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On generalized characteristics of smoothness of functions and on average ν-widths in the space $L_2(\mathbb{R})$

Нехай $\mathfrak{M}$ — клас комплекснозначних функцій $w : \mathbb{R} \to \mathbb{C}$, для яких $w(0) = 0$ й $|w|^2$ є неперервною, обмеженою на $\mathbb{R}$ функцією, яка відмінна від нуля майже всюди; $|w(t_*)|^2 = \sup\{|w(x)|^2 : 0 < x < \infty\}$. Якщо верхня межа досягається більш ніж при одному значенні аргументу, то у якості t_* беремо найменше з них. Функція $w \in \mathfrak{M}$ задовольняє властивість A, якщо функція $|w|^2$ є монотонно зростаючою на сегменті $[0, t_*]$. Нехай $\Delta_h^w : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $h \in \mathbb{R}$, є узагальненим диференціальним оператором для $f \in L_2(\mathbb{R})$ і

$$\Delta_h^w(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}(f,t)w(ht)\frac{e^{ixt}-1}{it}dt,$$

де $\mathcal{F}(f,t)$ є перетворенням Фур'є функції f. За допомогою величини $\Delta_h^w(f)$ для $f \in L_2(\mathbb{R})$ визначено узагальнений модуль неперервності $\omega^w(f,t) = \sup\{\|\Delta_h^w(f)\| : 0 < h \leqslant t\}$, $t > 0$. Символом $W^r(\omega^w, \Psi)$, $r \in \mathbb{N}$, позначається клас функцій $f \in L_2^r(\mathbb{R})$, для кожної з яких має місце нерівність $\omega^w(f^{(r)}, t) \leqslant \Psi(t)$, $0 < t < \infty$. Тут $\Psi(t)$, $0 \leqslant t < \infty$, — неперервна, зростаюча функція, така, що $\Psi(0) = 0$ (мажоранта).

Доведено, що коли комплекснозначна функція $w : \mathbb{R} \to \mathbb{C}$ належить класу $\mathfrak{M}$ і квадрат її модуля задовольняє властивості A, Ψ є довільною мажорантою, $\nu \in (0, \infty)$, $r \in \mathbb{N}$, а $\overline{\Pi}_\nu(\cdot)$ є будь-яким з середніх ν-поперечників: колмогорівським, лінійним, бернштейновським, то мають місце нерівності:

$$(\nu\pi)^{-r}\inf\Big\{\Psi(t)/|w(t\nu\pi)| : 0 < t \leqslant t_*/(\nu\pi)\Big\} \leqslant \overline{\Pi}_\nu(W^r(\omega^w, \Psi); L_2(\mathbb{R})) \leqslant$$

$$\leqslant \mathcal{A}_{\nu\pi}(W^r(\omega^w, \Psi)) \leqslant (\nu\pi)^{-r}\ \overline{\lim}\Big\{\Psi(t)/|w(t\nu\pi)| : t \to 0+\Big\}.$$

Вказано умову для мажоранти Ψ, коли вдається обчислити точні значення середніх ν-поперечників класу $W^r(\omega^w, \Psi)$ та наведено низку прикладів мажорант, для яких зазначена умова виконується.

Ключові слова: узагальнений модуль неперервності, мажоранта, ціла функція, середній ν-поперечник, перетворення Фур'є.

Estimates above and estimates below have been obtained for Kolmogorov, linear and Bernshtein average ν-widths on the classes of functions $W^r(\omega^w, \Psi)$, where $r \in \mathbb{N}$, $\omega^w(f)$

is the generalized characteristic of smoothness of a function $f \in L_2(\mathbb{R})$, Ψ is a majorant. Exact values of the enumerated extremal characteristics of approximation, following from the one condition on the majorant were obtained too.

Key words: generalized modulus of continuity, majorant, entire function, average ν-width, Fourier transform.

MSC2010: Pri 41A35, Sec 41A44, 42A38, 30D20

The beginning of research related to the approximation of functions defined on the whole real axis by entire functions of exponential type was laid down in the work of S. N. Bernstein [1]. This direction was developed in the subsequent works of N.I. Akhiezer, A.F.Timan, M.F.Timan, S.M.Nikolsky, I.I.Ibragimov, F.G.Nasibov, V.G.Ponomarenko, V.Yu.Popov, G.G.Magaril-Il'yaev, A.G.Babenko, S.N. Vasiliev and others (see, for example, [2] – [20]). In order to further generalize a number of the results we have mentioned S.N.Vasilyev, S.Yu.Artamonov, S.B.Vakarchuk which proposed a number of approaches that allowed him to come to one degree or another to solve this problem (see, for example, [13], [21] – [23]). In this article, which is a natural continuation of papers [22] – [23], the distribution of the results obtained in [19] to the more general case is given.

Further we shall formulate necessary concepts and definitions.

Let $L_2(\mathbb{R})$, where $\mathbb{R} = \{x : -\infty < x < \infty\}$, is the space of all measurable functions f given on all real axis $\mathbb{R}$, the square of the module of which is Lebesgue integrable on any finite interval and the norm is defined by the formula

$$\|f\| := \Big\{ \int_{-\infty}^{\infty} |f(x)|^2 dx \Big\}^{1/2} < \infty.$$

The following statement takes place.

Plancherel's theorem. [3, ch. III, point 3.11.21] *For any function $f \in L_2(\mathbb{R})$ the integral*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{e^{-itx} - 1}{-it} dt$$

has the final derivative almost everywhere on $\mathbb{R}$

$$\mathcal{F}(f, x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} f(t) \frac{e^{-itx} - 1}{-it} dt, \tag{1}$$

for which

$$\int_{-\infty}^{\infty} |\mathcal{F}(f, x)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx, \tag{2}$$

and almost everywhere on $\mathbb{R}$

$$f(x)=\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty}\mathcal{F}(f,t)\frac{e^{itx}-1}{it}dt. \tag{3}$$

In addition, when $k\to\infty$

$$\int\limits_{-\infty}^{\infty}\Big|\mathcal{F}(f,x)-\frac{1}{\sqrt{2\pi}}\int\limits_{-k}^{k}f(t)e^{-itx}dt\Big|^2dx\to 0, \tag{4}$$

$$\int\limits_{-\infty}^{\infty}\Big|f(x)-\frac{1}{\sqrt{2\pi}}\int\limits_{-k}^{k}\mathcal{F}(f,t)e^{itx}dt\Big|^2dx\to 0. \tag{5}$$

Function (1) is called the Fourier transform for $f\in L_2(\mathbb{R})$ and formulas (1) and (3) are called the inversion formulas. If in (1) and (3) one can change the operations of differentiation and integration, then these formulas go into the usual Fourier transform.

Relations (4), (5) show that the Fourier transform in $L_2(\mathbb{R})$ can be defined as the limit in the average. On this basis we writing the inversion formulas for $f\in L_2(\mathbb{R})$ in the form

$$\mathcal{F}(f,x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}f(t)e^{-itx}dt \qquad \text{и} \qquad f(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\mathcal{F}(f,t)e^{itx}dt.$$

We specifically stipulate that the integrals are understood to converge in the average square, i.e. that relations (4) and (5) hold respectively.

According to [22] by $\mathbb{G}$ we denote the set of all continuous, non-negative, even functions φ bounded on $\mathbb{R}$ which are non-zero almost everywhere on $\mathbb{R}$ and such that $\varphi(0)=0$. Since in parity of functions from the set $\mathbb{G}$ it is enough to consider them only on the semi-axis of $\mathbb{R}_+$. Then for an arbitrary element $\varphi\in\mathbb{G}$ we denote by $t_*\in(0,\infty)$ the value of the argument x for which

$$\varphi(t_*)=\sup\{\varphi(x):0<x<\infty\}.$$

If the upper bound is reached with more than one argument value, then the smallest of them is considered as t_*. Obviously, the value of t_* depends of φ.

We say that a function $\varphi\in\mathbb{G}$ satisfies the *property A* if on the segment $[0,t_*]$ it is monotonically increasing [22]. For an arbitrary element $\varphi\in\mathbb{G}$ satisfying *property A* we set

$$\varphi_*(x):=\{\varphi(x)\text{ if }0\leqslant x\leqslant t_*;\ \ \varphi(t_*)\text{ if }t_*\leqslant x<\infty\}.$$

By symbol $\mathfrak{M}$ we shall designate a class all complex-valued functions $w\colon\mathbb{R}\to\mathbb{C}$, for which $|w|^2\in\mathbb{G}$.

Let $f \in L_2(\mathbb{R})$; $\mathcal{F}(f)$ be the Fourier transform of the function f in $L_2(\mathbb{R})$; $w \in \mathfrak{M}$; $h \in \mathbb{R}$. Using (2) we write $\|\mathcal{F}(f,\cdot)w(h\cdot)\| \leqslant \|w\|_{C(\mathbb{R})}\|f\| < \infty$, that is $\mathcal{F}(f,x)w(hx) \in L_2(\mathbb{R})$. We consider the generalized differences operator $\Delta_h^w : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ which is defined by relation

$$\Delta_h^w(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)w(ht)\frac{e^{itx}-1}{it}dt, \tag{6}$$

almost everywhere on $\mathbb{R}$ [22].

According to formulas (3) and (6) for $f \in L_2(\mathbb{R})$ we have almost everywhere on $\mathbb{R}$

$$\mathcal{F}(\Delta_h^w(f),x) = \mathcal{F}(f,x)w(hx). \tag{7}$$

Let us demonstrate by several examples that the formula (6) can indeed be considered as a kind of generalization of the concept of finite difference in the space $L_2(\mathbb{R})$.

Let $w(x) = w_m(x) := (e^{ix}-1)^m$, $m \in \mathbb{N}$. Obviously, $w_m \in \mathfrak{M}$. Using the formula (3) we obtain almost everywhere on $\mathbb{R}$

$$\Delta_h^{w_m}(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)w_m(ht)\frac{e^{itx}-1}{it}dt =$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)\Big(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}e^{ijht}\Big)\frac{e^{itx}-1}{it}dt =$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)\Big(\sum_{j=0}^{m}\frac{e^{i(x+jh)t}-(e^{ijht}-1)-1}{it}(-1)^{m-j}\binom{m}{j}e^{ijht}\Big)dt =$$

$$= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)\frac{e^{i(x+jh)t}-1}{it}dt =$$

$$= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jh) = \Delta_h^m(f,x). \tag{8}$$

In formula (8) the value $\Delta_h^m(f,x)$ is the usual m^{th}-order finite difference of function $f \in L_2(\mathbb{R})$ with step $h \in \mathbb{R}$ defined almost everywhere on $\mathbb{R}$.

We further assume that $w(x) = \widetilde{w}_m(x) := (\operatorname{sinc}(x)-1)^m$, $m \in \mathbb{N}$, where $\operatorname{sinc}(x) := \{\sin(x)/x \text{ if } x \neq 0;\ 1 \text{ if } x = 0\}$. It's easy to make sure that $\widetilde{w}_m \in \mathfrak{M}$. By virtue of (6) we write for almost all $x \in \mathbb{R}$ and $f \in L_2(\mathbb{R})$

$$\Delta_h^{\widetilde{w}_m}(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)\widetilde{w}_m(ht)\frac{e^{itx}-1}{it}dt =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \mathcal{F}(f,t) \Big(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \operatorname{sinc}^j(ht) \Big) \frac{e^{itx}-1}{it} dt =$$

$$= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \mathcal{F}(f,t) \operatorname{sinc}^j(ht) \frac{e^{itx}-1}{it} dt. \tag{9}$$

We write the Steklov function for $f \in L_2(\mathbb{R})$

$$S_h(f,x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt, \quad x \in \mathbb{R}, \tag{10}$$

where $h > 0$. For $h \to 0+$ we have $S_h(f,x) \to f(x)$ almost everywhere on $\mathbb{R}$. Also we suppose $S_{h,0}(f) \equiv f$, $S_{h,j}(f) := S_h(S_{h,j-1}(f))$, $j \in \mathbb{N}$. The Steklov function (10) can be represented as a convolution of two functions $f \in L_2(\mathbb{R})$ and $q \in L_1(\mathbb{R})$, where $q(x) := \{1/(2h) \text{ if } |x| \leqslant h; \ 0 \text{ if } |x| > h\}$, i.e.

$$S_h(f,x) = \int_{-\infty}^{\infty} f(t)q(x-t)dt. \tag{11}$$

Since (see, for example, [24, ch.II, §2.3])

$$S_h(f,x) = \int_{-\infty}^{\infty} \mathcal{F}(f,t)\mathcal{F}(q,t)e^{ixt}dt,$$

the product $\sqrt{2\pi}\mathcal{F}(f,x)\mathcal{F}(q,x)$ is the Fourier transform of convolution (11). Considering that almost everywhere on $\mathbb{R}$

$$\mathcal{F}(q,x) = \frac{1}{\sqrt{2\pi}} \int_{-h}^{h} \frac{1}{2h} e^{-ixt} dt = \frac{\operatorname{sinc}(hx)}{\sqrt{2\pi}},$$

(see, for example, [2, ch.III, point 67]) we have

$$\mathcal{F}(S_h(f),x) = \sqrt{2\pi}\mathcal{F}(f,x)\mathcal{F}(q,x) = \mathcal{F}(f,x)\operatorname{sinc}(hx). \tag{12}$$

Then according to relations (3) and (12) we write for $f \in L_2(\mathbb{R})$ and for almost all $x \in L_2(\mathbb{R})$

$$S_h(f,x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \mathcal{F}(f,t)\operatorname{sinc}(ht) \frac{e^{ixt}-1}{it} dt. \tag{13}$$

Using the method of mathematical induction and (13) it can be shown that in the general case almost everywhere on $\mathbb{R}$ the relation holds

$$S_{h,j}(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty}\mathcal{F}(f,t)\mathrm{sinc}^{\,j}(ht)\frac{e^{ixt}-1}{it}dt, \quad (14)$$

where $j \in \mathbb{N}$ and $j \geqslant 2$.

Proceeding from (9) and (14) we get

$$\Delta^{\widetilde{w}_m}(f,x) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}S_{h,j}(f,x) = (S_h - \mathbb{I})^m(f,x) = \widetilde{\Delta}_h^m(f,x), \;\; m \in \mathbb{N}, \quad (15)$$

for $f \in L_2(\mathbb{R})$ and for almost all $x \in L_2(\mathbb{R})$. Here $\mathbb{I}$ is the unit operator in the space $L_2(\mathbb{R})$. Recall that earlier special finite differences $\widetilde{\Delta}_h^m(f,x)$ for $f \in L_2(\mathbb{R})$, $m \in \mathbb{N}$, were considered in [17].

Based on the work [22] we write the generalized modulus of continuity ω^w, $w \in \mathfrak{M}$, for a function f from the space $L_2(\mathbb{R})$. From the definition of class $\mathfrak{M}$ it follows that the function $|w|^2$ is even. When writing the value of ω^w we will take into account that the following relation takes place

$$\|\Delta_h^w(f)\|^2 = \|\mathcal{F}(\Delta_h^w(f))\|^2 = \int\limits_{-\infty}^{\infty}|\mathcal{F}(f,t)|^2\,|w(ht)|^2dt$$

on the basis of (2) and (6) – (7). From this equality it follows that only positive values of h can be considered, i.e.

$$\omega^w(f,t) := \sup\{\|\Delta_h^w(f)\| : 0 < h \leqslant t\}, \;\; t > 0. \quad (16)$$

It is obvious that $\omega^w(f,t) \to 0$ under $t \to 0+$ for any $w \in \mathfrak{M}$.

Believing, for example, $w = w_m$, $m \in \mathbb{N}$, and taking into account (8), (16) we obtain the usual modulus of continuity of m^{th} order for $f \in L_2(\mathbb{R})$

$$\omega_m(f,t) := \omega^{w_m}(f,t) = \sup\{\|\Delta_h^m(f)\| : 0 < h \leqslant t\}, \;\; t > 0, \quad (17)$$

which was used in many of the works listed above.

In the case when $w = \widetilde{w}_m$, $m \in \mathbb{N}$, on the basis of (15) - (16) we have the smoothness characteristic for $f \in L_2(\mathbb{R})$

$$\widetilde{\Omega}_m(f,t) := \omega^{\widetilde{w}_m}(f,t) = \sup\{\|\widetilde{\Delta}_h^m(f)\| : 0 < h \leqslant t\}, \;\; t > 0, \quad (18)$$

which was considered, for example, in papers [17], [22].

It should be noted that since $|w_m(x)|^2 = 2^m(1-\cos x)^m$ and $|\widetilde{w}_m(x)|^2 = (1 - \mathrm{sinc}\,(x))^{2m}$, the functions $|w_m|^2$ and $|\widetilde{w}_m|^2$ satisfy the *property A*. At the same time $t_* = t_*(|w_m|^2) = \pi$ and $t_* = t_*(|\widetilde{w}_m|^2) \in (4,49; 4,51)$ [17].

By symbol $\mathbb{B}_{\sigma,2}$, $\sigma \in (0, \infty)$, we denote the subspace consisting of entire functions of exponential type not exceeding σ whose extractions on $\mathbb{R}$ belong to the space $L_2(\mathbb{R})$. For an arbitrary function $f \in L_2(\mathbb{R})$ by $\mathcal{A}_\sigma(f)$, $\sigma \in (0, \infty)$, we denote its best average square approximation by elements of the subspace $\mathbb{B}_{\sigma,2}$, i.e. $\mathcal{A}_\sigma(f) = \inf\{\|f - g\| : g \in \mathbb{B}_{\sigma,2}\}$.

In [6] was showed by I.I.Ibragimov and F.G.Nasibov that for an arbitrary function $f \in L_2(\mathbb{R})$ the entire function

$$\mathcal{L}_\sigma(f, x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\sigma}^{\sigma} \mathcal{F}(f, t) e^{ixt} dt, \ \ \sigma \in (0, \infty), \tag{19}$$

belongs to the subspace $\mathbb{B}_{\sigma,2}$ and gives the least deviates from f in $L_2(\mathbb{R})$, i.e.

$$\mathcal{A}_\sigma(f) = \|f - \mathcal{L}_\sigma(f)\| = \Big\{ \int\limits_{|t| \geqslant \sigma} |\mathcal{F}(f, t)|^2 dt \Big\}^{1/2}. \tag{20}$$

Here $\mathcal{F}(f) \in L_2(\mathbb{R})$ is the Fourier transform of f in the sense of the space $L_2(\mathbb{R})$.

Let Q is a set of functions from $L_2(\mathbb{R})$. Then we assume $\mathcal{A}_\sigma(Q) := \sup\{\mathcal{A}_\sigma(f) : f \in Q\}$.

We next consider a series of extremal approximation characteristics introduced by G.G.Magaril-Il'yaev in papers [10] – [11] and based on the use of the concept of average dimension. This made it possible to determine the asymptotic characteristics of the subspaces similar to ordinary diameters when the average dimension was used as the dimension.

Let $BL_2(\mathbb{R})$ be the unit ball in $L_2(\mathbb{R})$. Let $Lin(L_2(\mathbb{R}))$ be the set of all linear subspaces in $L_2(\mathbb{R})$;

$$Lin_n(L_2(\mathbb{R})) := \{\mathcal{L} \in Lin(L_2(\mathbb{R})) : dim\mathcal{L} \leqslant n\}; n \in \mathbb{Z}_+,$$

and let

$$d(Q, A, L_2(\mathbb{R})) := \sup\{\inf\{\|x - y\| : y \in A\} : x \in Q\}$$

be the best approximation of a set $Q \subset L_2(\mathbb{R})$ by the set $A \subset L_2(\mathbb{R})$. By A_T, where $T > 0$, we denote the restriction of a set $A \subset L_2(\mathbb{R})$ to the closed interval $[-T, T]$, and by $Lin_C L_2(\mathbb{R})$ we denote the set of such subspaces $\mathcal{L} \in Lin(L_2(\mathbb{R}))$ for which the set $(\mathcal{L} \cap BL_2(\mathbb{R}))_T$ is precompact in $L_2([-T, T])$ for any $T > 0$.

If $\mathcal{L} \in Lin_C(L_2(\mathbb{R}))$ and $T, \varepsilon > 0$, then there exist $n \in \mathbb{Z}_+$ and $\mathcal{M} \in Lin_n(L_2(\mathbb{R}))$, for which

$$d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T, T])) < \varepsilon.$$

Let be

$$D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R})) := \min\{n \in \mathbb{Z}_+ : \exists \mathcal{M} \in Lin_n(L_2([-T, T])),$$
$$d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T, T])) < \varepsilon\}.$$

This function is nondecreasing in T and nonincreasing in ε. The quantity

$$\overline{dim}(\mathcal{L}, L_2(\mathbb{R})) := \lim\{\liminf\{D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R}))/(2T) : T \to \infty\} : \varepsilon \to 0\},$$

where $\mathcal{L} \in Lin_C(L_2(\mathbb{R}))$, is called the average dimension of the subspace $\mathcal{L}$ in $L_2(\mathbb{R})$. It was shown in [10]

$$\overline{dim}(\mathbb{B}_{\sigma,2}; L_2(\mathbb{R})) = \frac{\sigma}{\pi}. \tag{21}$$

Let Q be a centrally symmetric subset from $L_2(\mathbb{R})$ and let $\nu > 0$ be an arbitrary number. Then by the average Kolmogorov ν-width of a set Q in $L_2(\mathbb{R})$ we understand the quantity

$$\overline{d}_\nu(Q, L_2(\mathbb{R})) := \inf\{\sup\{\inf\{\|f - \varphi\| : \varphi \in \mathcal{L}\} : f \in Q\} :$$

$$: \mathcal{L} \in Lin_C(L_2(\mathbb{R})), \overline{dim}(\mathcal{L}, L_2(\mathbb{R})) \leqslant \nu\}.$$

The subspace on which the lower bound is attained is called extremal.

By the average linear ν-width of a set Q in $L_2(\mathbb{R})$ we understand the quantity

$$\overline{\delta}_\nu(Q, L_2(\mathbb{R})) := \inf\{\sup\{\|f - V(f)\| : f \in Q\} : (X, V)\},$$

where the lower bound is taken over all pairs (X, V) such that X is a normed space directly embedded in $L_2(\mathbb{R})$; $V : X \to L_2(\mathbb{R})$ is a continuous linear operator for which $ImV \in Lin_C(L_2(\mathbb{R}))$ and the following inequality holds $\overline{dim}(ImV, L_2(\mathbb{R})) \leqslant \nu$; $Q \subset X$. Here ImV is the image of the operator V. The pair on which the lower bound is attained is called extremal.

The quantity

$$\overline{b}_\nu(Q, L_2(\mathbb{R})) := \sup\{\sup\{\rho > 0 : \mathcal{L} \cap \rho BL_2(\mathbb{R}) \subset Q\} :$$

$$: \mathcal{L} \in Lin_C(L_2(\mathbb{R})), \overline{dim}(\mathcal{L}, L_2(\mathbb{R})) > \nu, \overline{d}_\nu(\mathcal{L} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1\}$$

is called the average Bernstein ν-width of a set Q in $L_2(\mathbb{R})$. The last condition imposed on $\mathcal{L}$ in calculating the outer upper bound means that we consider only subspaces for which the analog of Tihomirov's theorem on the width of the ball is valid. This requirement is satisfied, for example, for the subspace $\mathbb{B}_{\sigma,2}$ if $\sigma > \nu\pi$, i.e. $\overline{d}_\nu(\mathbb{B}_{\sigma,2} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1$.

For a set $Q \subset L_2(\mathbb{R})$ between its extremal characteristics indicated above the following inequalities hold

$$\overline{b}_\nu(Q, L_2(\mathbb{R})) \leqslant \overline{d}_\nu(Q, L_2(\mathbb{R})) \leqslant \overline{\delta}_\nu(Q, L_2(\mathbb{R})). \tag{22}$$

By the symbol $L_2^r(\mathbb{R})$, $r \in \mathbb{N}$, we denote the class of functions f belonging to the space $L_2(\mathbb{R})$ in which the $(r-1)^{th}$ order derivatives $f^{(r-1)}$ ($f^{(0)} \equiv f$) are locally absolutely continuous and the r^{th} order derivatives $f^{(r)} \in L_2(\mathbb{R})$. Note that $L_2^r(\mathbb{R})$ becomes a Banach space if the norm in it is defined as $\|f\| + \|f^{(r)}\|$.

Let $\Psi(t)$, $t \in [0, \infty)$, be a continuous increasing function such that $\Psi(0) = 0$, which we will further call majorant. By $W^r(\omega^w, \Psi)$, $r \in \mathbb{N}$, we denote the class of functions, consisting of the elements $f \in L_2^r(\mathbb{R})$ for each of which for any value $t \in (0, \infty)$ the inequality $\omega^w(f^{(r)}, t) \leqslant \Psi(t)$ holds. Here $w \in \mathfrak{M}$.

Theorem 1. *Suppose that a complex-valued function $w : \mathbb{R} \to \mathbb{C}$ belongs to the class $\mathfrak{M}$ and the square of its module satisfies the property A; Ψ is an arbitrary majorant; $\nu \in (0,\infty)$; $r \in \mathbb{N}$; $\overline{\Pi}_\nu(\cdot)$ is any of the considered average ν-widths. Then the following relation holds:*

$$\frac{1}{(\nu\pi)^r} \inf\Big\{\frac{\Psi(t)}{|w(t\nu\pi)|} : 0 < t \leqslant \frac{t_*}{\nu\pi}\Big\} \leqslant \overline{\Pi}_\nu(W^r(\omega^w,\Psi); L_2(\mathbb{R})) \leqslant$$

$$\leqslant \mathcal{A}_{\nu\pi}(W^r(\omega^w,\Psi)) \leqslant \frac{1}{(\nu\pi)^r} \overline{\lim_{t\to 0+}} \frac{\Psi(t)}{|w(t\nu\pi)|}, \qquad (23)$$

where $t_* = t_*(|w|^2)$.

Proof. Let f be an arbitrary function from the class $L_2^r(\mathbb{R})$. For it by virtue of [6] – [7] we have

$$\mathcal{A}_\sigma(f) \leqslant \frac{1}{\sigma^r}\, \mathcal{A}_\sigma(f^{(r)}), \qquad (24)$$

where $\sigma \in (0,\infty)$. Based on (20) we obtain

$$\mathcal{A}_\sigma^2(f^{(r)}) = \int\limits_{|t|\geqslant\sigma} |\mathcal{F}(f^{(r)},t)|^2\, dt. \qquad (25)$$

Based on the formula (21) we calculate the average dimension of the subspace $\mathbb{B}_{\sigma,2}$ in $L_2(\mathbb{R})$ when $\sigma = \nu\pi$, i.e. $\overline{dim}(\mathbb{B}_{\nu\pi,2}; L_2(\mathbb{R})) = \nu$. Using (25) for an arbitrary $j \in \mathbb{N}$ we write the relation

$$\int\limits_{|t|\geqslant\nu\pi} |\mathcal{F}(f^{(r)},t)|^2\, dt = \int\limits_{\nu\pi\leqslant|t|\leqslant(j+1)\nu\pi} |\mathcal{F}(f^{(r)},t)|^2\, dt + \varepsilon_{j,f}, \qquad (26)$$

where $\varepsilon_{j,f} \geqslant 0$. If a function f is not integer, then the set of numbers $\{\varepsilon_{j,f}\}_{j\in\mathbb{N}}$ forms a non-increasing sequence of positive numbers such that $\varepsilon_{j,f} \to 0$ for $j \to \infty$. In the case when f is an entire function of exponential type $\sigma_0 \in (j_0\nu\pi, (j_0+1)\nu\pi]$, $j_0 \in \mathbb{N}$, then for $j_0 > 1$ the sequence $\{\varepsilon_{j,f}\}_{j\in\mathbb{N}}$ will have non-increasing positive values $\varepsilon_{j,f} > 0$ for $1 \leqslant j \leqslant j_0 - 1$ and zero values $\varepsilon_{j,f} = 0$ for all $j \geqslant j_0$. In the case when $j_0 = 1$ we will have $\varepsilon_{j,f} = 0$ for any $j \in \mathbb{N}$. Since the course of the proof of theorem is almost the same for each of these two cases, we will not separate them in the following arguments.

We construct by arbitrarily way a numerical sequence of positive numbers $\{h_j\}_{j\in\mathbb{N}}$ so that the following conditions are satisfied: $h_j \in (0, t_*/((j+1)\nu\pi)]$; $h_j > h_{j+1}$ for any $j \in \mathbb{N}$; $h_j \to 0$ if $j \to \infty$.

For $f \in L_2^r(\mathbb{R})$, proceeding from (16), we have

$$\omega^w(f^{(r)},t) = \sup\Big\{\Big(\int\limits_{-\infty}^{\infty} |\mathcal{F}(f^{(r)},t)|^2\, |w(ht)|^2\, dt\Big)^{1/2} : 0 < h \leqslant t\Big\}, t > 0. \qquad (27)$$

Considering that $w \in \mathfrak{M}$ and w satisfies the *property A*, for any $j \in \mathbb{N}$ proceeding from (27) we write

$$\int\limits_{\nu\pi\leqslant|t|\leqslant(j+1)\nu\pi} |\mathcal{F}(f^{(r)},t)|^2\,dt = \int\limits_{\nu\pi\leqslant|t|\leqslant(j+1)\nu\pi} |\mathcal{F}(f^{(r)},t)|^2 \frac{|w(h_jt)|^2}{|w(h_jt)|^2}\,dt \leqslant$$

$$\leqslant \frac{1}{|w(h_j\nu\pi)|^2} \int\limits_{\nu\pi\leqslant|t|\leqslant(j+1)\nu\pi} |\mathcal{F}(f^{(r)},t)|^2 |w(h_jt)|^2\,dt \leqslant$$

$$\leqslant \frac{1}{|w(h_j\nu\pi)|^2} \int\limits_{-\infty}^{\infty} |\mathcal{F}(f^{(r)},t)|^2\,|w(h_jt)|^2\,dt \leqslant \frac{(\omega^w(f^{(r)},h_j))^2}{|w(h_j\nu\pi)|^2}. \tag{28}$$

Proceeding from (24) – (28) for an arbitrary function $f \in L_2^r(\mathbb{R})$ and for any $j \in \mathbb{N}$ we write

$$\mathcal{A}^2_{\nu\pi}(f) \leqslant \frac{1}{(\nu\pi)^{2r}} \Big\{ \int\limits_{\nu\pi\leqslant|t|\leqslant(j+1)\nu\pi} |\mathcal{F}(f^{(r)},t)|^2\,dt + \varepsilon_{j,f} \Big\} \leqslant \frac{1}{(\nu\pi)^{2r}} \Big\{ \frac{(\omega^w(f^{(r)},h_j))^2}{|w(h_j\nu\pi)|^2} + \varepsilon_{j,f} \Big\}.$$

On the basis of this relation we obtain for $f \in W^r(\omega^w,\Psi)$

$$\mathcal{A}_{\nu\pi}(f) \leqslant \frac{1}{(\nu\pi)^r} \overline{\lim_{j\to\infty}} \frac{\omega^w(f^{(r)},h_j)}{|w(h_j\nu\pi)|} \leqslant$$

$$\leqslant \frac{1}{(\nu\pi)^r} \overline{\lim_{t\to 0+}} \frac{\omega^w(f^{(r)},t)}{|w(t\nu\pi)|} \leqslant \frac{1}{(\nu\pi)^r} \overline{\lim_{t\to 0+}} \frac{\Psi(t)}{|w(t\nu\pi)|}. \tag{29}$$

Using the relations (22) and (29), we write down the upper bounds

$$\overline{\Pi}_\nu(W^r(\omega^w,\Psi); L_2(\mathbb{R})) \leqslant \mathcal{A}_{\nu\pi}(W^r(\omega^w,\Psi)) \leqslant \frac{1}{(\nu\pi)^r} \overline{\lim_{t\to 0+}} \frac{\Psi(t)}{|w(t\nu\pi)|}, \tag{30}$$

where $\overline{\Pi}_\nu(W^r(\omega^w,\Psi); L_2(\mathbb{R}))$ is any of the average ν-widths listed above.

Further we shall obtain the lower estimates for the considered extremal characteristics of the class $W^r(\omega^w,\Psi)$ in the space $L_2(\mathbb{R})$. Let $\widehat{\sigma} := \nu\pi(1+\varepsilon)$, where $\varepsilon \in (0,\widetilde{\nu})$, $\widetilde{\nu} := \min(\nu, 1/\nu)$. Proceeding from (21) we have $\overline{dim}(\mathbb{B}_{\widehat{\sigma},2}; L_2(\mathbb{R})) = \nu(1+\varepsilon)$. Putting

$$\rho := \frac{1}{(\widehat{\sigma})^r} \inf\Big\{ \frac{\Psi(t)}{|w(t\widehat{\sigma})|_*} : 0 < t \leqslant \frac{t_*}{\nu\pi} \Big\}, \tag{31}$$

where

$$|w(x)|_* := \{|w(x)| \text{ if } 0 \leqslant x \leqslant t_*;\ \ |w(t_*)| \text{ if } t_* \leqslant x < \infty\},$$

we consider the set of functions $\mathcal{B}_{\widehat{\sigma}}(\rho) = \mathbb{B}_{\widehat{\sigma},2} \cap \rho BL_2(\mathbb{R}) = \{g \in \mathbb{B}_{\widehat{\sigma},2} : \|g\| \leqslant \rho\}$ and we show that the inclusion $\mathcal{B}_{\widehat{\sigma}}(\rho) \subset W^r(\omega^w,\Psi)$ is valid.

For an arbitrary function $g \in \mathbb{B}_{\widehat{\sigma},2}$ the inequality $\|g^{(r)}\| \leqslant (\widehat{\sigma})^r \|g\|$ holds and the representation

$$g(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} \mathcal{F}(g,t) e^{ixt} dt$$

is valid, where $\mathcal{F}(g) \in L_2(-\sigma,\sigma)$ (see, for example, [24, ch. II, §2.5]). In the case when $g \in \mathcal{B}_{\widehat{\sigma}}(\rho)$ we obtain on the basis of (27)

$$\omega^w(g^{(r)},x) = \sup\Big\{\Big(\int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} |\mathcal{F}(g^{(r)},t)|^2 |w(ht)|^2 dt\Big)^{1/2} : 0 < h \leqslant x\Big\} \leqslant |w(\widehat{\sigma}x)|_* \|g^{(r)}\| \leqslant$$

$$\leqslant (\widehat{\sigma})^r |w(\widehat{\sigma}x)|_* \|g\| \leqslant |w(\widehat{\sigma}x)|_* \inf\Big\{\frac{\Psi(t)}{|w(t\widehat{\sigma})|_*} : 0 < t \leqslant \frac{t_*}{\nu\pi}\Big\}. \tag{32}$$

Let first $0 < x < t_*/\widehat{\sigma}$. Assuming on the right-hand side of the relation (32) $t = x$, we write down

$$\omega^w(g^{(r)},x) \leqslant \Psi(x).$$

Let now $t_*/\widehat{\sigma} \leqslant x < \infty$. Taking into account that in this case $|w(\widehat{\sigma}x)|_* = |w(t_*)|$ and assuming on the right-hand side of the relation (32) $t = t_*/\widehat{\sigma}$, we obtain

$$\omega^w(g^{(r)},x) \leqslant \Psi(t_*/\sigma) \leqslant \Psi(x).$$

Consequently, the set $\mathcal{B}_{\widehat{\sigma}}(\rho)$ belongs to the class $W^r(\omega^w,\Psi)$.

Proceeding from the definition of the average Bernstein ν-width, we write down the chain of inequalities

$$\overline{b}_\nu(W^r(\omega^w,\Psi); L_2(\mathbb{R})) \geqslant \overline{b}_\nu(\mathcal{B}_{\widehat{\sigma}}(\rho), L_2(\mathbb{R})) \geqslant \rho. \tag{33}$$

Putting

$$\mathcal{K}_{\nu,r,t}(\varepsilon) := (1+\varepsilon)^r |w(t\nu\pi(1+\varepsilon))|_* \tag{34}$$

from (33) and (31) we get

$$\overline{b}_\nu(W^r(\omega^w,\Psi); L_2(\mathbb{R})) \geqslant \frac{1}{(\nu\pi)^r} \inf\Big\{\frac{\Psi(t)}{\mathcal{K}_{\nu,r,t}(\varepsilon)} : 0 < t \leqslant \frac{t_*}{\nu\pi}\Big\}. \tag{35}$$

It follows from (34) that for fixed values of the quantities ν, r, t the function $\mathcal{K}_{\nu,r,t}(\varepsilon)$ is monotonously increasing from ε and $\lim\{\mathcal{K}_{\nu,r,t}(\varepsilon) : \varepsilon \to 0+\} = |w(t\nu\pi)|$. Then on the basis of (35) we write down

$$\overline{b}_\nu(W^r(\omega^w,\Psi); L_2(\mathbb{R})) \geqslant \frac{1}{(\nu\pi)^r} \overline{\lim}\Big\{ \inf\Big\{\frac{\Psi(t)}{\mathcal{K}_{\nu,r,t}(\varepsilon)} : 0 < t \leqslant \frac{t_*}{\nu\pi}\Big\} : \varepsilon \to 0+\Big\} \geqslant$$

$$\geqslant \frac{1}{(\nu\pi)^r} \lim\Big\{\inf\Big\{\frac{\Psi(t)}{\mathcal{K}_{\nu,r,t}(1/n)} : 0 < t \leqslant \frac{t_*}{\nu\pi}\Big\} : n \to \infty\Big\} =$$

$$= \frac{1}{(\nu\pi)^r} \inf\left\{ \frac{\Psi(t)}{|w(t\nu\pi)|} : 0 < t \leqslant \frac{t_*}{\nu\pi} \right\}. \tag{36}$$

Using relations (22), (29) and (36), we obtain the required result (23). Theorem 1 is proved.

Corollary 1. *Let $\nu \in (0, \infty)$; $r \in \mathbb{N}$; a majorant Ψ satisfies the condition*

$$\inf\left\{ \frac{\Psi(t)}{|w(t\nu\pi)|} : 0 < t \leqslant \frac{t_*}{\nu\pi} \right\} = \varlimsup_{t\to 0+} \frac{\Psi(t)}{|w(t\nu\pi)|} \tag{37}$$

and the remaining requirements of Theorem 1 are satisfied. Then the following equalities hold:

$$\overline{\Pi}_\nu(W^r(\omega^w, \Psi); L_2(\mathbb{R})) = \mathcal{A}_{\nu\pi}(W^r(\omega^w, \Psi)) =$$

$$= \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(\omega^w, \Psi)\} = \frac{1}{(\nu\pi)^r} \inf\left\{ \frac{\Psi(t)}{|w(t\nu\pi)|} : 0 < t \leqslant \frac{t_*}{\nu\pi} \right\}, \tag{38}$$

where $\overline{\Pi}_\nu(W^r(\omega^w, \Psi); L_2(\mathbb{R}))$ is any of the average n-widths considered earlier. Moreover a pair of $(L_2^r(\mathbb{R}), \mathcal{L}_{\nu\pi})$, where the operator $\mathcal{L}_{\nu\pi}$ is defined by formula (19) for $\sigma = \nu\pi$, is extremal for the average linear ν-width $\overline{\delta}_\nu(W^r(\omega^w, \Psi); L_2(\mathbb{R}))$; the subspace $\mathbb{B}_{\nu\pi,2}$ will be extremal for the average Kolmogorov ν-width $\overline{d}_\nu(W^r(\omega^w, \Psi); L_2(\mathbb{R}))$.

Let us give several examples of the implementation of the results obtained in Corollary 1. Further let $\xi(t)$, $0 \leqslant t < \infty$, be an arbitrary continuous non-decreasing function such that $\xi(0) > 0$.

At first we assume that $w = w_m$, $m \in \mathbb{N}$. Then $|w_m(x)| = 2^m|\sin(x/2)|^m$ and $t_* = t_*(|w_m|^2) = \pi$. We consider the majorant $\Psi_m(\xi, t) := t^m\xi(t)$, $0 \leqslant t < \infty$. It can be shown that condition (37) will be satisfied, since

$$\inf\left\{ \frac{(t/2)^m}{\sin^m(t\nu\pi/2)}\xi(t) : 0 < t \leqslant \frac{1}{\nu} \right\} = \varlimsup_{t\to 0+} \frac{(t/2)^m}{\sin^m(t\nu\pi/2)}\xi(t) = \frac{\xi(0)}{(\nu\pi)^m}$$

Then by virtue of (38) we have

$$\overline{\Pi}_\nu(W^r(\omega_m, \Psi_m(\xi)); L_2(\mathbb{R})) = \mathcal{A}_{\nu\pi}(W^r(\omega_m, \Psi_m(\xi))) =$$

$$= \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(\omega_m, \Psi_m(\xi))\} = \frac{\xi(0)}{(\nu\pi)^{r+m}}. \tag{39}$$

In the case when $\xi(t) \equiv 1$ from (39) we obtain one of the results of the authors' work [19].

Now let $w = \widetilde{w}_m$, $m \in \mathbb{N}$. It can be shown that the condition (37) holds for the majorant $\Psi_{2m}(\xi, t)$, since

$$\inf\left\{ \frac{t^{2m}}{(1 - \operatorname{sinc}(t\nu\pi))^m}\xi(t) : 0 < t \leqslant \frac{t_*}{\nu\pi} \right\} =$$

$$= \overline{\lim_{t\to 0+}} \frac{t^{2m}}{(1-\mathrm{sinc}\,(t\nu\pi))^m}\xi(t) = \frac{6^m}{(\nu\pi)^{2m}}\xi(0).$$

Here $t_* = t_*(|\widetilde{w}_m|^2)$. Then on the basis of (38) we write down

$$\overline{\Pi}_\nu(W^r(\widetilde{\Omega}_m, \Psi_{2m}(\xi)); L_2(\mathbb{R})) = \mathcal{A}_{\nu\pi}(W^r(\widetilde{\Omega}_m, \Psi_{2m}(\xi))) =$$

$$= \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in W^r(\widetilde{\Omega}_m, \Psi_{2m}(\xi))\} = \frac{6^m}{(\nu\pi)^{r+2m}}\xi(0).$$

From the given examples it follows that there is a wide class of majorants satisfying the condition (37). This condition was not as rigid as the conditions imposed on the majorants in the 2π-periodic case or as the restrictions imposed on the majorants in the case of analytic functions in the unit circle (see, for example, [28] – [29]).

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