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On the Landau-Kolmogorov inequality between $\|f'\|_\infty$, $\|f\|_\infty$ and $\|f'''\|_1$

Класична задача теорії наближення про знаходження точних нерівностей для норм похідних функцій вперше виникає на початку XX сторіччя в роботах Е. Ландау, Ж. Адамара, Г. Г. Ґарді, Дж. І. Літлвуда. Глибокі дослідження цієї задачі проводили багато визначних математиків, серед яких А. М. Колмогоров, С. Б. Стєчкін, Е. М. Стейн, Л. В. Тайков, Ю. І. Любіч, М. П. Купцов, І. Дж. Шонберг, С. Карлін, А. Пінкус, В. Ф. Бабенко, В. В. Арєстов, С. О. Пічугов, В. О. Кофанов, Б. Д. Боянов та інші. Проте для функцій, визначених на скінченному відрізку, ця задача залишається малодослідженою.

Задача про знаходження точних нерівностей для норм похідних функцій, визначених на скінченному відрізку, називається задачею Ландау-Колмогорова та має дві, взагалі кажучи, різні постановки. Одна з них полягає в знаходженні множини всіх пар (A, B) невід'ємних чисел, для яких виконується точна адитивна нерівність $\left\|f^{(k)}\right\|_q \leqslant A\|f\|_p + B\left\|f^{(r)}\right\|_s$, $f \in L^r_{p,s}$. Інша, більш загальна постановка, полягає у відшуканні модуля неперервності оператора диференціювання $D^k : L_p \to L_q$ на множині W^r_s функцій f, для яких $\left\|f^{(r)}\right\|_s \leqslant 1$.

В даній роботі задача Ландау-Колмогорова розв'язана у ситуації $p = q = \infty$, $s = 1$, $k = 1$ та $r = 3$. Для її розв'язання застосовано відомий метод Стєчкіна проміжного наближення оператора диференціювання за допомогою лінійних обмежених операторів. Для цього було побудовано скінченно-різницевий обмежений оператор $S_N : L_\infty \to L_\infty$. Обраний метод доведення дозволив розв'язати споріднені задачі про найкраще наближення оператора диференціювання першого порядку D^1 лінійними обмеженими операторами на множині W^3_1 та про найкраще відновлення оператора D^1 на елементах множини W^3_1, які задано з похибкою.

Ключові слова: задача Ландау-Колмогорова, задача Стечкіна, найкраще відновлення оператора, модуль неперервності оператора.

Решена задача Ландау-Колмогорова о нахождении точных аддитивных неравенств, оценивающих $\|f'\|_\infty$ через $\|f\|_\infty$ и $\|f'''\|_1$. Также решены родственные задачи о наилучшем приближении оператора дифференцирования первого порядка D^1 линейными ограниченными и наилучшем восстановлении оператора D^1 на элементах класса, заданных с ошибкой.

Ключевые слова: задача Ландау-Колмогорова, задача Стечкина, наилучшее восстановление оператора, модуль непрерывности оператора.

We solve the Landau-Kolmogorov problem on finding sharp additive inequalities that estimate $\|f'\|_\infty$ in terms of $\|f\|_\infty$ and $\|f'''\|_1$. Simalteneously we solve related problems of the best approximation of first order differentiation operator D^1 by linear bounded ones and the best recovery of operator D^1 on elements of a class given with error.

Key words: the Landau-Kolmogorov problem, the Stechkin problem, best recovery of operator, modulus of continuity of operator.

MSC2010: Pri 41A17, Sec 41A35, 26D10

1. Introduction

Let $k, r \in \mathbb{N}$, $r > k$, and $1 \leqslant p, q, s \leqslant \infty$. By L_p we denote the space of measurable functions $f : [0,1] \to \mathbb{R}$ endowed with the standard norm

$$\|f\|_p = \begin{cases} \left(\int_0^1 |f(t)|^p \, \mathrm{d}t\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname{esssup}\{|f(t)| \, : \, t \in [0,1]\}, & p = \infty. \end{cases}$$

Let L_p^r be the space of $(r-1)$-times differentiable functions $f : [0,1] \to \mathbb{R}$ having absolutely continuous on $[0,1]$ derivative $f^{(r-1)}$ and $f^{(r)} \in L_p$. Inequalities of the form

$$\left\|f^{(k)}\right\|_q \leqslant A\|f\|_p + B\left\|f^{(r)}\right\|_s \tag{1.1}$$

that hold true for every function $f \in L_s^r$ with constants $A, B \geqslant 0$ independent of f, are called *additive Kolmogorov type inequalities*. First investigations of inequalities (1.1) appeared in pioneer works of E. Landau [21], J. Hadamard [16], G. Hardy and J. Littlewood [17]. Naturally, the problem of finding sharp additive Kolmogorov type inequalities presents the most interest.

Problem 1. Find the set $\Gamma\left(D^k; L_s^r\right) = \Gamma\left(D^k : L_p \to L_q; L_s^r\right)$ of all pairs (A, B) of non-negative numbers satisfying conditions:
1) inequality (1.1) holds true for every function $f \in L_s^r$;
2) for every $\varepsilon > 0$, there exists a function $f_\varepsilon \in L_s^r$ such that

$$\left\|f_\varepsilon^{(k)}\right\|_q > A\left\|f_\varepsilon\right\|_p + (B - \varepsilon)\left\|f_\varepsilon^{(r)}\right\|_s.$$

The above problem is usually called *the Landau-Kolmogorov problem.* Its more general setting (see [26, 5]) consists in finding the modulus of continuity of differentiation operator $D^k : L_p \to L_q$ on the class

$$W_s^r := \left\{f \in L_s^r \, : \, \left\|f^{(r)}\right\|_s \leqslant 1\right\}.$$

Problem 2. For every $\delta > 0$, find

$$\Omega\left(\delta; D^k; W_s^r\right) := \sup\left\{\left\|f^{(k)}\right\|_q \, : \, \|f\|_p \leqslant \delta, \, \left\|f^{(r)}\right\|_s \leqslant 1\right\}.$$

At present, the Landau-Kolmogorov problem is solved completely in the following situations:

1. $r = 2$, $k = 1$, $p = q = s = \infty$ – E. Landau [21], C. Chui and P. Smith [14];

2. $r = 3$, $k = 1, 2$, $p = q = s = \infty$ – M. Sato [25], A. Zviagintsev and A. Lepin [31];

3. $r = 4$, $k = 1, 2, 3$, $p = q = s = \infty$ – A. Zviagintsev [30], N. Naidenov [24];

4. $r \in \mathbb{N} \setminus \{1\}$, $k = r - 1$, $q = \infty$, $1 \leqslant p \leqslant \infty$, $s = 1$ – V. Burenkov [12];

5. $r = 2$, $k = 1$, $p = q = \infty$, $1 \leqslant s < \infty$ – Yu. Babenko [6, 7], V. Burenkov and V. Gusakov [13];

6. $r = 2$, $k = 1$, $p = s = \infty$, $1 \leqslant q < \infty$ – N. Naidenov [23].

Partial solutions of Problems 1 and 2 can be found in the papers [11, 18, 26, 4, 15, 10, 27]. For the overview of other results in this and closely related directions we refer the reader to books [20, 5] and surveys [2, 26]. In this paper we will solve Problems 1 and 2 in the case $k = 1$, $r = 3$, $p = q = \infty$ and $s = 1$.

Let us now consider several extremal problems of Approximation Theory that are closely related to the Landau-Kolmogorov problem.

The Markov-Nikolskii problem. For $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the set of algebraic polynomials of degree at most n. Inequality of the form

$$\left\|Q^{(k)}\right\|_q \leqslant M \cdot \|Q\|_p\,, \tag{1.2}$$

that holds true for every polynomial $Q \in \mathcal{P}_n$ with some constant $M > 0$ independent of Q, is called *the Markov-Nikolskii inequality.* The smallest possible constant $M = M_{k,n}^{p,q}$ in (1.2) is called *the Markov-Nikolskii constant.* Detailed overview of cases when sharp constant in inequality (1.2) is known can be found in books [19, 22] and references therein.

In [5, Theorem 4.6.2] it was proved that $M_{k,r-1}^{p,q}$ coincides with the smallest possible constant A in inequality (1.1). More rigorously, there holds true the following statement.

Proposition 1. Let $1 \leqslant p, q, s \leqslant \infty$ and $k, r \in \mathbb{N}$, $r > k$. Then, for every $A \geqslant M_{k,r-1}^{p,q}$, there exists $B = B(A)$ such that $(A, B) \in \Gamma\left(D^k; L_s^r\right)$. Moreover,

$$M_{k,r-1}^{p,q} = \inf\left\{A : (A, B) \in \Gamma\left(D^k; L_s^r\right)\right\}.$$

Also, for convenience, we recall that $M_{1,2}^{\infty,\infty} = 8$ (see, *e.g.* [19, 22]).

The Stechkin problem. Following [28, 29], we remind the statement of the problem on the best approximation of operators by linear bounded ones. Let X, Y be Banach spaces, $A : X \to Y$ be unbounded operator with domain $\mathcal{D}(A)$, $W \subset \mathcal{D}(A)$ be some class. For every linear bounded operator $S : X \to Y$ denote the error of approximation of operator A by operator S on the class W:

$$U\,(A, S; W) := \sup_{x \in W} \|Ax - Sx\|_Y.$$

For $N > 0$, let $\mathcal{L}(N)$ be the space of linear bounded operators $S : X \to Y$.

Problem 3. For every $N > 0$, find the error of *the best approximation of operator* A by linear bounded operators on the class W

$$E_N(A;W) = \inf_{S\in\mathcal{L}(N)} \sup_{x\in W} \|Ax - Sx\|_Y\,, \tag{1.3}$$

and find extremal operators $S^* \in \mathcal{L}(N)$ (if any exists) for which the infimum in the right hand part of (1.3) is achieved.

In [29, §2] S. B. Stechkin established simple but powerful lower bound for (1.3) in terms of modulus of continuity of the operator A on the class W. Good overview of known results on the problem of the best approximation of unbounded operators by linear bounded ones as well as discussion of related problems can be found in surveys [3, 2] and book [5].

For spaces $X = L_p$, $Y = L_q$, differential operator $A = D^k$ and class $W = W^r_s$, Problem 3 was solved in the following situations: (1) $r = 2$, $k = 1$, $q = s = \infty$, $1 \leqslant p \leqslant \infty$ – in [8]; (2) $r = 3$, $k = 1, 2$, $p = q = s = \infty$ – in [9].

For convenience, we formulate the following simple corollary from [29, §2] establishing the relation between the Stechkin and Landau-Kolmogorov problems.

Proposition 2. Let $1 \leqslant p, q, s \leqslant \infty$ and $k, r \in \mathbb{N}$, $r > k$. Then, for every $(A, B) \in \Gamma\left(D^k; L^r_s\right)$, we have $B \leqslant E_N\left(D^k; W^r_s\right)$.

In this paper we will solve Problem 3 for operator $A = D^k$ and class $W = W^r_s$ in the case $r = 3$, $k = 1$, $p = q = \infty$ and $s = 1$.

Problem of the best recovery of operators. Let X, Y be Banach spaces. Following [1] (see also [2] and [5, §7.1]), we consider the problem of the best recovery of operator $A : X \to Y$ with domain $\mathcal{D}(A)$ on the class $W \subset \mathcal{D}(A)$. For recovery of operator A we will use the set $\mathcal{R}$ of operators (or single-valued mappings) $S : X \to Y$. Usually one of the following sets is considered as the choice of the set $\mathcal{R}$: the set $\mathcal{O} = \mathcal{O}(X, Y)$ of all linear operators acting from X to Y, the set $\mathcal{L} = \mathcal{L}(X, Y)$ of all linear operators acting from X to Y, the set $\mathcal{B} = \mathcal{B}(X, Y)$ of all linear bounded operators acting from X to Y.

For $\delta \geqslant 0$ and operator $S \in \mathcal{R}$, we define

$$U_\delta(A, S; W) := \sup\{\|Ax - S\eta\|_Y \;:\; x \in W,\, \eta \in X,\, \|x - \eta\|_X \leqslant \delta\}.$$

Problem 4. For every $\delta > 0$, find the error of the best recovery of operator A with the help of the set of operators (called methods of recovery) $\mathcal{R}$ on the elements of the class W given with the error δ:

$$\mathcal{E}_\delta(\mathcal{R}; A; W) := \inf_{S\in\mathcal{R}} U_\delta(A, S; W)\,, \tag{1.4}$$

and the best methods of recovery $S^* \in \mathcal{R}$ (if any exists) delivering the infimum in the right hand part of (1.4).

The detailed survey of existing results on the problem of the best recovery of operators on elements given with an error and further references can be found in [2, 5].

The following statement is a corollary from Theorem 7.1.2 in [5] for the case when $X = L_p$, $Y = L_q$, $A = D^k$ and $W = W_s^r$ and establishes the relation between the Stechkin, the Landau-Kolmogorov problems and Problem 4.

Proposition 3. Let $1 \leqslant p, q, s \leqslant \infty$ and $k, r \in \mathbb{N}$, $r > k$. Then, for every $\delta \geqslant 0$,

$$\begin{aligned}\Omega\left(\delta; D^k; W_s^r\right) &\leqslant \mathcal{E}_\delta\left(\mathcal{O}; D^k; W_s^r\right) \leqslant \mathcal{E}_\delta\left(\mathcal{B}; D^k; W_s^r\right) = \mathcal{E}_\delta\left(\mathcal{L}; D^k; W_s^r\right) \\ &\leqslant \inf_{N\geqslant 0}\left(E_N\left(D^k; W_s^r\right) + N\delta\right).\end{aligned}$$

The rest of the paper is organized as follows. In the next section we present main results. We devote section 3 to the proof of auxiliary lemmas and section 4 to the proof of main results.

2. Main results

We start with formulating the solution to Problem 1.

Theorem 1. *For $\sigma \in (0, 1/2]$ and $f \in L_1^3$, there holds true sharp inequality*

$$\|f'\|_\infty \leqslant \frac{2}{\sigma(1-\sigma)}\|f\|_\infty + \frac{\sigma}{2(1+\sigma)}\|f'''\|_1. \tag{2.1}$$

Furthermore,

$$\Gamma\left(D^1 : L_\infty \to L_\infty; L_1^3\right) = \left\{\left(N, \frac{N+4-\sqrt{N(N-8)}}{8(N+1)}\right) \;:\; N \geqslant 8\right\}. \tag{2.2}$$

The solution to Problem 2 is given by the following consequence from Theorem 1.

Corollary 1. *For every $\delta > 0$, let σ be a unique solution to the equation $4\delta(1+\sigma)^2(1-2\sigma) = \sigma^2(1-\sigma)^2$ on the interval $(0, 1/2)$. Then*

$$\Omega\left(\delta; D^1 : L_\infty \to L_\infty; W_1^3\right) = \frac{\sigma(1-\sigma-\sigma^2)}{(1+\sigma)^2(1-2\sigma)}.$$

To formulate the results on the solution to the Stechkin problem of the best approximation of differentiation operator by linear bounded ones, we first start with the construction of approximation operator S_N. Consider the set $\Pi = [0, 1/2] \times [8, +\infty)$ and subdivide it into two non-intersecting parts:

$$\mathcal{I} = \{(t, N) \in \Pi \;:\; Nt > 1\}, \qquad \mathcal{J} = \{(t, N) \in \Pi \;:\; Nt \leqslant 1 \text{ and } N \geqslant 8\}.$$

For every $(t, N) \in \Pi$, we denote

$$\rho := \rho(t, N) = \begin{cases} t + \frac{1}{N}, & (t, N) \in \mathcal{I}, \\ \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8(1-2t)}{N}}, & (t, N) \in \mathcal{J}, \end{cases} \tag{2.3}$$

and define the functional $S_{t,N} : L_\infty \to \mathbb{R}$ with the help of the rule: for every $f \in L_\infty$,

$$S_{t,N}f := \begin{cases} \frac{1}{2(\rho-t)}\left(f(\rho) - f(2t-\rho)\right), & (t,N) \in \mathcal{I}, \\ -\frac{1+\rho-2t}{\rho} f(0) + \frac{1-2t}{\rho(1-\rho)} f(\rho) - \frac{\rho-2t}{1-\rho} f(1), & (t,N) \in \mathcal{J}. \end{cases} \tag{2.4}$$

Next, for a function $f : [0,1] \to \mathbb{R}$, we set $\tilde{f}(\cdot) := f(1-\cdot)$, and define the operator $S_N : L_\infty \to L_\infty$ as follows: for every $f \in L_\infty$, set

$$S_N f := \begin{cases} S_{t,N} f, & 0 \leqslant t \leqslant \frac{1}{2}, \\ S_{1-t,N}\tilde{f}, & \frac{1}{2} < t \leqslant 1. \end{cases}$$

The solution to Problem 3 is given by the following proposition.

Theorem 2. *For every $N \geqslant 8$, the operator S_N is the extremal operator in the problem of the best approximation of operator $D^1 : L_\infty \to L_\infty$ by linear bounded operators on the class W_1^3. Moreover,*

$$E_N\left(D^1; W_1^3\right) = \begin{cases} \dfrac{N+4-\sqrt{N(N-8)}}{8(N+1)}, & N \geqslant 8, \\ +\infty, & N \in (0,8). \end{cases}$$

Finally, combining Corollary 1 and Theorem 2 we can formulate the result giving the solution to the problem of the best recovery of differentiation operator $D^1 : L_\infty \to L_\infty$ on elements of the class W_1^3 given with an error.

Theorem 3. *For every $\delta > 0$, let σ be a unique solution to the equation $4\delta(1+\sigma)^2(1-2\sigma) = \sigma^2(1-\sigma)^2$ on the interval $(0, 1/2)$. Then*

$$\mathcal{E}_\delta\left(\mathcal{B}; D^1; W_1^3\right) = \mathcal{E}_\delta\left(\mathcal{L}; D^1; W_1^3\right) = \mathcal{E}_\delta\left(\mathcal{O}; D^1; W_1^3\right) = \frac{\sigma\left(1-\sigma-\sigma^2\right)}{(1+\sigma)^2(1-2\sigma)}.$$

3. Auxiliary results

We start with estimating the deviation of the first order differentiation functional D_t^1 at the point t from the functional $S_{t,N}$ defined by (2.4) on the class W_1^3.

Lemma 1. *Let $(t,N) \in \mathcal{I}$ and $\rho = \rho(t,N)$ be defined by relation (2.3). Then $\|S_{t,N}\| = N$ and*

$$U\left(D_t^1, S_{t,N}; W_1^3\right) = \left\|\psi_{t,N}\right\|_\infty,$$

where

$$\psi_{t,N}(u) = \frac{1}{4(\rho-t)}\left(\rho - t - |t-u|\right)_+^2, \quad u \in [0,1].$$

Proof. Evidently, equality $\|S_{t,N}\| = N$ follows immediately from definition (2.4). Next, since $(t,N) \in \mathcal{I}$, we have $2t - \rho = t - \frac{1}{N} > 0$ and $\rho = t + \frac{1}{N} < 2t \leqslant 1$. Hence, using the Taylor expansion formula of the second order for a function $f \in L_1^3$ at the point t with the remainder in the integral form, we obtain

$$\begin{aligned} S_{t,N}f &= f'(t) + \frac{1}{4(\rho - t)} \left(\int_t^\rho (\rho - u)^2 f'''(u)\,\mathrm{d}u - \int_t^{2t-\rho} (2t - \rho - u)^2 f'''(u)\,\mathrm{d}u \right) \\ &= f'(t) + \int_{2t-\rho}^{\rho} \psi_{t,N}(u) f'''(u)\,\mathrm{d}u = f'(t) + \int_0^1 \psi_{t,N}(u) f'''(u)\,\mathrm{d}u. \end{aligned}$$

Hence, we conclude that

$$U\left(D_t^1, S_{t,N}; W_1^3\right) = \sup_{f \in W_1^3} |f'(t) - S_{t,N}f| = \sup_{f \in W_1^3} \left| \int_0^1 \psi_{t,N}(u) f'''(u)\,du \right| = \|\psi_{t,N}\|_\infty ,$$

which finishes the proof.

Lemma 2. *Let* $(t,N) \in \mathcal{J}$ *and* $\rho = \rho(t,N)$ *be defined by relation* (2.3)*. Then* $\|S_{t,N}\| = N$ *and*

$$U\left(D_t^1, S_{t,N}; W_1^3\right) = \|\psi_{t,N}\|_\infty ,$$

where

$$\psi_{t,N}(u) = \begin{cases} \frac{1+\rho-2t}{2\rho}\, u^2, & u \in [0,t], \\ \frac{1-2t}{2\rho(1-\rho)}\, (\rho - u)_+^2 - \frac{\rho-2t}{2(1-\rho)} (1-u)^2, & u \in [t,1]. \end{cases} \tag{3.1}$$

Proof. The validity of equality $\|S_{t,N}\| = N$ is clear. Next, we observe that for every $(t,N) \in \mathcal{J}$, we have $t \leqslant 1/N \leqslant 1/8$. Hence,

$$\rho = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8(1-2t)}{N}} \geqslant \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8(1-2t)t} = \frac{1}{2} - \frac{(1-4t)}{2} = 2t.$$

The rest of the proof is similar to the proof of Lemma 1.

Let us show that L_∞-norm of function $\psi_{t,N}$ achieves its maximum on variable t at the point $t = 0$. More accurately, there holds true the following proposition.

Lemma 3. *For every* $(t,N) \in \Pi$,

$$\|\psi_{t,N}\|_\infty \leqslant \|\psi_{0,N}\|_\infty . \tag{3.2}$$

Proof. For convenience, we fix an arbitrary point $(t,N) \in \Pi_p$ and use notations: $\rho = \rho(t,N)$ and $\sigma = \rho(0,N)$. Straightforward computations show that

$$\|\psi_{0,N}\|_\infty = -\psi_{0,N}\left(\frac{\sigma}{1+\sigma}\right) = \frac{\sigma}{2(1+\sigma)}.$$

First, we consider the case $(t, N) \in \mathcal{I}$. By definition (2.3) and Lemma 1, we have

$$\|\psi_{t,N}\|_\infty = \psi_{t,N}(t) = \frac{\rho - t}{4} = \frac{1}{4N} = \frac{\sigma(1-\sigma)}{8} < \frac{\sigma}{2(1+\sigma)} = \|\psi_{0,N}\|_\infty .$$

Next, we consider the case $(t, N) \in \mathcal{J}$. It is clear that

$$2t \leqslant \rho = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8(1-2t)}{N}} \leqslant \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8}{N}} = \sigma < 1, \tag{3.3}$$

where the first inequality in the above chain was proved in Lemma 2. By definition of the function $\psi_{t,N}$ (see Lemma 2), we have

$$\|\psi_{t,N}\|_\infty = \max\left\{\psi_{t,N}(t); -\psi_{t,N}\left(\frac{\rho}{1+\rho-2t}\right)\right\} = \max\left\{\frac{1+\rho-2t}{2\rho}\cdot t^2; \frac{(1-2t)(\rho-2t)}{2(1+\rho-2t)}\right\}.$$

Applying inequalities (3.3), we see that

$$\psi_{t,N}(t) = \frac{1+\rho-2t}{2\rho}\cdot t^2 \leqslant \frac{\rho(1+\rho)}{8} \leqslant \frac{\sigma}{4} \leqslant \frac{\sigma}{2(1+\sigma)},$$

$$-\psi_{t,N}\left(\frac{\rho}{1+\rho-2t}\right) = \frac{(1-2t)(\rho-2t)}{2(1+\rho-2t)} \leqslant \frac{1}{2\left(1+\frac{1}{\rho-2t}\right)} \leqslant \frac{1}{2\left(1+\frac{1}{\sigma}\right)} = \frac{\sigma}{2(1+\sigma)}.$$

Hence, $\|\psi_{t,N}\|_\infty \leqslant \|\psi_{0,N}\|_\infty$, which finishes the proof.

4. Proofs of main results

Proof of Theorem 1. For $\sigma \in (0, \frac{1}{2}]$, let $N = \frac{2}{\sigma(1-\sigma)}$. By Lemmas 1, 2, 3, for every $t \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} |f'(t)| &\leqslant |S_{t,N}f| + |f'(t) - S_{t,N}f| \leqslant |S_{t,N}f| + U\left(D_t^1, S_{t,N}; W_1^3\right)\cdot \|f'''\|_1 \\ &\leqslant N\|f\|_\infty + \|\varphi_{0,N}\|_\infty \|f'''\|_1 = \frac{2}{\sigma(1-\sigma)}\|f\|_\infty + \frac{\sigma}{2(1+\sigma)}\|f'''\|_1 . \end{aligned} \tag{4.1}$$

The same upper bound as in (4.1) also holds true in the case $t \in \left[\frac{1}{2}, 1\right]$, and can be verified by approximating $f'(t)$ with the help of $S_{1-t,N}\tilde{f}$, where $\tilde{f}(\cdot) = f(1-\cdot)$. Hence, taking supremum over $t \in [0,1]$, we obtain the desired inequality (2.1).

Now, we prove sharpness of inequality (2.1). Let us first consider the function

$$g(t) := \frac{(t-\sigma)^2}{2} + \frac{(1+\sigma)^2(1-2\sigma)}{2\sigma^2}\cdot\left(t - \frac{\sigma}{1+\sigma}\right)_+^2 - \frac{(1-\sigma)^2}{4}, \quad t \in [0,1].$$

Straightforward calculations show that

$$\|g\|_\infty = \frac{(1-\sigma)^2}{4}, \quad |g'(0)| = \frac{1-\sigma-\sigma^2}{\sigma}, \quad \bigvee_0^1 g'' = \frac{(1+\sigma)^2(1-2\sigma)}{\sigma^2}.$$

For sufficiently small $h>0$, we consider the Steklov average g_h of function g:

$$g_h(t) := \frac{1}{h}\int_t^{t+h} g(u)\,\mathrm{d}u, \qquad t\in[0,1],$$

where we set $g(t) := \frac{(t-\sigma)^2}{2} - \frac{(1-\sigma)^2}{4}$, for $t>1$. It is not difficult to see that

$$\lim_{h\to 0^+}\|g_h\|_\infty = \|g\|_\infty, \qquad \lim_{h\to 0^+}|g_h'(0)| = |g'(0)|, \qquad \lim_{h\to 0^+}\|g_h'''\|_1 = \bigvee_0^1 g''.$$

From this we conclude that

$$\lim_{h\to 0^+}\frac{\|g_h'\| - N\|g_h\|_\infty}{\|g_h'''\|_1} \geqslant \lim_{h\to 0^+}\frac{|g_h'(0)| - N\|g_h\|_\infty}{\|g_h'''\|_1} = \frac{|g'(0)| - N\|g\|_\infty}{\bigvee_0^1 g''} = \frac{\sigma}{2(1+\sigma)}.$$

Since $M_{1,2}^{\infty;\infty} = 8$ (see, *e.g.* [19, 22]), observing that $N = \frac{2}{\sigma(1-\sigma)}$ attains all values on the interval $[8,+\infty)$, we conclude from above considerations and Proposition 1 that equality (2.2).

Proof of Corollary 1. First, we observe that a function $\xi(\sigma) = \frac{\sigma^2(1-\sigma)^2}{(1+\sigma)^2(1-2\sigma)}$ increases on $\left(0,\frac12\right)$. Indeed,

$$\xi'(\sigma) = \frac{2\sigma(1-\sigma)\left((1-\sigma)^3+2\sigma^3\right)}{(1+\sigma)^3(1-2\sigma)^2} > 0.$$

Hence, for every $\delta>0$, equation $4\delta(1+\sigma)^2(1-2\sigma) = \sigma^2(1-\sigma)^2$ has a unique solution $\sigma=\sigma_\delta$ on the interval $\left(0,\frac12\right)$. Then by Theorem 1, we have

$$\Omega\left(\delta; D^1; W_1^3\right) \leqslant \inf_{\sigma>0}\left(\frac{2\delta}{\sigma(1-\sigma)} + \frac{\sigma}{2(1+\sigma)}\right) = \frac{\sigma_\delta\,(1-\sigma_\delta-\sigma_\delta^2)}{(1+\sigma_\delta)^2(1-2\sigma_\delta)}.$$

On the other hand, let functions g_h and g be defined in the proof of Theorem 1 and g correspond to $\sigma=\sigma_d$. For every $\varepsilon>0$, we obtain

$$\Omega\left(\delta; D^1; W_1^3\right) \geqslant \frac{1}{1+\varepsilon}\lim_{h\to 0^+}\frac{\|g_h'\|_\infty}{\|g_h'''\|_1} = \frac{\sigma_\delta\,(1-\sigma_\delta-\sigma_\delta^2)}{(1+\varepsilon)(1+\sigma_\delta)^2(1-2\sigma_\delta)}.$$

Taking supremum over $\varepsilon>0$, we finish the proof of Corollary 1.

Proof of Theorem 2. For every $N \geqslant 8$, let $\sigma = \rho(0, N)$. By definition and Lemma 3, it is clear that

$$E_N\left(D^1; W_1^3\right) \leqslant U\left(D^1, S_N; W_1^3\right) = \sup_{t\in\left[0,\frac{1}{2}\right]} U\left(D_t^1, S_{t,N}; W_1^3\right) \leqslant \frac{\sigma}{2(1+\sigma)}.$$

On the other hand, by Proposition 2 and Theorem 1 we have

$$E_N\left(D^1; W_1^3\right) \geqslant \frac{\sigma}{2(1+\sigma)}.$$

Combining the latter two estimates we finish the proof of Theorem 2.

Proof of Theorem 3. In Corollary 1 we have shown that, for every $\delta > 0$, there exists a unique $\sigma = \sigma_\delta \in \left(0, \frac{1}{2}\right)$ such that $4\delta(1+\sigma)^2(1-2\sigma) = \sigma^2(1-\sigma)^2$. Then by Theorem 2,

$$\inf_{N\geqslant 0}\left(E_N\left(D^1; W_1^3\right) + N\delta\right) = \inf_{\sigma\in\left(0,\frac{1}{2}\right]}\left(\frac{2}{\sigma(1-\sigma)} + \frac{2\delta}{\sigma(1-\sigma)}\right) = \frac{\sigma_\delta\left(1-\sigma_\delta-\sigma_\delta^2\right)}{\left(1+\sigma_\delta\right)^2\left(1-2\sigma_\delta\right)}.$$

Combining Corollary 1 and Proposition 3 with the later equality, we finish the proof.

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