{I}$.

Thus, in generic case we have $\mathcal{I} = \mathbb{C}[a_1,...,a_n, I_1,...,I_r] \subset \mathbb{C}[A,B]^{\mathbb{S}L}$. In order that the equality $\mathcal{I} = \mathbb{C}[a_1,...,a_n, I_1,...,I_r] = \mathbb{C}[A,B]^{\mathbb{S}L}$ takes place, the ideal $\mathcal{I}$ must be radical. (In this case, the conditions of Theorem 7.4 will be fulfilled.)

A verification of the last condition is a difficult problem. Therefore, in the present paper of question about construction of ring of invariants for arbitrary systems was not be considered. (The exception is made by the systems of types $(2p,2)$ and $(2p{+}1,2)$, for which the rings of invariants were indicated in Subsections 7.2 and 7.3.)

Nevertheless, we can build the generators of quotient field $\mathbb{C}(A,B)^{\mathbb{S}L}$ for the systems (A,B), which were considered in Section 5. Knowledge of these generators already allows to solve the equivalence problem. (However, their number is not minimal; it is a main lack of conception of null-forms.)

8. Description of invariants for system (2.1), (2.2)

Below, we will use the ring of invariants of matrix pair (C,A) with respect to action of group $\mathbb{S}L$. This ring can be got from algebra $\mathbb{C}[A,B]^{\mathbb{S}L}$ by replacements

$B \to C^T$ and $A \to A^T$. We will designate this ring by the symbol $\mathbb{C}[C, A]^{\mathbb{S}L}$, where $\mathbb{S}L = \{S \times W\} = \mathbb{S}L(n, \mathbb{C}) \times \mathbb{S}L(p, \mathbb{C})$. In addition, for system (2.1),(2.2) we will use the designation (C, A, B). We will also call the system (C, A, B) by a system of type (p, n, m), where numbers p, n, and m are dimensions of the output space, state space, and input space.

Note that results of this subsection can be got from the theorem about structure of generators of ring of invariants with respect to action (2.3) of group $\mathbb{S}L$ for one $(n \times n)$-matrix, m column vectors and p row vectors of dimension n [37].

Introduce the matrices

$$R_1(A, B) = (B, AB, ..., A^{n-1}B) \in \mathbb{C}^{n \times nm},$$

$$R_2(C, A) = (C^T, (CA)^T, ..., (CA^{n-1})^T)^T \in \mathbb{C}^{pn \times n},$$

$$R_3(C, A, B) = (CB, CAB, ..., CA^{n-1}B) \in \mathbb{C}^{p \times mp},$$

$$R_4(C, A, B) = ((CB)^T, (CAB)^T, ..., (CA^{n-1})^T B^T)^T \in \mathbb{C}^{mp \times m}.$$

Then action (2.3) of group $\mathbb{S}L$ on the space $\mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{p \times n}$ induce actions of the same group on spaces $\mathbb{C}^{n \times nm}$, $\mathbb{C}^{pn \times n}$, $\mathbb{C}^{p \times mp}$, and $\mathbb{C}^{mp \times m}$, which are given by the following formulas:

$$1) R_1(S^{-1}AS, S^{-1}BT) = S^{-1} \cdot R_1(A, B) \cdot \begin{pmatrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{pmatrix},$$

$$2) R_2(S^{-1}AS, WCS) = \begin{pmatrix} W & & 0 \\ & \ddots & \\ 0 & & W \end{pmatrix} \cdot R_2(C, A) \cdot S,$$

$$3) R_3(WCS, S^{-1}AS, S^{-1}BT) = W \cdot R_3(C, A, B) \cdot \begin{pmatrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{pmatrix},$$

$$4) R_4(WCS, S^{-1}AS, S^{-1}BT) = \begin{pmatrix} W & & 0 \\ & \ddots & \\ 0 & & W \end{pmatrix} \cdot R_4(C, A, B) \cdot T.$$

Further, as well as in Section 5, it is necessary to find decompositions of the representations

$$\bigwedge^n (\bigoplus_{i=1}^n T), \ \bigwedge^n (\bigoplus_{i=1}^n W), \ \bigwedge^p (\bigoplus_{i=1}^n T), \ \bigwedge^m (\bigoplus_{i=1}^n W)$$

on irreducible components. These decompositions are given by the following formulas:

$$\bigwedge^n\left(\bigoplus_{i=1}^{n} T\right) = \bigoplus_{\omega_1=(n_{11},...,n_{1d})} r_{\omega_1} \wedge^{n_{11}} T \otimes \wedge^{n_{12}} T \otimes ... \otimes \wedge^{n_{1d}} T,$$

$$\bigwedge^n\left(\bigoplus_{i=1}^{n} W\right) = \bigoplus_{\omega_2=(n_{21},...,n_{2d})} r_{\omega_2} \wedge^{n_{21}} W \otimes \wedge^{n_{22}} W \otimes ... \otimes \wedge^{n_{2d}} W,$$

$$\bigwedge^p\left(\bigoplus_{i=1}^{n} T\right) = \bigoplus_{\omega_1=(n_{31},...,n_{3d})} r_{\omega_3} \wedge^{n_{31}} T \otimes \wedge^{n_{32}} T \otimes ... \otimes \wedge^{n_{3d}} T,$$

$$\bigwedge^m\left(\bigoplus_{i=1}^{n} W\right) = \bigoplus_{\omega_4=(n_{41},...,n_{4d})} r_{\omega_4} \wedge^{n_{41}} W \otimes \wedge^{n_{42}} W \otimes ... \otimes \wedge^{n_{4d}} W.$$

(Here a meaning of denotations the same that in Section 5.)

In future we will be restricted only to the case $m = p$.

Introduce the following invariants of group $\mathbb{S}L$:

$$K_j = \det(CA^{r-1}B), \; j = 1, ..., r = Bin(n, m). \tag{8.1}$$

Theorem 8.1. *Let (C, A, B) be a system of type (n, m, p), where $m = p \leq n$. Then the ring of invariants $\mathbb{C}[C, A, B]^{\mathbb{S}L}$ of this system is generated:*

– by the coefficients $a_n, ..., a_n$ of characteristic polynomial of matrix A;

– by the generators of ring $\mathbb{C}[A, B]^{\mathbb{S}L}$ essentially depending on B;

– by the generators of ring $\mathbb{C}[C, A]^{\mathbb{S}L}$ essentially depending on C;

– by invariants $K_j = \det(CA^{r-1}B), \; j = 1, ..., r = Bin(n, m)$.

Proof. We assume that we know rings of invariants $\mathbb{C}[A, B]^{\mathbb{S}L}$ of system (A, B) and $\mathbb{C}[C, A]^{\mathbb{S}L}$ of systems (C, A). Assume also that the matrix A resulted to the diagonal form: $A = \mathtt{diag}\,(()\,\alpha_1, ..., \alpha_n)$. Then invariants (8.1) can be rewritten in the form:

$$K_j = \sum_{1\leq i_1<...<i_m\leq n} (\alpha_{i_1}^j \cdot ... \cdot \alpha_{i_n}^j) B_j C_j, \tag{8.2}$$

where $B_j (C_j)$ are coordinates of polyvector $\wedge^m B$ (of polyvector $\wedge^m C$); $j = 1, ..., r$.

Rewrite equations (8.2) in the matrix form:

$$\mathbf{K} = H(\alpha_1, ..., \alpha_n)\mathbf{D}, \tag{8.3}$$

where $H \in \mathbb{C}^{r\times r}$, $\mathbf{K} = (K_1, ..., K_r)^T$, $\mathbf{D} = (B_1C_1, ..., B_rC_r)^T$.

If $\mathrm{disc}(H(\alpha_1, ..., \alpha_n)) = 0$, then we will replace the matrix A by the matrix $A + \lambda E$, where E is the identity matrix. Then by the choice λ it is possible to obtain $\mathrm{disc}(H(\alpha_1 + \lambda, ..., \alpha_n + \lambda)) \neq 0$ (see Theorem 6.2).

Thus, we can consider that $\mathrm{disc}(H(\alpha_1, ..., \alpha_n)) \neq 0$. In this case the vector $\mathbf{D}$ is uniquely determined from equations (8.3). Since the function $\mathrm{disc}(H(\alpha_1, ..., \alpha_n))$

is regular, then from solvability of system (8.2) we get products of coordinates B_j and C_j; $j = 1, ..., r$.

We can consider that the solutions of system (8.3) are seeking on some open set $\mathcal{L} \subset \mathbb{C}^r$ such that if $\wedge^m B \in \mathcal{L}$ and $\wedge^m C^T \in \mathcal{L}$, then $B_j C_j = W_j \neq 0$; $j = 1, ..., r$. From here it follows that if coordinates of vector $\wedge^m B$ (or $\wedge^m C^T$) are known, then coordinates of vector $\wedge^m C^T$ (or $\wedge^m B$) are uniquely determined from the system of equations $B_j C_j = W_j \neq 0$; $j = 1, ..., r$.

Further, in the same way as in the proof of Theorem 7.2, we use the Hilbert theorem [35] and the method of construction of invariants (8.1). The proof is finished. □

It is possible to specify the results of Theorem 8.1 if to take advantage of Theorems 7.2 and 7.3. Two next theorems are the obvious corollaries of Theorem 8.1.

Denote by $\mathbb{O}^{\mathbb{G}}(C, A, B) \subset \mathbb{S}$ an orbit of system (C, A, B) with respect to action (2.3) of group $\mathbb{S}L$.

Note that the dimension of space of orbits $\mathbb{O}^{\mathbb{G}}(\mathbb{S})$ for system of type (p, n, m) is given by the formula:

$$\dim_{\mathbb{C}} \mathbb{O}^{\mathbb{S}L}(\mathbb{S}) = n(n + m + p) - \dim_{\mathbb{C}} \mathbb{S}L(n, \mathbb{C}) - \dim_{\mathbb{C}} \mathbb{S}L(m, \mathbb{C}) - \dim_{\mathbb{C}} \mathbb{S}L(p, \mathbb{C})$$

$$= n(p + m) + 3 - m^2 - p^2.$$

Let (C, A, B) be a system of type $(2, 2p, 2)$. In this case $\dim_{\mathbb{C}} \mathbb{O}^{\mathbb{S}L}(\mathbb{S}) = 8p - 5$.

Theorem 8.2. *Let (C, A, B) be a system of type $(2, 2p, 2)$, where $p \geq 1$. Then the ring of invariants $\mathbb{C}[C, A, B]^{\mathbb{S}L}$ of this system is generated:*

– *by $2p$ coefficients $a_1, ..., a_{2p}$ of characteristic polynomial of matrix A;*

– *by $2p - 2$ polynomials $\det(B, AB, ..., A^{p-1}B)$,...,$\det(B, AB, ..., A^{2p-2}B)$;*

– *by $2p - 2$ polynomials $\det(C^T, (CA)^T, ..., (CA^{p-1})^T)$,...,$\det(C^T, (CA)^T, ..., (CA^{2p-2})^T)$;*

– *by $2p - 1$ polynomials $\det(CB), \det(CAB), ..., \det(CA^{2p-2}B)$.*

The number $8p - 5$ of generators of ring $\mathbb{C}[C, A, B]^{\mathbb{S}L}$ is minimal.

Let (C, A, B) be a system of type $(2, 2p + 1, 2)$. In this case $\dim_{\mathbb{C}} \mathbb{O}^{\mathbb{S}L}(\mathbb{S}) = 8p - 1$.

Theorem 8.3. *Let (C, A, B) be a system of type $(2, 2p+1, 2)$, where $p \geq 1$. Then the ring of invariants $\mathbb{C}[C, A, B]^{\mathbb{S}L}$ of this system is generated:*

– *by $2p + 1$ coefficients $a_1, ..., a_{2p+1}$ of characteristic polynomial of matrix A;*

– *by $2p - 1$ polynomials*

$$\det \begin{pmatrix} \det(B, AB, ..., A^{p-1}B, A^i b_1) & \det(B, AB, ..., A^{p-1}B, A^i b_2) \\ \det(B, AB, ..., A^{p-1}B, A^j b_1) & \det(B, AB, ..., A^{p-1}B, A^j b_2) \end{pmatrix},$$

where $i = p, p + 1 \leq j \leq 2p$ and $i = p + 1, p + 2 \leq j \leq 2p$;

– by $2p-1$ polynomials

$$\det\begin{pmatrix} \det(C^T,...,(CA^{p-1})^T,(c_1A^i)^T)^T & \det(C^T,...,(CA^{p-1})^T,(c_2A^i)^T)^T \\ \det(C^T,...,(CA^{p-1})^T,(c_1A^j)^T)^T & \det(C^T,...,(CA^{p-1})^T,(c_2A^j)^T)^T \end{pmatrix},$$

where $i=p, p+1 \le j \le 2p$ and $i=p+1, p+2 \le j \le 2p$;

– by $2p$ polynomials $\det(CB), \det(CAB), ..., \det(CA^{2p-1}B)$.

The number $8p-1$ of generators of ring $\mathbb{C}[C,A,B]^{\mathbb{S}L}$ is minimal.

Let (C,A,B) be a system of type $(m, m+1, m)$ and let $a_1, ..., a_{m+1}$ be coefficients of characteristic polynomial of matrix A. In this case $\dim_{\mathbb{C}} \mathbb{O}^{\mathbb{S}L}(\mathbb{S}) = 2m+3$.

Theorem 8.4. *Let (C,A,B) be a system of type $(m, m+1, m)$, where $m \ge 1$. Let also $R(A,B)$ be matrix (5.3). Then we have*

$$\mathbb{C}[C,A,B]^{\mathbb{S}L}$$

$$= \mathbb{C}[a_1, ..., a_{m+1}, \det R(A,B), \det R(A^T,C^T), \det(CB), ..., \det(CA^{(m-1)}B],$$

where the number $2m+3$ of generators is minimal. In addition, the set of all $\mathbb{S}L$-stable systems is given by the condition $\det(R(A,B)\cdot R(A^T,C^T)) \ne 0$.

Proof of the last assertion follows from Subsection 6.2. Here we have

$$\det(R(A,B)\cdot R(A^T,C^T)) = I(A,B)\cdot I(A^T,C^T)$$

$$= \mathrm{disc}(A)\,\underbrace{\Delta_{1...m}(B)\cdot ... \cdot \Delta_{2...m+1}(B)}_{m+1}\,\underbrace{\Delta_{1...m}(C)\cdot ... \cdot \Delta_{2...m+1}(C)}_{m+1} \ne 0.\Box$$

9. Equivalence of linear control systems

The most invariants, which considered in the present article, are described in Theorem 6.1. It means that these invariants are minors of appropriate matrix $G(A,B)$ (or $R_i(A,B)$). From here it follows that all syzygies, which exist between invariants, always are automatically satisfied. It facilitates the search of minimal base of invariants (see Theorems 8.2 – 8.4).

Although the problem of description of generators of ring of invariants in general case is not solved, nevertheless the equivalence problem, which is important for applications, got the solution. (It is here enough to know the generators of quotient field only.)

All said before we can sum up by the following obvious theorem.

Let (C,A,B) be a system of type (p,n,m). Let also the numbers $a_1(A), ..., a_n(A)$ be the coefficients of characteristic polynomial of matrix A. Denote by $I_j(A,B), j=1,...,r_B$, the generators of quotient field $\mathbb{C}(A,B)^{\mathbb{S}L}$ depending on B and denote by $P_q(C,A), q=1,...,r_C$, the generators of quotient field $\mathbb{C}(C,A)^{\mathbb{S}L}$ depending on C. If $p=m$, then we denote by $K_l(C,A,B)=\det(CA^{l-1}B), l=1,...,r_{BC}$ the invariants depending on B and C.

Theorem 9.1. *Let the positive integers* (n, m)*, where* $n > m$*, be coprime. Let also* $p = m$*. Then two systems* $(C_1, A_1, B_1) \in \mathbb{S}_{open}$ *and* $(C_2, A_2, B_2) \in \mathbb{S}_{open}$ *of type (p,n,m) are equivalent if and only if there exist nonzero numbers* v *and* t *such that*

$$a_1(A_1) = a_1(A_2), ..., a_n(A_1) = a_n(A_2),$$

$$I_j(A_1, B_1) = vI_j(A_2, B_2), j = 1, ..., r_B = r_C,$$

$$P_q(C_1, A_1) = tP_q(C_2, A_2), q = 1, ..., r_C = r_B,$$

$$K_l(C_1, A_1, B_1) = (vt)^{1/r} K_l(C_2, A_2, B_2), l = 1, ..., r_{BC},$$

where $r = n \cdot Bin(m, s)$*,* $1 < s < m$*,* $s = n - dm$*,* d *is the integer part of* n/m*, and as the set* $\mathbb{S}_{open}$ *the set* $\mathbb{F}_s$ *of all* $\mathbb{S}L$*-stable systems can be taken .*

Proof. We have that the numbers n and m are coprime and even one of invariants $I_j(A, B)$ ($P_q(C, A)$) is not equal to zero. Then from Theorems 6.1 and 6.3 it follows that there exists some subset of all $\mathbb{S}L$-stable systems of type (n, m) ((p, n)), for which the equivalence problem has solution. □

For some types of systems the generators of quotient field can be taken from Theorems 5.1, 5.4, and 5.5.

Another important problem, which was not solved, it is the search problem of set of all $\mathbb{S}L$-stable systems. (The solution was got only for systems (A, B) of types $(4, 2)$ and $(m + 1, m)$.) However, it should be said that the set of all $\mathbb{S}L$-stable systems contains a subset, which is defined by the condition $\mathrm{disc}(A) \prod_{i=1}^{h_B} (\wedge^m B)_i \cdot \prod_{j=1}^{h_C} (\wedge^m C)_j \neq 0$, where $(\wedge^m B)_i$ ($(\wedge^p C)_j$) are coordinates of polyvector $\wedge^m B$ (of polyvector $\wedge^p C$); $h_B = Bin(n, m), h_C = Bin(n, p)$. In future the authors hope to solve the problem of description of all $\mathbb{S}L$-stable systems of type (p, n, m), $n > m$, $n > p$.

10. Reconstruction of differential equations system on the known multivariate time series

We assume that there are n characteristics (measurements and computations) of some dynamic process : $z_1(t_i), ..., z_n(t_i), i = 1, 2, ..., N$. In addition, we also suppose that these measurements are noisy. Thus, we have multivariate time series

$$z_1(t_i) = x_1(t_i) + \theta_1(t_i), ..., z_n(t_i) = x_n(t_i) + \theta_n(t_i), \tag{10.1}$$

which defined for $\forall t_i \in (t_1, t_N)$. Here $\forall i = 1, 2, ..., N$, we have $t_i = i\Delta t$ and $\Delta t = (t_N - t_1)/N$. In addition, we suppose that $\theta_1(t_i), ..., \theta_n(t_i)$ are Gaussian (white) noises, unable by definition to produce statistically systematical errors [29] – [31], [34], [49].

Finally, we assume that $x_1(t_i), ..., x_n(t_i)$ is a discrete approximation of some curve $\mathbf{x}(t) = (x_1(t), ..., x_n(t))^T \in \mathbb{R}^n$ [16], [29] – [31], [34], [49]. In the turn, it is assumed that the curve $\mathbf{x}(t)$ is a solution of some autonomous differential equations system. (The necessity of such description is dictated by the considerations resulted higher.)

Further, we use the procedure of determining unknown right-hand side of the system of differential equations (1.3), which was suggested in [16], [29] – [31], [34], [36], [49]. This procedure is based on the least squares method and the fact that we know sufficient precision the components of $\mathbf{x}(t)$ and its derivative $\dot{\mathbf{x}}(t)$.

In view of the fact that number N may be chosen arbitrary large, a high precision reconstruction may be achieved. Thus, we can expect that the solution of reconstructed system will be near the purified solution $\mathbf{x}(t)$.

However, it should be said that one important circumstance, which can arise up at a reconstruction, remained outside the attention of authors of article [34], [49]. The point is that in [34], [49] it is assumed that this interval (t_1, t_N) is finite. If the problem of long-term prediction is considered, it is necessary to assume that $t_N \to \infty$. In this case a reconstruction must be fulfilled so that system (1.3) had the bounded solutions [2] – [6].

Now suppose that the dimension n of phase space in which the dynamic process under study, is known. Assume also that m different variables $(m < n)$ describing this process, for which time series can be measured, are also known. Then the matrices A and B of system (1.3) can be represented in the following form [5, 20]:

$$S^{-1}AS = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & A_{ii} & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}, S^{-1}BT = \begin{pmatrix} B_{11} & \cdots & 0 \\ \vdots & B_{ii} & \vdots \\ 0 & \cdots & B_{mm} \end{pmatrix}, \quad (10.2)$$

where

$$A_{ii} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{i,\eta_{i-1}+1} & -a_{i,\eta_{i-1}+2} & \cdots & -a_{i,\eta_i} \end{pmatrix} \in \mathbb{R}^{\nu_i \times \nu_i}; B_{ii} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{\nu_i};$$

$$\eta_0 = 0,\ \eta_1 = \nu_1,\ \eta_2 = \nu_1 + \nu_2, ...,\ \eta_i = \nu_1 + ... + \nu_i;$$

$$A_{ij} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ -a_{i,\delta_{j-1}+1} & \cdots & -a_{i,\delta_j} \end{pmatrix} \in \mathbb{R}^{\nu_i \times \nu_j},\ i \neq j;$$

$$\delta_0 = 0,\ \delta_1 = \nu_1,\ \delta_2 = \nu_1 + \nu_2 ..., \delta_i = \nu_1 + ... + \nu_j.$$

Here integer positive numbers ν_i accept two values: $\nu_i = k+1$, if $i = 1, ..., n-km$ and $\nu_i = k$, if $i = n-km+1, ..., m$, where k is the integer part of number n/m; $i, j = 1, ..., m$.

All theory described higher allows to form the following procedure of reconstruction of differential equations system on the known multivariate time series (1.2) of process $\mathbf{P}(t)$:

1. Let m be the number of measured variables. By the embedding technique ([34, 49]) define the embedding dimension space $n > m$.

2. By a procedure of approximation of higher derivatives, compute a different composition from $n - m$ derivatives (these are missing or hidden variables) of the complementary m known variables to the basis of the phase space (for example, $x(t) = x_1, \dot{x}(t) = x_2, \ddot{x}(t) = x_3, ...$) ([16], [36]).

3. Select the activation functions for system (1.3). After this, by the least square method ([7], [16], [36]) compute the indeterminate coefficients of matrices A, B, and C of system (1.3) reconsructed from the known multivariate time series (1.2).

4. Compose the linear control system (1.9) and by methods of Sections 5 – 9 compute the polynomial invariants $a_1, ..., a_n; I_1(A, B) \neq 0, ..., I_q(A, B); P_1(C, A) \neq 0, ..., P_r(C, A);\ K_1(C, A, B) \neq 0, ..., K_s(C, A, B)$ of this system.

5. Repeat the steps 1 – 4 of algorithm and compute new polynomial invariants

$$a_1^{'}, ..., a_n^{'};\ I_1^{'}(A^{'}, B^{'}) \neq 0,\ dots,\ I_q^{'}(A^{'}, B^{'});\ P_1^{'}(C^{'}, A^{'}) \neq 0, \dots,$$
$$P_r^{'}(C^{'}, A^{'});\ K_1^{'}(C^{'}, A^{'}, B^{'}) \neq 0, \dots,\ K_s^{'}(C^{'}, A^{'}, B^{'})$$

of system (1.5) reconsructed from the known multivariate time series (1.4).

6. Verify the restrictions

$$|a_i - a_i^{'}| < \epsilon, |I_k/I_1 - I_k^{'}/I_1^{'}| < \epsilon, |P_j/P_1 - P_j^{'}/P_1^{'}| < \epsilon, |K_l/K_1 - K_l^{'}/K_1^{'}| < \epsilon, \quad (10.3)$$

where $i = 1, ..., n; k = 2, ..., q; j = 2, ..., r; l = 2, ..., s$. (It is here taken into account that among invariants I_k there is even one nonzero. We assumed that it is the invariant I_1. The same assumption must be satisfied for invariants P_j and K_l.) If they are faithful, then the problem of adequacy of neural network models (1.3) and (1.5) can considered solved.

10.1. Reconstruction of dynamic processes in a contact electric network

In article [43] a question about construction of regulator for stabilization of voltage in a contact network was studied. In this article the system of differential equations for the design of behavior of current (I) and voltage (U) in the mentioned netwowk was constructed. However, the problem of adequacy of the got model and real dynamics of electric processes in the contact network was not investigated.

Below, we will remove the indicated lack. For this purpose we apply the method of invariant reconstruction of differential equations to the process shown on the following Fig.2(a1).

In most practical cases of the reconstruction of differential equations according to the results of measurements of certain variables, it is impossible to change the composition of the measuring instruments. This means that the representation of the constructed model in other variables is also impossible. Therefore, the question of checking the adequacy of the model and the real system remains open.

In this paper, we propose the following method for checking the adequacy of the model and the real system.

The dynamics of arbitrary autonomous system of differential equations is fully determined by the parameters of this system.

Let $\mathbb{K}$ be a set of restrictions on the parameters of system (1.3), which determines its desired behavior. In the general case the restrictions are included in $\mathbb{K}$ should be the functions of invariants of system (1.3). (For example, the behavior of any linear dynamic system can be described using the coefficients of the characteristic polynomial, which, in turn, are invariants with respect to changes of variables of this system.)

At present, the construction of basis of polynomial invariants of an arbitrary polynomial system of differential equations is an unsolved problem. Therefore, we can use only a part of the polynomial invariants that are constructed in this article. In this regard, we propose the following methodology for checking the adequacy of the model and the real system.

1. Using the method of least squares to restore some system ($\mathcal{S}_1$) of differential equations according to the measurement results.

2. Repeat measurements of the same dynamic characteristics of the process under study. After that, perform a reconstruction of new system differential equations ($\mathcal{S}_2$).

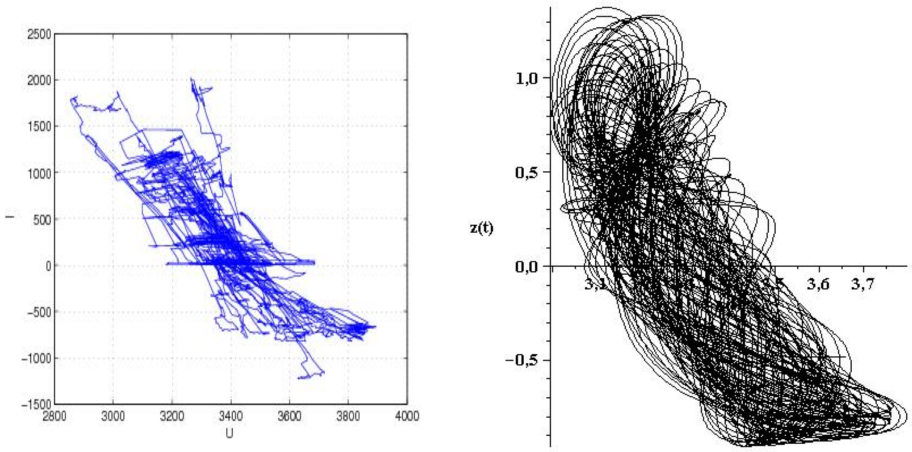
3. For systems ($\mathcal{S}_1$), ($\mathcal{S}_2$) check the implementation of inequalities of set $\mathbb{K}$.

4. If the current-voltage characteristics of systems ($\mathcal{S}_1$) and ($\mathcal{S}_2$) are equivalent, then it can be argued that the system reconstructed from measurement results adequately describes the dynamics of the real process in the contact network.

We will assume that we can measure the voltage and current, and also if it is possible other dynamic characteristics of contact electric network. We also suppose that among these characteristics can be derivatives with respect to t from the voltage and current. (If the derivatives can not be measured, it is assumed that there exist smooth enough approximations of these derivatives.)

In [43] a structure of the process of represented on Fig. 2 (a2) was described by the following system of differential equations:

$$\begin{cases} \dot{x}_1(t) = x_2, \\ \dot{x}_2(t) = a_{20} + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + b_{11}x_1^2 + b_{12}x_1x_2, \\ \dot{x}_3(t) = x_4, \\ \dot{x}_4(t) = a_{40} + a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + b_{22}x_1^2 + b_{21}x_1x_2. \end{cases} \tag{10.4}$$



(a1) (a2)

Fig. 10.2. A current-voltage characteristic ($U - I$ characteristic) of contact network: (a1) experimental data; (a2) modeling after14000 measurements for system (10.5) in Kv and Ka [7,43]; here $x = U, z = I$.

The system got as a result of reconstruction has the following form:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.0193 - 0.0072x(t) + 0.0218y(t) - 0.000814z(t) + 0.0057u(t) \\ \qquad -0.0039x(t)y(t) + 0.000422x^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.0294 - 0.0145x(t) - 0.8506y(t) - 0.0019z(t) - 0.0095u(t) \\ \qquad +0.2380x(t)y(t) + 0.0017x^2(t). \end{cases} \tag{10.5}$$

For verification of conditions Theorem 9.1 we will carry beginning of coordinates of system (10.5) in the equilibrium point $(3.335950, 0, -0.027746, 0)$. In the total we obtain such system:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -0.004384x(t) + 0.008790y(t) - 0.000814z(t) + 0.005700u(t) \\ \qquad -0.00097(x(t) + y(t))^2 + 0.00097(x(t) - y(t))^2 + 0.000422x^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = -0.003158x(t) - 0.056644y(t) - 0.0019z(t) - 0.0095u(t) \\ \qquad +0.05950(x(t) + y(t))^2 - 0.05950(x(t) - y(t))^2 + 0.0017x^2. \end{cases} \tag{10.6}$$

Let the activation function for system (10.6) be $u_i = (x \pm y)^2; i = 1, 2; u_3 = x^2$. In this case for system (1.9) the matrices A , B , and C have the following forms:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.004384 & 0.008790 & -0.000814 & 0.005700 \\ 0 & 0 & 0 & 1 \\ -0.003158 & -0.056644 & -0.001900 & -0.009500 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -0.00097 & 0.00097 & 0.000422 \\ 0 & 0 & 0 \\ 0.059500 & -0.059500 & 0.001700 \end{pmatrix}, C^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (10.7)$$

Note that for system (10.7) we have $m = p = 3$ and $\mathtt{rank}B = \mathtt{rank}C = 2 < 3$. Therefore, invariant analysis of such system cannot be performed only using Theorem 9.1. In this case, we restrict ourselves to a visual comparison of phase portraits (see Fig.2(a2)) of system (10.5) with real process presented on Fig.2(a1).

10.2. Example

In accordance with the results given in [7, 43], the process presented on Fig. 2(a1) can be described by the following system of differential equations:

$$\begin{cases} \dot{x}_1(t) = x_2, \\ \dot{x}_2(t) = a_{20} + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + b_{11}x_2^2 + b_{12}x_4^2, \\ \dot{x}_3(t) = x_4, \\ \dot{x}_4(t) = a_{40} + a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + b_{22}x_2^2 + b_{21}x_4^2. \end{cases} \quad (10.8)$$

Here $x_1 = U, x_2 = \dot{U}, x_3 = I, x_4 = \dot{I}$.

The reconstructed system obtained from the results of 14000 measurements can be presented in the following form:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.00676 - 0.0018348x(t) + 0.005384y(t) - 0.00005z(t) + 0.001827u(t) \\ \qquad +0.40864y^2(t) - 0.02409u^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.028596 - 0.000646x(t) - 0.03005y(t) - 0.001789z(t) - 0.00628u(t) \\ \qquad -2.50578y^2(t) + 0.06470u^2(t). \end{cases} \quad (10.9)$$

Here $x = x_1 = U, y = x_2 = \dot{x} = \dot{U}, z = x_3 = I, u = x_4 = \dot{z} = \dot{I}$.

For verification of conditions Theorem 9.1 we will carry beginning of coordinates of system (10.9) in the equilibrium point $(3.67857, 0, 0.027012, 0)$. In the total we obtain such system:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -0.0018348x(t) + 0.005384y(t) - 0.00005z(t) + 0.001827u(t) \\ \qquad +0.40864y^2(t) - 0.02409u^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = -0.000646x(t) - 0.03005y(t) - 0.001789z(t) - 0.00628u(t) \\ \qquad -2.50578y^2(t) + 0.06470u^2(t). \end{cases} \quad (10.10)$$

Let the activation function for system (10.10) be $u_i = z_i^2; i = 1, ..., 3$. In this case for system (1.9) the matrices A , B , and C have the following forms:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.001835 & 0.005384 & -0.00005 & 0.001827 \\ 0 & 0 & 0 & 1 \\ -0.000646 & -0.030050 & -0.001789 & -0.006280 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0.408640 & -0.024090 \\ 0 & 0 \\ -2.505780 & 0.064700 \end{pmatrix}, C^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

System (10.9) has the following invariants:

$$a_1 = -0.000914, a_2 = 0.001367, a_3 = 0., a_4 = 0.; I_1 = -0.001612, I_2 = 0.;$$

$$P_1 = 0.000003, P_2 = 0.; K_1 = -0.034036, K_2 = K_3 = K_4 = K_5 = 0.$$

Now we will consider the modeling of the dynamic process in Fig.2(a1), but using time series only the first 10000 measurements. In this case, we get the following system:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.006 - 0.0017348x(t) + 0.005784y(t) - 0.00004z(t) + 0.001927u(t) \\ \qquad +0.30864y^2(t) - 0.02409u^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.028596 - 0.000546x(t) - 0.02005y(t) - 0.001789z(t) - 0.00528u(t) \\ \qquad -2.50578y^2(t) + 0.06370u^2(t). \end{cases} \tag{10.11}$$

For this system the equilibrium point is $(3.44600, 0, 0.54672, 0)$. Now repeating the procedure for system (10.9) described above in Subsection 10.2, we arrive at such matrices of system (1.9):

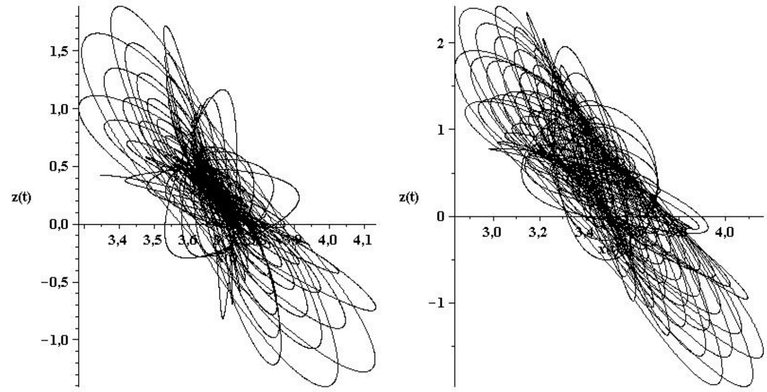
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.001735 & 0.005738 & -0.00004 & 0.001927 \\ 0 & 0 & 0 & 1 \\ -0.000546 & -0.020050 & -0.001789 & -0.006658 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0.308640 & -0.024090 \\ 0 & 0 \\ -2.505780 & 0.063700 \end{pmatrix}, C^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

System (10.11) has the following invariants:

$$a_1 = -0.000921, a_2 = 0.003532, a_3 = 0., a_4 = 0.; I_1 = -0.001657, I_2 = 0.;$$

$$P_1 = 0.000003, P_2 = 0.; K_1 = -0.034016, K_2 = K_3 = K_4 = K_5 = 0.$$



(a1) (a2)

Fig. 10.3. The current-voltage characteristics of contact electric network modeling experimental characteristic on Fig.2 at different lengths of time series: (a1) 14000 measurements for system (10.9); (a2) 10000 measurements for system (10.11)

The verification of the inequalities (10.3) shows that they are valid at $\epsilon = 0.1 \cdot 10^{-2}$. This circumstance means that for describing the process shown in Fig.2 can be used any from systems (10.9) or (10.11). The last statement can be confirmed by Fig.3 (here $z = I$ and $x = U$).

11. Conclusion

The results given above allow us to draw the following conclusions.

1. The problem of description of algebraic invariants for the linear control system is solved.

2. With the help of these invariants, the equivalence problem of two nonlinear systems obtained from results of studies of the corresponding time series is also solved.

Indeed, consider two nonlinear systems of 4th order:

$$\begin{cases} \dot{x}_1(t) = x_2, \\ \dot{x}_2(t) = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + v_1(x_1, ..., x_4), \\ \dot{x}_3(t) = x_4, \\ \dot{x}_4(t) = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + v_2(x_1, ..., x_4), \end{cases} \tag{11.1}$$

and

$$\begin{cases} \dot{x}_1(t) = x_2, \\ \dot{x}_2(t) = x_3, \\ \dot{x}_3(t) = a'_{31}x_1 + a'_{32}x_2 + a'_{33}x_3 + a'_{34}x_4 + v'_1(x_1, ..., x_4), \\ \dot{x}_4(t) = a'_{41}x_1 + a'_{42}x_2 + a'_{43}x_3 + a'_{34}x_4 + v'_2(x_1, ..., x_4), \end{cases} \tag{11.2}$$

where $v_1(...), v_2(...)$ and $v'_1(...), v'_2(...)$ are nonlinear functions.

Invariants of systems (11.1) and (11.2) in accordance with Theorem 7.2 look like: $a_1, ..., a_4, I_1 = -1, I_2 = -a_{22}a_{44} + a_{24}a_{42}$, and $a_1^{'}, ..., a_4^{'}, I_1^{'} = 0, I_2^{'} = -a_{34}^{'}$.

Since $I_1 \neq I_1^{'}$, then these systems are not equivalent for any nonlinearities within them. From here it follows that the structure of any system of differential equations obtained as a result of reconstruction is uniquely determined by the embedding dimension space n and the number of independent measured variables m $(n > m)$ [7]. This structure is represented by matrices A and B (see (10.2)).

3. Note that using the invariants of system (1.9), it is impossible to fully describe system (1.3). (For this purpose, it is necessary to find the basis of all polynomial invariants of this system.) However, information about the behavior of measured variables already contains information about non-linearities that determine real process in the contact electric network. Therefore, the matrices A, B and C of system (1.9) determine the linear part and the structure of the nonlinear part of system (1.3).

4. As the considered examples show, the invariant reconstruction method is also suitable when we are dealing with chaotic processes taking place in the contact network.

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