Journal of Optimization, Differential Equations and Their Applications

ISSN (print) 2617-0108

ISSN (online) 2663-6824

Volume 27

Issue 1

2019

Online Edition at http://model-dnu.dp.ua

Journal of Optimization, Differential Equations and Their Applications

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JODEA is published twice a year (in June and December).
ISSN (print) 2617–0108, ISSN (on-line) 2663–6824, DOI 10.15421/14192701.

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **27**, Issue **1**, June **2019**, pp. 1–20, DOI 10.15421/141901

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

FOURIER PROBLEM FOR WEAKLY NONLINEAR EVOLUTION INCLUSIONS WITH FUNCTIONALS

Mykola M. Bokalo*, Iryna V. Skira†

Abstract. The Fourier problem or, in other words, the problem without initial conditions for evolution equations and inclusions arise in modeling different nonstationary processes in nature, that started a long time ago and initial conditions do not affect on them in the actual time moment. Thus, we can assume that the initial time is $-\infty$, while 0 is the final time, and initial conditions can be replaced with the behaviour of the solution as time variable turns to $-\infty$. The Fourier problem for evolution variational inequalities (inclusions) with functionals is considered in this paper. The conditions for existence and uniqueness of weak solutions of the problem are set. Also the estimates of weak solutions are obtained.

Key words: Fourier problem, problem without initial condition, evolution inclusion, subdifferential of functional.

2010 Mathematics Subject Classification: 26D10, 47J20, 47J22, 49J40.

Communicated by Prof. P. I. Kogut

1. Introduction

In this paper we consider problem without initial conditions, or, in other words, the Fourier problem for evolution variational inequalities (inclusions) with functionals. Let us introduce an example of the problem being studied here.

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \in \mathbb{N})$, $\partial\Omega$ be the boundary of Ω, which is piecewise surface. We put $Q := \Omega \times (-\infty, 0]$, $\Sigma := \partial\Omega \times (-\infty, 0]$, $\Omega_t := \Omega \times \{t\}$ $\forall\, t \in \mathbb{R}$. For an arbitrary measurable set $F \subset \mathbb{R}^k$, where $k = n$ or $k = n+1$, let $L^2(F)$ be the standard Lebesgue space. Let $L^2_{\rm loc}(\overline{Q})$ be the space of functions defined on Q such that their restrictions on any bounded measurable set $Q' \subset Q$ belong to $L^2(Q')$. Denote by $H^1(\Omega)$ the standard Sobolev space, e.i., $H^1(\Omega) = \{v \in L^2(\Omega) \mid v_{x_i} \in L^2(\Omega),\ i = \overline{1,n}\}$ with scalar product $(v, w)_{H^1(\Omega)} = \int_\Omega [\nabla v \nabla w + vw]\, dx$, where $\nabla u := (u_{x_1}, \dots, u_{x_n})$, $\nabla w := (w_{x_1}, \dots, w_{x_n})$.

Let K be a convex closed set in $H^1(\Omega)$ which contains 0. Let us consider the problem of finding a function $u \in L^2_{\rm loc}(\overline{Q})$ such that $u_{x_i} \in L^2_{\rm loc}(\overline{Q})$, $i = \overline{1,n}$, $u_t \in L^2_{\rm loc}(\overline{Q})$, and, for a.e. $t \in (-\infty, 0]$, $u(\cdot, t) \in K$ and

$$\int_{\Omega_t} \big\{ u_t(v-u) + \nabla u \nabla (v-u) + u(v-u) + (v-u)\int_\Omega b(x,y,t) u(y,t)\, dy \big\}\, dx \geq \int_{\Omega_t} f(v-u)\, dx \quad \forall\, v \in K, \tag{1.1}$$

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$$\lim_{t\to-\infty} ||u(\cdot,t)||_{L^2(\Omega)} = 0, \tag{1.2}$$

where $f \in L^2_{\mathrm{loc}}(\overline{Q})$, $b \in L^\infty(\Omega\times\Omega\times(-\infty,0))$.

As it will be shown in the sequel, if

$$f \in L^2(Q), \qquad \operatorname*{ess\,sup}_{(x,y,t)\in\Omega\times\Omega\times(-\infty,0]} |b(x,y,t)|\sqrt{\mathrm{mes}_n\Omega} < K,$$

where $K > 0$ is a constant from inequality $K\|v\|_{L^2(\Omega)} \le \|v\|_{H^1(\Omega)}$, $\forall v \in H^1(\Omega)$, then this problem, which we call problem (1.1),(1.2), has unique solution.

We remark that problem (1.1),(1.2) can be written in more abstract way. Indeed, after appropriate identification of functions and functionals, we have continuous and dense imbedding

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))',$$

where $(H^1(\Omega))'$ is dual to $H^1(\Omega)$ space. Clearly, for any $h \in L^2(\Omega)$ and $v \in H^1(\Omega)$ we have $\langle h, v\rangle = (h, v)$, where $\langle\cdot,\cdot\rangle$ is the notation for scalar product on dual pair $[(H^1(\Omega))', H^1(\Omega)]$, and $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$. Thus, we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$.

Now, we denote $S := (-\infty, 0]$, $V := H^1(\Omega)$, $H := L^2(\Omega)$ and define an operator $A : V \to V'$ as follows

$$(Av, w) = \int_\Omega \big[\nabla v\nabla w + vw\big]\,dx, \quad v, w \in V.$$

For all $t \in S$ define an operator $B(t,\cdot) : H \to H$ as follows

$$B(t, v)(\cdot) = \int_\Omega b(\cdot, y, t)v(y)\,dy, \quad v \in H.$$

Then problem (1.1),(1.2) can be rewritten as following: find a function $u \in L^2_{\mathrm{loc}}(S;V)$ such that $u' \in L^2_{\mathrm{loc}}(S;H)$, condition (1.2) holds, and, for a.e. $t \in S$, $u(t) \in K$ and

$$(u'(t) + Au(t) + B(t, u(t)), v - u(t)) \ge (f(t), v - u(t)) \quad \forall\, v \in K. \tag{1.3}$$

Here $f \in L^2_{\mathrm{loc}}(S;H)$ is a given function.

We remark that variational inequality (1.3) can be written as a subdifferential inclusion. For this purpose we put $I_K(v) := 0$ if $v \in K$, and $I_K(v) := +\infty$ if $v \in V \setminus K$, and also

$$\Phi(v) = \frac{1}{2}\int_\Omega \big(|\nabla v|^2 + |v|^2\big)\,dx + I_K(v), \quad v \in V.$$

It is easy to verify that the functional $\Phi : V \to \mathbb{R}\cup\{+\infty\}$ is convex and semi-lower-continuous. By the known results (see, e.g., [22, p. 83]) it follows that the problem of finding a solution of variational inequality (1.3) can be written as such subdifferential inclusion: to find a function $u \in L^2_{\rm loc}(S;V)$ such that $u' \in L^2_{\rm loc}(S;H)$, condition (1.2) holds and, for a.e. $t \in S$, $u(t) \in D(\partial\Phi)$ and

$$u'(t) + \partial\Phi(u(t)) + B(t,u(t)) \ni f(t) \quad \text{in} \quad H. \tag{1.4}$$

The aim of this paper is to investigate problems for inclusions of type (1.4).

Problem without initial conditions or, in other words, the Fourier problem for evolution equations and inclusions arise in modeling different nonstationary processes in nature, that started a long time ago and initial conditions do not affect on them in the actual time moment. Thus, we can assume that the initial time is $-\infty$, while 0 is the final time, and initial conditions can be replaced with the behaviour of the solution as time variable turns to $-\infty$. Such problem appear in modeling in many fields of science such as ecology, economics, physics, cybernetics, etc. The research of the problem without initial conditions for the evolution equations and variational inequalities were conducted in the monographs [16, 18, 22], the papers [3, 6–8, 13, 15, 17, 19, 21], and others. In particular, R.E. Showalter in the paper [21] proved the existence of a unique solution $u \in \mathrm{e}^{2\omega\cdot}H^1(S;H)$, where H is a Hilbert space, of the problem without initial condition

$$u'(t) + \mu u(t) + A\big(u(t)\big) \ni f(t), \quad t \in S,$$

for $\omega + \mu > 0$ and $f \in \mathrm{e}^{2\omega\cdot}H^1(S;H)$, in case when $A : H \to 2^H$ is maximal monotone operator such that $0 \in A(0)$. Moreover, if $A = \partial\varphi$, where $\varphi : H \to \mathbb{R}\cup\{+\infty\}$ is proper, convex and lower-semi-continuous functional such that $\varphi(0) = 0 = \inf\{\varphi(v) : v \in H\}$, then this problem is uniquely solvable for each $\mu > 0$, $f \in L^2(S;H)$ and $\omega = 0$.

As is well known the uniqueness of the solutions of problem without initial conditions for linear parabolic equations and variational inequalities is possible only under some restrictions on the behavior of solutions as time variable terns to $-\infty$. For the first time it was strictly justified by A.N. Tikhonov [23] in the case of heat equation. However, as it was shown by M.M. Bokalo [3], problem without initial conditions for some nonlinear parabolic equations has a unique solution in the class of functions without behavior restriction as time variable terns to $-\infty$. Similar result for evolutionary variational inequalities were also obtained in the paper [4].

Note that in inclusion (1.4) the unknown function can enter both in the differential part and in functional part. Previously, the Fourier problem for evolution integro-differential equations were studied in [5, 9, 10] (see also references therein). Let us note that problems without initial conditions for variational inequalities or inclusions with functionals have not been considered in the literature, and this serves as one of the motivations for the study of such problems.

The outline of this paper is as follows. In Section 2, we give notation, definitions of needed function spaces and auxiliary results. In Section 3, we formulate the problem and main result. We prove the main result in Section 4.

2. Preliminaries

Set $S := (-\infty, 0]$. Let V and H be separable Hilbert spaces with the scalar products $(\cdot,\cdot)_V$, $(\cdot,\cdot)$ and norms $\|\cdot\|$, $|\cdot|$, respectively. Suppose that $V \subset H$ with dense, continuous and compact injection, i.e., the closure of V in H coincides with H, and there exists a constant $\lambda > 0$ such that

$$\lambda |v|^2 \le \|v\|^2 \quad \text{for all } v \in V, \tag{2.1}$$

and for every sequence $\{v_k\}_{k=1}^{\infty}$ bounded in V there exist an element $v \in V$ and a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ such that $v_{k_j} \underset{j\to\infty}{\longrightarrow} v$ strongly in H.

Let V' and H' be the dual spaces to V and H, respectively. We suppose (after appropriate identification of functionals), that the space H' is a subspace of V'. Identifying the spaces H and H' by the Riesz-Fréchet representation theorem, we obtain dense and continuous embeddings

$$V \subset H \subset V'. \tag{2.2}$$

Note that in this case $\langle g, v\rangle_V = (g, v)$ for every $v \in V, g \in H$, where $\langle\cdot,\cdot\rangle_V$ is the scalar product for the duality $[V', V]$. Therefore, further we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle_V$.

We introduce some spaces of functions and distributions. Let X be an arbitrary Hilbert space with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$. By $C(S; X)$ we mean the linear space of continuous functions defined on S with values in X. We say that $w_m \underset{m\to\infty}{\longrightarrow} w$ in $C(S; X)$ if for each $t_1, t_2 \in S$, $t_1 < t_2$, we have $\max\limits_{t\in[t_1,t_2]} \|w(t) - w_m(t)\|_X \underset{m\to\infty}{\longrightarrow} 0$.

Denote by $L^2_{\mathrm{loc}}(S; X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[t_1, t_2] \subset S$ belong to the space $L^2(t_1, t_2; X)$. We say that a sequence $\{w_m\}$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to w) in $L^2_{\mathrm{loc}}(S; X)$, if for each $t_1, t_2 \in S$, $t_1 < t_2$, the sequence of restrictions of $\{w_m\}$ on the segment $[t_1, t_2]$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to the restriction of w on this segment) in $L^2(t_1, t_2; X)$.

Let $\nu \in \mathbb{R}$. Put by definition

$$L^2_\nu(S; X) := \Big\{ f \in L^2_{\mathrm{loc}}(S; X) \;\Big|\; \int_S e^{2\nu t} \|f(t)\|_X^2 \, dt < \infty \Big\}.$$

This space is a Hilbert space with the scalar product

$$(f, g)_{L^2_\nu(S;X)} = \int_S e^{2\nu t} (f(t), g(t))_X \, dt$$

and the corresponding norm

$$\|f\|_{L^2_\nu(S;X)} := \Big(\int_S e^{2\nu t} \|f(t)\|_X^2 \, dt \Big)^{1/2}.$$

Also we introduce the space

$$L^\infty_\nu(S;X) := \{f \in L^\infty_{\mathrm{loc}}(S;X) \mid \operatorname*{ess\,sup}_{t\in S} \big[e^{\nu t}\|f(t)\|_X\big] < \infty\}.$$

By $D'(-\infty,0;V')$ we mean the space of continuous linear functionals on $D(-\infty,0)$ with values in V'_w (hereafter $D(-\infty,0)$ is space of test functions, that is, the space of infinitely differentiable on $(-\infty,0)$ functions with compact supports, equipped with the corresponding topology, and V'_w is the linear space V' equipped with weak topology). It is easy to see (using (2.2)), that spaces $L^2_{\mathrm{loc}}(S;V)$, $L^2_{\mathrm{loc}}(S;H)$, $L^2_{\mathrm{loc}}(S;V')$ can be identified with the corresponding subspaces of $D'(-\infty,0;V')$. In particular, this allows us to talk about derivatives w' of functions w from $L^2_{\mathrm{loc}}(S;V)$ or $L^2_{\mathrm{loc}}(S;H)$ in the sense of distributions $D'(-\infty,0;V')$ and belonging of such derivatives to $L^2_{\mathrm{loc}}(S;H)$ or $L^2_{\mathrm{loc}}(S;V')$.

Let us define the spaces

$$H^1_{\mathrm{loc}}(S;H) := \{w \in L^2_{\mathrm{loc}}(S;H) \,\big|\, w' \in L^2_{\mathrm{loc}}(S;H)\},$$
$$W_{2,\mathrm{loc}}(S;V) := \{w \in L^2_{\mathrm{loc}}(S;V) \,\big|\, w' \in L^2_{\mathrm{loc}}(S;V')\}.$$

From known results (see., for example, [14, pp. 177–179]) it follows that

$$H^1_{\mathrm{loc}}(S;H) \subset C(S;H) \quad \text{and} \quad W_{2,\mathrm{loc}}(S;V) \subset C(S;H).$$

Moreover, for every w in $H^1_{\mathrm{loc}}(S;H)$ or $W_{2,\mathrm{loc}}(S;V)$ the function $t \to |w(t)|^2$ is absolutely continuous on any segment of the interval S and the following equality holds

$$\frac{d}{dt}|w(t)|^2 = 2(w'(t),w(t)) \quad \text{for a.e.} \quad t \in S. \tag{2.3}$$

Denote

$$H^1_\nu(S;H) := \{w \in L^2_\nu(S;H) \,\big|\, w' \in L^2_\nu(S;H)\}, \quad \nu \in \mathbb{R}. \tag{2.4}$$

In this paper we use the following well-known facts.

Lemma 2.1 (Cauchy-Schwarz inequality [14, p. 158]). *Suppose that t_1, $t_2 \in \mathbb{R}$, $t_1 < t_2$, and X is a Hilbert space with the scalar product $(\cdot,\cdot)_X$. Then, for $v, w \in L^2(t_1,t_2;X)$, we have $(w(\cdot),v(\cdot))_X \in L^1\big(t_1,t_2\big)$ and*

$$\int_{t_1}^{t_2} (w(t),v(t))_X\,dt \le \|w\|_{L^2(t_1,t_2;X)}\|v\|_{L^2(t_1,t_2;X)}.$$

Lemma 2.2 ([27, pp. 173,179]). *Let Y be a Banach space with the norm $\|\cdot\|_Y$, and $\{v_k\}_{k=1}^\infty$ be a sequence of elements of Y, which is weakly or $*$-weakly convergent to v in Y. Then $\varliminf_{k\to\infty} \|v_k\|_Y \ge \|v\|_Y$.*

Lemma 2.3 (Aubin theorem [1], [2, p. 393]). *Let $q>1, r>1$, $t_1,t_2 \in \mathbb{R}$, $t_1 < t_2$, and $\mathcal{W},\mathcal{L},\mathcal{B}$ are Banach spaces such that $\mathcal{W} \overset{c}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$ (here $\overset{c}{\subset}$ means compact embedding, and $\circlearrowleft$ means continuous embedding). Then*

$$\{w \in L^q(t_1,t_2;\mathcal{W}) \mid w' \in L^r(t_1,t_2;\mathcal{B})\} \overset{c}{\subset} \Big(L^q(t_1,t_2;\mathcal{L}) \cap C([t_1,t_2];\mathcal{B})\Big). \tag{2.5}$$

Note that, we understand embedding (2.5) as follows: if a sequence $\{w_m\}$ is bounded in the space $L^q(t_1,t_2;\mathcal{W})$ and the sequence $\{w_m'\}$ is bounded in the space $L^r(t_1,t_2;\mathcal{B})$, then there exist a function $w\in C([t_1,t_2];\mathcal{B})\cap L^q(t_1,t_2;\mathcal{L})$ and a subsequence $\{w_{m_j}\}$ of the sequence $\{w_m\}$ such that $w_{m_j}\underset{j\to\infty}{\longrightarrow} w$ in $C([t_1,t_2];\mathcal{B})$ and strongly in $L^q(t_1,t_2;\mathcal{L})$.

Lemma 2.4. *If a sequence $\{w_m\}$ is bounded in the space $L^2_{\rm loc}(S;V)$ and the sequence $\{w_m'\}$ is bounded in the space $L^2_{\rm loc}(S;H)$, then there exist a function $w\in L^2_{\rm loc}(S;V)$, $w'\in L^2_{\rm loc}(S;H)$, and a subsequence $\{w_{m_j}\}$ of the sequence $\{w_m\}$ such that $w_{m_j}\underset{j\to\infty}{\longrightarrow} w$ in $C(S;H)$ and weakly in $L^2_{\rm loc}(S;V)$, and $w_{m_j}'\underset{j\to\infty}{\longrightarrow} w'$ weakly in $L^2_{\rm loc}(S;H)$.*

Proof of Lemma 2.4. Lemma 2.3 for $q=2$, $r=2$, $\mathcal{W}=V$, $\mathcal{L}=\mathcal{B}=H$ and reflexiveness of Hilbert spaces yield, for every $t_1,t_2\in S$, $t_1<t_2$, from the sequence of restrictions of the elements $\{w_m\}$ to the segment $[t_1,t_2]$ one can choose a subsequence which is convergent in $C([t_1,t_2];H)$ and weakly in $L^2(t_1,t_2;V)$, and the sequence of derivatives of the elements of this subsequence is weakly convergent in $L^2(t_1,t_2;H)$. For each $k\in\mathbb{N}$ we choose a subsequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$ of the given sequence which is convergent in $C([-k,0];H)$ and weakly in $L^2(-k,0;V)$ to some function $\widehat{w}_k\in C([-k,0];H)\cap L^2(-k,0;V)$, and the sequence $\{w_{m_{k,j}}'\}_{j=1}^{\infty}$ is weakly convergent to the derivative $\widehat{w}_k'$ in $L^2(-k,0;H)$. Making this choice we ensure that the sequence $\{w_{m_{k+1,j}}\}_{j=1}^{\infty}$ was a subsequence of the sequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$. Now, according to the diagonal process we select the desired subsequence as $\{w_{m_{j,j}}\}_{j=1}^{\infty}$, and we define the function w as follows: for each $k\in\mathbb{N}$ we take $w(t):=\widehat{w}_k(t)$ for $t\in(-k,-k+1]$. □

3. Statement of the problem and main result

Let $\Phi: V\to\mathbb{R}_\infty:=(-\infty,+\infty]$ be a proper functional, i.e.,

$$\mathrm{dom}(\Phi):=\{v\in V:\ \Phi(v)<+\infty\}\neq\emptyset,$$

which satisfies the conditions:

$(\mathcal{A}_1)$ $\quad \Phi\big(\alpha v+(1-\alpha)w\big)\le\alpha\Phi(v)+(1-\alpha)\Phi(w)\quad\forall\, v,w\in V,\ \forall\,\alpha\in[0,1],$

i.e., the functional Φ is *convex*,

$(\mathcal{A}_2)$ $\quad v_k\underset{k\to\infty}{\longrightarrow} v$ in $V \quad\Longrightarrow\quad \underset{k\to\infty}{\underline{\lim}}\ \Phi(v_k)\ge\Phi(v),$

i.e., the functional Φ is *lower semicontinuous.*

Recall that the *subdifferential* of functional Φ is a mapping $\partial\Phi : V \to 2^{V'}$, defined as follows

$$\partial\Phi(v) := \{v^* \in V' \mid \Phi(w) \geq \Phi(v) + (v^*, w - v) \quad \forall\, w \in V\}, \quad v \in V,$$

and the *domain* of the subdifferential $\partial\Phi$ is the set $D(\partial\Phi) := \{v \in V \mid \partial\Phi(v) \neq \emptyset\}$. We identify the subdifferential $\partial\Phi$ with its graph, assuming that $[v, v^*] \in \partial\Phi$ if and only if $v^* \in \partial\Phi(v)$, i.e., $\partial\Phi = \{[v, v^*] \mid v \in D(\partial\Phi),\ v^* \in \partial\Phi(v))\}$. R. Rockafellar in paper [20, Theorem A] proves that the subdifferential $\partial\Phi$ is a *maximal monotone operator*, that is,

$$(v_1^* - v_2^*, v_1 - v_2) \geq 0 \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi,$$

and for every element $[v_1, v_1^*] \in V \times V'$ we have the implication

$$(v_1^* - v_2^*, v_1 - v_2) \geq 0 \quad \forall\, [v_2, v_2^*] \in \partial\Phi \quad \Longrightarrow \quad [v_1, v_1^*] \in \partial\Phi.$$

Let, for each $t \in S$, $B(t, \cdot) : H \to H$ be an operator which satisfies the condition:

($\mathcal{B}$) for any $v \in H$ the mapping $B(\cdot, v) : S \to S$ is measurable, and there exists a constant $L \geq 0$ such that following inequality holds

$$|B(t, v_1) - B(t, v_2)| \leq L|v_1 - v_2| \tag{3.1}$$

for a.e. $t \in S$, and for all $v_1, v_2 \in H$; in addition, $B(t, 0) = 0$ for a.e. $t \in S$.

Remark 3.1. From the condition ($\mathcal{B}$) it follows that for a.e. $t \in S$, and for every $v \in H$ the following estimate is valid:

$$|B(t, v)| \leq L|v|. \tag{3.2}$$

Let us consider the evolutionary variational inequality

$$u'(t) + \partial\Phi\big(u(t)\big) + B(t, u(t)) \ni f(t), \quad t \in S, \tag{3.3}$$

where $f : S \to V'$ is a given measurable function and $u : S \to V$ is an unknown function.

Definition 3.1. Let conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{B}$) hold, and $f \in L^2_{\mathrm{loc}}(S; V')$. The *solution* of variational inequality (3.3) is a function $u : S \to V$ that satisfies the following conditions:

1) $u \in W_{2,\mathrm{loc}}(S; V)$;

2) $u(t) \in D(\partial\Phi)$ for a.e. $t \in S$;

3) there exists a function $g \in L^2_{\mathrm{loc}}(S;V')$ such that, for a.e. $t \in S$, $g(t) \in \partial\Phi\big(u(t)\big)$ and
$$u'(t) + g(t) + B(t,u(t)) = f(t) \quad \text{in } V'.$$

For variational inequality (3.3) consider the problem: find its solution which satisfies the condition
$$\lim_{t\to-\infty} e^{\gamma t}|u(t)| = 0, \tag{3.4}$$
where $\gamma \in \mathbb{R}$ is given.

The problem of finding a solution of variational inequality (3.3) (for given Φ,B,f) satisfying the condition (3.4) for given γ, is called the Fourier problem or, in other words, the problem without initial conditions for the evolution variational inequality (3.3). This problem, in short, be called the problem $\mathbf{P}(\Phi, B, f, \gamma)$, and the function u is called its solution.

Additionally, assume that the following conditions hold:

$(\mathcal{A}_3)$ there exists a constant $K_1 > 0$ such that
$$(v_1^* - v_2^*, v_1 - v_2) \ge K_1|v_1 - v_2|^2 \quad \forall\ [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi;$$

$(\mathcal{A}_4)$ there exists a constant $K_2 > 0$ such that
$$\Phi(v) \ge K_2\|v\|^2 \quad \forall\ v \in \mathrm{dom}(\Phi);$$
moreover, $\Phi(0) = 0$.

Remark 3.2. Condition $(\mathcal{A}_4)$ implies that $\Phi(v) \ge \Phi(0) + (0, v - 0)$ $\forall v \in V$, hence $[0,0] \in \partial\Phi$. From this and condition $(\mathcal{A}_3)$ we have
$$(v^*, v) \ge K_1|v|^2 \quad \forall\,[v, v^*] \in \partial\Phi. \tag{3.5}$$

Now we shall formulate the main result.

Theorem 3.1. *Let conditions $(\mathcal{A}_1)$ – $(\mathcal{A}_3)$, $(\mathcal{B})$ hold, and $\gamma \in \mathbb{R}$ is such that*
$$\gamma < K_1 - L. \tag{3.6}$$
Then the problem $\boldsymbol{P}(\Phi, B, f, \gamma)$ has at most one solution.

Theorem 3.2. *Let conditions $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$, $(\mathcal{B})$ hold, and*

$(\mathcal{F})$ $\quad f \in L^2_\gamma(S;H)$,

where $\gamma \in \mathbb{R}$ satisfies inequality (3.6). *Then the problem $\boldsymbol{P}(\Phi, B, f, \gamma)$ has a unique solution, it belongs to the space $L^\infty_\gamma(S;V) \cap L^2_\gamma(S;V) \cap H^1_\gamma(S;H)$ and satisfies the estimate:*
$$e^{2\gamma\sigma}\|u(\sigma)\|^2 + \int_{-\infty}^{\sigma} e^{2\gamma t}\|u(t)\|^2\,dt + \int_{-\infty}^{\sigma} e^{2\gamma t}|u'(t)|^2\,dt$$
$$+ \int_{-\infty}^{\sigma} e^{2\gamma t}\Phi(u(t))dt \le C_1 \int_{-\infty}^{\sigma} e^{2\gamma t}|f(t)|^2\,dt, \quad \sigma \in S, \tag{3.7}$$
where C_1 is a positive constant depending on K_1, K_2, L and γ only.

Remark 3.3. The problem $\mathbf{P}(\Phi, B, f, \gamma)$ can be replaced by the following problem. Let K be a convex and closed set in V, $A : V \to V'$ be a monotone, bounded and semi-continuous operator such that $(A(v), v) \geq \widetilde{K}_1 \|v\|^2 \quad \forall v \in V$, where $\widetilde{K}_1 = \text{const} > 0$. The problem is to find a function $u \in W_{2,\text{loc}}(S; V)$ satisfying the condition (3.4) and, for a.e. $t \in S$, $u(t) \in K$ and

$$(u'(t) + A(u(t)) + B(t, u(t)), v - u(t)) \geq (f(t), v - u(t)) \quad \forall\, v \in K.$$

4. Proof of the main result

Proof of the Theorem 3.1. Assume the contrary. Let u_1, u_2 be two solutions of the problem $\mathbf{P}(\Phi, B, f, \gamma)$. Then for every $i \in \{1, 2\}$ there exists function $g_i \in L^2_{\text{loc}}(S; V')$ such that, for a.e. $t \in S$, $g_i(t) \in \partial\Phi\big(u_i(t)\big)$ and

$$u_i'(t) + g_i(t) + B(t, u_i(t)) = f(t) \quad \text{in } V'. \tag{4.1}$$

We put $w(t) := u_1(t) - u_2(t)$, $t \in S$. From equalities (4.1) for a.e. $t \in S$ we obtain

$$w'(t) + g_1(t) - g_2(t) + B(t, u_1(t)) - B(t, u_2(t)) = 0 \quad \text{in } V'. \tag{4.2}$$

From (3.4) it follows that the following condition holds

$$e^{2\gamma t}|w(t)|^2 \to 0 \quad \text{as} \quad t \to -\infty. \tag{4.3}$$

Let $\sigma_1, \sigma_2 \in S$ be arbitrary numbers such that $\sigma_1 < \sigma_2$. Multiplying equality (4.2) by $w(t)e^{2\gamma t}$, and integrating from σ_1 to σ_2 we obtain

$$\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(w'(t), w(t))\, dt + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(g_1(t) - g_2(t), u_1(t) - u_2(t)) dt \\ + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big(B(t, u_1(t)) - B(t, u_2(t)), w(t)\big) dt = 0. \tag{4.4}$$

By condition $(\mathcal{A}_3)$ and the fact that $g_i(t) \in \partial\Phi(u_i(t))$, $i = 1, 2$, for a.e. $t \in S$ we have the inequality

$$(g_1(t) - g_2(t), u_1(t) - u_2(t)) \geq K_1 |w(t)|^2. \tag{4.5}$$

Consider the last term from left-hand side of equality (4.4). Using (3.1) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \Big| \int_{\sigma_1}^{\sigma_2} & e^{2\gamma t}\big(B(t, u_1(t)) - B(t, u_2(t)), w(t)\big) dt \Big| \\ & \leq \int_{\sigma_1}^{\sigma_2} e^{2\gamma t} \big|B(t, u_1(t)) - B(t, u_2(t))\big| |w(t)|\, dt \\ & \leq L \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\, |w(t)|^2\, dt. \end{aligned} \tag{4.6}$$

By (2.3), (4.5), (4.6), from (4.4) we obtain the following inequality

$$\frac{1}{2}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\frac{d|w(t)|^2}{dt}\,dt + \big(K_1 - L\big)\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|w(t)|^2 dt \le 0. \tag{4.7}$$

Using the integration-by-parts formula, from (4.7) we have

$$e^{2\gamma t}|w(t)|^2\Big|_{\sigma_1}^{\sigma_2} + 2\big(K_1 - L - \gamma\big)\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|w(t)|^2\,dt \le 0. \tag{4.8}$$

Since condition (3.6) holds, from (4.8) we obtain

$$e^{2\gamma\sigma_2}|w(\sigma_2)|^2 \le e^{2\gamma\sigma_1}|w(\sigma_1)|^2. \tag{4.9}$$

Let us fix an arbitrary σ_2 in (1.1), and pass to the limit as $\sigma_1 \to -\infty$. According to condition (4.3), the the right side of inequality (1.1) turns to 0. Thus, we get the equality $e^{2\gamma\sigma_2}|w(\sigma_2)|^2 = 0$. Since $\sigma_2 \in S$ is an arbitrary number, we have $w(t) = 0$ for a.e. $t \in S$, this contradicts our assumption. Therefore, a solution of the problem $\mathbf{P}(\Phi, B, f, \gamma)$ is unique. $\square$

Proof of the Theorem 3.2. We divide the proof into five steps.

Step 1 (*auxiliary statements*). Under assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ we define the functional $\Phi_H : H \to \mathbb{R}_\infty$ by the rule: $\Phi_H(v) := \Phi(v)$, if $v \in V$, and $\Phi_H(v) := +\infty$ otherwise. Note that conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, Lemma IV.5.2 and Proposition IV.5.2 of the monograph [22] imply that Φ_H is a proper, convex and lower-semi-continuous functional on H, $\mathrm{dom}(\Phi_H) = \mathrm{dom}(\Phi) \subset V$ and $\partial\Phi_H = \partial\Phi \cap (V \times H)$, where $\partial\Phi_H : H \to 2^H$ is the subdifferential of the functional Φ_H.

The following statements will be used in the sequel.

Lemma 4.1 ([22, Lemma IV.4.3]). *Let $-\infty < a < b < +\infty$, $w \in H^1(a,b;H)$, and there exists $g \in L^2(a,b;H)$ such that $g(t) \in \partial\Phi_H\big(w(t)\big)$ for a.e. $t \in (a,b)$. Then the function $\Phi_H\big(w(\cdot)\big)$ is absolutely continuous on the interval $[a,b]$ and for any function $h : [a,b] \to H$ such that $h(t) \in \partial\Phi_H\big(w(t)\big)$ the following equality holds*

$$\frac{d}{dt}\Phi_H\big(w(t)\big) = (h(t), w'(t)) \quad \textit{for a.e. } t \in (a,b).$$

Lemma 4.2 ([11, Proposition 3.12], [22, Proposition IV.5.2]). *Let $T > 0$, $\widetilde{f} \in L^2(0,T;H)$ and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function*

$$w \in C([0,T];H) \cap H^1(0,T;H)$$

such that $w(0) = w_0$ and, for a.e. $t \in (0,T]$, $w(t) \in D(\partial\Phi_H)$ and

$$w'(t) + \partial\Phi_H\big(w(t)\big) \ni \widetilde{f}(t) \quad \textit{in } H. \tag{4.10}$$

Lemma 4.3. *Let $t_0 < 0$, $\widetilde{f} \in L^2(t_0, 0; H)$, and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function $w \in C([t_0, 0]; H) \cap H^1(t_0, 0; H)$ such that $w(t_0) = w_0$ and, for a.e. $t \in (t_0, 0]$, $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) + B(t, w(t)) \ni \widetilde{f}(t) \quad \text{in } H, \tag{4.11}$$

that is, there exists $\widetilde{g} \in L^2(t_0, 0; H)$ such that, for a.e. $t \in (t_0, 0]$, we have $\widetilde{g}(t) \in \partial\Phi_H(w(t))$ and

$$w'(t) + \widetilde{g}(t) + B(t, w(t)) = \widetilde{f}(t) \quad \text{in } H. \tag{4.12}$$

Proof of Lemma 4.3. Let $\alpha > 0$ be an arbitrary fixed number and set

$$M := \{w \in C([t_0, 0]; H) \mid w(t_0) = w_0\}.$$

Consider M with the metric

$$\rho(w_1, w_2) = \max_{t \in [t_0, 0]} \big[e^{-\alpha(t-t_0)} |w_1(t) - w_2(t)|\big], \quad w_1, w_2 \in M.$$

It is obvious that the metric space (M, ρ) is complete. Now let us consider an operator $A : M \to M$ defined as follows: for any given function $\widetilde{w} \in M$, it defines a function $\widehat{w} \in M \cap H^1(t_0, 0; H)$ such that, for a.e. $t \in (t_0, 0]$, $\widehat{w}(t) \in D(\partial\Phi_H)$ and

$$\widehat{w}'(t) + \partial\Phi_H(\widehat{w}(t)) \ni \widetilde{f}(t) - B(t, \widetilde{w}(t)) \quad \text{in} \quad H. \tag{4.13}$$

Clearly, variational inequality (4.13) coincides with variational inequality (4.10) after replacing $[0, T]$ by $[t_0, 0]$, $\widetilde{f}(t)$ by $\widetilde{f}(t) - B(t, \widetilde{w}(t))$, the condition $w(0) = w_0$ by the condition $\widehat{w}(t_0) = w_0$. Thus, using Lemma 4.2, we get that operator A is well-defined. Let us show that the operator A is a contraction for some $\alpha > 0$. Indeed, let $\widetilde{w}_1, \widetilde{w}_2$ be arbitrary functions from M and $\widehat{w}_1 := A\widetilde{w}_1$, $\widehat{w}_2 := A\widetilde{w}_2$. According to (4.13) there exist functions $\widehat{g}_1$ and $\widehat{g}_2$ from $L^2(t_0, 0; H)$ such that for every $k \in \{1, 2\}$ and for a.e. $t \in (t_0, 0]$ we have $\widehat{g}_k(t) \in \partial\Phi_H(\widehat{w}_k(t))$ and

$$\widehat{w}_k'(t) + \widehat{g}_k(t) = \widetilde{f}(t) - B(t, \widetilde{w}_k(t)), \tag{4.14}$$

while $\widehat{w}_k(t_0) = w_0$.

Subtracting identity (4.14) for $k = 2$ from identity (4.14) for $k = 1$, and, for a.e. $t \in (t_0, 0]$, multiplying the obtained identity by $\widehat{w}_1(t) - \widehat{w}_2(t)$, we get

$$\begin{aligned}&\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) + (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t)) \\ &= -(B(t, \widetilde{w}_1(t)) - B(t, \widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t)) \text{ for a.e. } t \in (t_0, 0],\end{aligned} \tag{4.15}$$

$$\widehat{w}_1(t_0) - \widehat{w}_2(t_0) = 0. \tag{4.16}$$

We integrate equality (4.15) by t from t_0 to $\sigma \in (t_0, 0]$, taking into account that for a.e. $t \in (t_0, 0]$ we have

$$\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) = \frac{1}{2}\frac{d}{dt}|\widehat{w}_1(t) - \widehat{w}_2(t)|^2.$$

As a result we get the equality

$$\frac{1}{2}|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + \int_{t_0}^{\sigma} (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t))\, dt$$
$$= -\int_{t_0}^{\sigma} \big(B(t, \widetilde{w}_1(t)) - B(t, \widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t)\big) dt. \quad (4.17)$$

By condition $(\mathcal{A}_3)$, for a.e. $t \in (t_0, 0]$ we have the inequality

$$(\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t)) \geq K_1|\widehat{w}_1(t) - \widehat{w}_2(t)|^2. \quad (4.18)$$

Taking into account condition $(\mathcal{B})$ and the Cauchy inequality, for a.e. $t \in (t_0, 0]$ we obtain

$$\begin{aligned} \big|\big(B(t, \widetilde{w}_1(t)) - B(t, \widetilde{w}_2(t)), &\widehat{w}_1(t) - \widehat{w}_2(t)\big)\big| \\ &\leq \big|B(t, \widetilde{w}_1(t)) - B(t, \widetilde{w}_2(t))\big| \cdot \big|\widehat{w}_1(t) - \widehat{w}_2(t)\big| \\ &\leq L|\widetilde{w}_1(t) - \widetilde{w}_2(t)| \cdot |\widehat{w}_1(t) - \widehat{w}_2(t)| \\ &\leq \varepsilon|\widehat{w}_1(t) - \widehat{w}_2(t)|^2 + \frac{L^2}{4\varepsilon}|\widetilde{w}_1(t) - \widetilde{w}_2(t)|, \end{aligned} \quad (4.19)$$

where $\varepsilon > 0$ is an arbitrary.

From (4.17), according to (4.18) and (4.19), we have

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + 2(K_1 - \varepsilon) \int_{t_0}^{\sigma} |\widehat{w}_1(t) - \widehat{w}_2(t)|^2\, dt$$
$$\leq (2\varepsilon)^{-1} L^2 \int_{t_0}^{\sigma} \big|\widetilde{w}_1(t) - \widetilde{w}_2(t)\big|^2 dt. \quad (4.20)$$

Choosing $\varepsilon = 2^{-1}K_1$, from (4.20) we obtain

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 \leq C_2 \int_{t_0}^{\sigma} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2\, dt, \quad \sigma \in (t_0, 0], \quad (4.21)$$

where $C_2 > 0$ is the constant.

After multiplying inequality (4.21) by $e^{-2\alpha(\sigma - t_0)}$ we obtain

$$\begin{aligned} &e^{-2\alpha(\sigma-t_0)}|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 \\ &\leq C_2 e^{-2\alpha(\sigma-t_0)} \int_{t_0}^{\sigma} e^{2\alpha(t-t_0)} e^{-2\alpha(t-t_0)} |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2\, dt \\ &\leq C_2 e^{-2\alpha(\sigma-t_0)} \max_{t\in[t_0,0]} \big[- e^{\alpha(t-t_0)}|\widetilde{w}_1(t) - \widetilde{w}_2(t)|\big]^2 \int_{t_0}^{\sigma} e^{2\alpha(t-t_0)}\, dt \\ &= \frac{C_2}{2\alpha}\big(1 - e^{-2\alpha(\sigma-t_0)}\big)\big[\rho(\widetilde{w}_1, \widetilde{w}_2)\big]^2 \leq \frac{C_2}{2\alpha}\big[\rho(\widetilde{w}_1, \widetilde{w}_2)\big]^2, \quad \sigma \in (t_0, 0]. \end{aligned} \quad (4.22)$$

From (4.22) it easily follows that

$$\rho(\widehat{w}_1, \widehat{w}_2) \leq \sqrt{C_2/(2\alpha)}\rho(\widetilde{w}_1, \widetilde{w}_2).$$

From this, choosing $\alpha > 0$ such that inequality $C_2/(2\alpha) < 1$ holds, we obtain that operator A is a contraction. Hence, we may apply the Banach fixed-point theorem [12, Theorem 5.7] and deduce that there exists a unique function $w \in M$ such that $Aw = w$, i.e., we have proved Lemma 4.3. □

Step 2 (*solution approximation*). We construct a sequence of functions which, in some sense, approximate the solution of the problem $\mathbf{P}(\Phi, B, f, \gamma)$.

For each $k \in \mathbb{N}$, let $\widehat{f}_k(t) := f(t)$ for $t \in S_k := (-k, 0]$ and let us consider the problem of finding a function $\widehat{u}_k \in C(\overline{S_k}; H) \cap H^1(S_k; H)$, where $H^1(S_k; H) := \big\{w \in L^2(S_k; H) \bigm| w' \in L^2(S_k; H)\big\}$, such that, for a.e. $t \in S_k$, we have $\widehat{u}_k(t) \in D(\partial\Phi_H)$ and

$$\widehat{u}_k'(t) + \partial\Phi_H\big(\widehat{u}_k(t)\big) + B(t, \widehat{u}_k(t)) \ni \widehat{f}_k(t) \quad \text{in } H, \tag{4.23}$$

$$\widehat{u}_k(-k) = 0. \tag{4.24}$$

Inclusion (4.23) means that there exists a function $\widehat{g}_k \in L^2(S_k; H)$ such that, for a.e. $t \in S_k$, we have $\widehat{g}_k(t) \in \partial\Phi_H(\widehat{u}_k(t))$ and

$$\widehat{u}_k'(t) + \widehat{g}_k(t) + B(t, \widehat{u}_k(t)) = \widehat{f}_k(t) \quad \text{in} \quad H. \tag{4.25}$$

Since $D(\partial\Phi_H) \subset \mathrm{dom}(\Phi_H) \subset V$, thus $\widehat{u}_k(t) \in V$ for a.e. $t \in S_k$. According to the definition of the subdifferential of a functional and the fact that $\widehat{g}_k(t) \in \partial\Phi(\widehat{u}_k(t))$ for a.e. $t \in S_k$, we have

$$\Phi(0) \geq \Phi(\widehat{u}_k(t)) + (\widehat{g}_k(t), 0 - \widehat{u}_k(t)) \quad \text{for a.e.} \quad t \in S_k.$$

Using this and condition $(\mathcal{A}_4)$ we obtain

$$(\widehat{g}_k(t), \widehat{u}_k(t)) \geq \Phi(\widehat{u}_k(t)) \geq K_2\|\widehat{u}_k(t)\|^2 \quad \text{for a.e.} \quad t \in S_k. \tag{4.26}$$

Since the left side of this chain of inequalities belongs to $L^1(S_k)$, then $\widehat{u}_k$ belongs to $L^2(S_k; V)$.

For each $k \in \mathbb{N}$ we extend functions $\widehat{f}_k, \widehat{u}_k$ and $\widehat{g}_k$ by zero for the entire interval S, and denote these extensions by f_k, u_k and g_k respectively. From the above it follows that, for each $k \in \mathbb{N}$, the function u_k belongs to $L^2(S; V)$, its derivative u_k' belongs to $L^2(S; H)$ and, for a.e. $t \in S$, the inclusion $g_k(t) \in \partial\Phi_H\big(u_k(t)\big)$ and the following equality (see (4.25)) hold

$$u_k'(t) + g_k(t) + B(t, u_k(t)) = f_k(t) \quad \text{in} \quad H. \tag{4.27}$$

In order to show the convergence $\{u_k\}_{k=1}^{\infty}$ to the solution of the problem $\mathbf{P}(\Phi, B, f, \gamma)$ we need some estimates of the functions u_k, $k \in \mathbb{N}$.

Step 3 (*estimates of solution approximations*).

Let $\sigma_1, \sigma_2 \in S$ be arbitrary numbers such that $\sigma_1 < \sigma_2$, and $k \in \mathbb{N}$. Multiplying identity (4.27), for a.e. $t \in S$, by $e^{2\gamma t} u_k(t)$ and integrating from σ_1 to σ_2, we obtain

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(u_k'(t), u_k(t))\,dt + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(g_k(t), u_k(t))\,dt&\\ + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big(B(t, u_k(t)), u_k(t)\big)\,dt&\\ = \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(f_k(t), u_k(t))\,dt.&\end{aligned}$$

From this taking into account (2.3) and using the integration-by-parts formula, we obtain

$$\begin{aligned}e^{2\gamma t}|u_k(t)|^2\Big|_{\sigma_1}^{\sigma_2} - 2\gamma \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2\,dt + 2\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(g_k(t), u_k(t))\,dt&\\ + 2\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big(B(t, u_k(t)), u_k(t)\big)\,dt = 2\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(f_k(t), u_k(t))\,dt.&\end{aligned} \tag{4.28}$$

Acccording to the definition of u_k and (4.26), we obtain

$$(g_k(t), u_k(t)) \geq \Phi\big(u_k(t)\big) \geq K_2\|u_k(t)\|^2 \quad \text{for a.e. } t \in S. \tag{4.29}$$

Let us estimate the third term on the left-hand side of inequality (4.28). From (3.5) and (4.29) for arbitrary $\delta \in (0, 1)$ we obtain

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(g_k(t), u_k(t))\,dt = (\delta + (1-\delta))\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(g_k(t), u_k(t))\,dt&\\ \geq \delta K_1 \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2\,dt&\\ + 2^{-1}(1-\delta)K_2 \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\|u_k(t)\|^2\,dt&\\ + 2^{-1}(1-\delta)\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\Phi\big(u_k(t)\big)\,dt.&\end{aligned} \tag{4.30}$$

Now, let us estimate the last item on the left-hand side of inequality (4.28). Using the Cauchy-Shwarz inequality, (3.2) we have

$$\begin{aligned}\Big|\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big(B(t, u_k(t)), u_k(t)\big)dt\Big| \leq \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big|B(t, u_k(t))\big||u_k(t)|\,dt&\\ \leq L\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2\,dt.&\end{aligned} \tag{4.31}$$

Using the Cauchy inequality we estimate the right-hand side of (4.28) as follows

$$\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(f_k(t), u_k(t))\,dt \leq \varepsilon \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2\,dt + (4\varepsilon)^{-1}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|f_k(t)|^2\,dt, \tag{4.32}$$

where $\varepsilon > 0$ is arbitrary.

From (4.28), taking into account (4.30), (4.31) and (4.32), we obtain

$$
\begin{aligned}
e^{2\gamma t}|u_k(t)|^2\Big|_{\sigma_1}^{\sigma_2} &+ 2[\delta K_1 - L - \gamma - \varepsilon]\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2\,dt \\
&+ (1-\delta)K_2\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\|u_k(t)\|^2\,dt \\
&+ (1-\delta)\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\Phi\big(u_k(t)\big)\,dt \\
&\le (2\varepsilon)^{-1}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|f_k(t)|^2\,dt, \quad \delta \in (0,1),\ \varepsilon \in (0,+\infty). \qquad (4.33)
\end{aligned}
$$

Since $K_1 > 0$, γ satisfies (3.6), we first choose δ from $(0,1)$ such that $\delta K_1 - L - \gamma > 0$, and then we choose $\varepsilon = 2^{-1}[\delta K_1 - L - \gamma] > 0$. As a result, from (4.33) we obtain the estimate

$$
\begin{aligned}
e^{2\gamma t}|u_k(t)|^2\Big|_{\sigma_1}^{\sigma_2} &+ \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big[|u(t)|^2 + \|u_k(t)\|^2\big]\,dt + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\Phi\big(u_k(t)\big)\,dt \\
&\le C_3\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|f_k(t)|^2\,dt, \qquad (4.34)
\end{aligned}
$$

where C_3 is a positive constant depending on K_1, K_2, L and γ only.

We take $\sigma_2 = \sigma \in S$ is arbitrary, and pass to the limit in (4.34) as $\sigma_1 \to -\infty$. Taking into account $(\mathcal{F})$ and the definition of u_k and f_k, we obtain

$$
\begin{aligned}
e^{2\gamma\sigma}|u_k(\sigma)|^2 &+ \int_{-\infty}^{\sigma} e^{2\gamma t}\big[|u(t)|^2 + \|u_k(t)\|^2\big]\,dt \\
&+ \int_{-\infty}^{\sigma} e^{2\gamma t}\Phi\big(u_k(t)\big)\,dt \le C_3\int_{-\infty}^{\sigma} e^{2\gamma t}|f_k(t)|^2\,dt, \quad \sigma \in S. \qquad (4.35)
\end{aligned}
$$

Since $\sigma \in S$ is arbitrary, from (4.35) it follows that

the sequence $\{u_k(\cdot)\}_{k=1}^{+\infty}$ is bounded in $L^\infty_\gamma(S;H)$, $L^2_\gamma(S;H)$ and $L^2_\gamma(S;V)$, (4.36)

the sequence $\big\{e^{2\gamma\cdot}\Phi\big(u_k(\cdot)\big)\big\}_{k=1}^{+\infty}$ is bounded in $L^1(S)$. (4.37)

Now let us find estimates of u_k', $k \in \mathbb{N}$. For arbitrary fixed $k \in \mathbb{N}$ and almost every $t \in S$ we multiply equality (4.27) by $e^{2\gamma t}u_k'(t)$ and integrate the resulting equality from σ_1 to σ_2, where $\sigma_1, \sigma_2 \in S$ are arbitrary numbers, $\sigma_1 < \sigma_2$. From this we obtain

$$
\begin{aligned}
&\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k'(t)|^2\,dt + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(g_k(t), u_k'(t))\,dt \\
&= \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(f_k(t), u_k'(t))\,dt - \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big(B(t,u_k(t)), u_k'(t)\big)\,dt. \qquad (4.38)
\end{aligned}
$$

Since $g_k \in L^2(\sigma_1,\sigma_2;H)$, Lemma 4.1 implies that the function $\Phi_H\big(u_k(\cdot)\big)$ is absolutely continuous on $[\sigma_1,\sigma_2]$ and

$$\frac{d}{dt}\Phi_H\big(u_k(t)\big) = (g_k(t), u_k'(t)) \quad \text{for a.e. } t \in (\sigma_1,\sigma_2). \tag{4.39}$$

Taking into account (4.39), we can rewrite the second term on the left side of (2.3) as follows

$$\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}(g_k(t), u_k'(t))\,dt = \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\frac{d}{dt}\Phi_H\big(u_k(t)\big)\,dt \\ = e^{2\gamma t}\Phi_H\big(u_k(t)\big)\Big|_{\sigma_1}^{\sigma_2} - 2\gamma\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\Phi_H\big(u_k(t)\big)\,dt. \tag{4.40}$$

By the Cauchy inequality and (3.2) we have

$$\Big|\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big(f_k(t), u_k'(t)\big)\,dt\Big| \le \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|f_k(t)||u_k'(t)|\,dt \\ \le \frac{1}{4}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k'(t)|^2\,dt + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|f_k(t)|^2\,dt, \tag{4.41}$$

$$\Big|\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big(B(t,u_k(t)), u_k'(t)\big)\,dt\Big| \le \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\big|B(t,u_k(t))\big||u_k'(t)|\,dt \\ \le L\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)||u_k'(t)|\,dt \\ \le L^2\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2 dt + \frac{1}{4}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k'(t)|^2\,dt. \tag{4.42}$$

From (2.3), taking into account (4.40), (4.41), (4.42), we obtain

$$\frac{1}{2}\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k'(t)|^2\,dt + e^{2\gamma t}\Phi_H\big(u_k(t)\big)\Big|_{\sigma_1}^{\sigma_2} \\ \le L^2\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2\,dt \\ + 2\gamma\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\Phi_H\big(u_k(t)\big)\,dt + \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|f_k(t)|^2\,dt. \tag{4.43}$$

By the definitions of u_k and f_k we pass to the limit in (4.43) when $\sigma_1 \to -\infty$. From obtained inequality, taking into account estimate (4.35), setting $\sigma_2 = \sigma \in S$, we have

$$e^{2\gamma\sigma}\Phi_H\big(u_k(\sigma)\big) + \int_{-\infty}^{\sigma} e^{2\gamma t}|u_k'(t)|^2\,dt \le C_4\int_{-\infty}^{\sigma} e^{2\gamma t}|f_k(t)|^2\,dt, \tag{4.44}$$

where C_4 is a positive constant depending on K_1, K_2, L and γ only.

According to the definitions of the functional Φ_H and the function f_k, and condition $(\mathcal{A}_4)$ (recall that $u_k(t) \in V$ for a.e. $t \in S$), from (4.44) we obtain

$$e^{2\gamma\sigma}\|u_k(\sigma)\|^2 + \int_{-\infty}^{\sigma} e^{2\gamma t}|u_k'(t)|^2\,dt \le C_5 \int_{-\infty}^{\sigma} e^{2\gamma t}|f(t)|^2\,dt, \tag{4.45}$$

where $C_5 > 0$ is a constant depending on K_1, K_2, L, and γ only.

Estimate (4.45) imply that

$$\text{the sequence } \{u_k\}_{k=1}^{+\infty} \text{ is bounded in } L^\infty_\gamma(S;V), \tag{4.46}$$

$$\text{the sequence } \{u_k'\}_{k=1}^{+\infty} \text{ is bounded in } L^2_\gamma(S;H). \tag{4.47}$$

Let us show that

$$\text{the sequence } \{g_k\}_{k=1}^{+\infty} \text{ is bounded in } L^2_\gamma(S;H). \tag{4.48}$$

Indeed, using (3.2) and (4.35) we have

$$\int_{\sigma_1}^{\sigma_2} e^{2\gamma t}\Big|B(t,u_k(t))\Big|^2 dt \le L^2 \int_{\sigma_1}^{\sigma_2} e^{2\gamma t}|u_k(t)|^2\,dt \le C_6, \tag{4.49}$$

where $C_6 > 0$ is a constant independing on $k \in \mathbb{N}, \sigma_1, \sigma_2 \in S$.

Therefore, from (4.27), (4.47), (4.49), $(\mathcal{F})$ and the definition of f_k we obtain (4.48)

Step 4 (*passing to the limit*). Since V and H are Hilbert spaces, and V embeds in H by compact injection, from (4.36), (4.46), (4.47), (4.48) and Lemma 2.4 we have that there exist functions

$$u \in L^\infty_\gamma(S;V) \cap L^2_\gamma(S;V) \cap H^1_\gamma(S;H), \quad g \in L^2_\gamma(S;H)$$

and a subsequence of the sequence $\{u_k, g_k\}_{k=1}^{+\infty}$ (still denoted by $\{u_k, g_k\}_{k=1}^{+\infty}$) such that

$$e^{\gamma\cdot}u_k(\cdot) \underset{k\to\infty}{\longrightarrow} e^{\gamma\cdot}u(\cdot) \quad *\text{-weakly in } L^\infty(S;V), \tag{4.50}$$

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{weakly in } L^2_\gamma(S;V) \text{ and weakly in } H^1_\gamma(S;H), \tag{4.51}$$

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{in } C(S;H), \tag{4.52}$$

$$g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in } L^2_\gamma(S;H). \tag{4.53}$$

Note that (4.51) and (4.53) imply

$$u_k \underset{k\to\infty}{\longrightarrow} u, \quad u_k' \underset{k\to\infty}{\longrightarrow} u', \quad g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in } \quad L^2_{\text{loc}}(S;H). \tag{4.54}$$

Using (3.1) and (4.52), for each $\sigma < 0$ we obtain

$$\int_{\sigma}^{0} \big|B(t,u_k(t)) - B(t,u(t))\big|^2 dt \le L^2 \int_{\sigma}^{0} |u_k(t)-u(t)|^2 dt \underset{k\to\infty}{\longrightarrow} 0. \tag{4.55}$$

Thus, we obtain

$$B(\cdot, u_k(\cdot)) \underset{k\to\infty}{\longrightarrow} B(\cdot, u(\cdot)) \quad \text{strongly in} \quad L^2_{\mathrm{loc}}(S;H). \tag{4.56}$$

Let $v \in H, \varphi \in D(-\infty, 0)$ be arbitrary. For a.e. $t \in S$ we multiply equality (4.27) by v, and then we multiply the obtained equality by φ and integrate in t on S. As a result, we obtain the equality

$$\int_S (u_k'(t), v\varphi(t))\,dt + \int_S (g_k(t), v\varphi(t))\,dt + \int_S \big(B(t, u_k(t)), v\varphi(t)\big)\,dt$$
$$= \int_S (f_k(t), v\varphi(t))\,dt, \quad k \in \mathbb{N}. \tag{4.57}$$

We pass to the limit in (4.57) as $k \to \infty$, taking into account (4.54), (4.56) and convergence of $\{f_k\}$ to f in $L^2_{\mathrm{loc}}(S;H)$. As a result, since $v \in H, \varphi \in D(-\infty, 0)$ are arbitrary, for a.e. $t \in S$ we obtain the equality

$$u'(t) + g(t) + B(t, u(t)) = f(t) \quad \text{in } H.$$

Step 5 (*completion of proof*). In order to complete the proof of the theorem it remains only to show that $u(t) \in D(\partial\Phi)$ and $g(t) \in \partial\Phi\big(u(t)\big)$ for a.e. $t \in S$.

Let $k \in \mathbb{N}$ be an arbitrary number. Since $u_k(t) \in D(\partial\Phi_H)$ and $g_k(t) \in \partial\Phi_H\big(u_k(t)\big)$ for every $t \in S \setminus \widetilde{S}_k$, where $\widetilde{S}_k \subset S$ is a set of measure zero, applying the monotonicity of the subdifferential $\partial\Phi_H$, we obtain that for every $t \in S \setminus \widetilde{S}_k$ the following equality holds

$$(g_k(t) - v^*, u_k(t) - v) \geq 0, \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{4.58}$$

Let $\sigma \in S$, $h > 0$ be arbitrary numbers. We integrate (4.58) on $(\sigma - h; \sigma)$:

$$\int_{\sigma-h}^{\sigma} (g_k(t) - v^*, u_k(t) - v)\,dt \geq 0, \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{4.59}$$

Now according to (4.52) and (4.53) we pass to the limit in (4.59) as $k \to \infty$. As a result we obtain

$$\int_{\sigma-h}^{\sigma} (g(t) - v^*, u(t) - v)\,dt \geq 0, \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{4.60}$$

The monograph [27, Theorem 2, p. 192] and (4.60) imply that for every $[v, v^*] \in \partial\Phi_H$ there exists a set $R_{[v,v_*]} \subset S$ of measure zero such that for all $\sigma \in S \setminus R_{[v,v_*]}$ we have

$$0 \leq \lim_{h\to+0} \frac{1}{h} \int_{\sigma-h}^{\sigma} \big(g(t) - v^*, u(t) - v\big)\,dt = \big(g(\sigma) - v^*, u(\sigma) - v\big). \tag{4.61}$$

Let us show that there exists a set of measure zero $R \subset S$ such that

$$\forall \sigma \in S \setminus R: \qquad \big(g(\sigma) - v^*, u(\sigma) - v\big) \geq 0, \quad \forall [v, v^*] \in \partial\Phi_H. \tag{4.62}$$

Since V and H are separable spaces, there exists a countable set $F \subset \partial\Phi_H \subset V \times H$ which is dense in $\partial\Phi_H$. Let us denote $R := \bigcup_{[v,v^*]\in F} R_{[v,v_*]}$. Since the set F is countable, and any countable union of sets of measure zero is a set of measure zero, R is a set of measure zero. Therefore, for any $\sigma \in S \setminus R$ inequality $\big(g(\sigma) - v^*, u(\sigma) - v\big) \geq 0$ holds for every $[v, v^*] \in F$. Let $[\widehat{v}, \widehat{v}^*]$ be an arbitrary element from $\partial\Phi_H$. Then from the density F in $\partial\Phi_H$ we have the existence of a sequence $\{[v_l, v_l^*]\}_{l=1}^{\infty}$ such that $v_l \to \widehat{v}$ in V, $v_l^* \to \widehat{v}^*$ in H and

$$\forall \sigma \in S \setminus R: \qquad (g(\sigma) - v_l^*, u(\sigma) - v_l) \geq 0 \quad \forall\, l \in \mathbb{N}. \tag{4.63}$$

Thus, passing to the limit in this equality as $l \to \infty$, we get $(g(\sigma) - \widehat{v}^*, u(\sigma) - \widehat{v}) \geq 0$ $\forall \sigma \in S \setminus R$. Therefore, inequality (4.62) holds. From this, according to maximal monotonicity of $\partial\Phi_H$, we obtain that $[u(t), g(t)] \in \partial\Phi_H$ for a.e. $t \in S$.

Estimate (3.7) of the solution of the problem $\mathbf{P}(\Phi, B, f, \gamma)$ follows directly from (4.35), (4.45), (4.50), (4.51) and (4.52), Lemma 2.2, Fatou's Lemma and the fact that Φ_H is lower semicontinuous in H.

From (4.35) we have

$$e^{2\gamma\sigma} |u(\sigma)|^2 \leq C_3 \int_{-\infty}^{\sigma} e^{2\gamma t} |f(t)|^2 \, dt.$$

This inequality and condition $(\mathcal{F})$ imply that u satisfies condition (3.4). Thus Theorem 3.2 is proved. □

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Received 22.04.2019

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **27**, Issue **1**, June **2019**, pp. 21–55, DOI 10.15421/141902

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

A NEW MATHEMATICAL MODEL OF DYNAMIC PROCESSES IN DIRECT CURRENT TRACTION POWER SUPPLY SYSTEM

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Abstract. A new autonomous 4D nonlinear model with two nonlinearities describing the dynamics of change of voltage and current in the contact railway electric network is offered. This model is a connection of two 2D oscillatory circuits for current and voltage in the contact electric network. In the found system for the defined values of parameters an existence of limit cycles is proved. By introduction of new variables this system can be reduced to 5D system only with one quadratic nonlinearity. The constructed model may be used for the control by voltage stability in a direct current power supply system.

Key words: Time series, polynomial models, chaos, four-dimensional chaotic system, time-delayed embedding, multidimensional recurrence quantification analysis, voltage stabilization.

2010 Mathematics Subject Classification: 34C28, 37G35, 37N35.

Communicated by Prof. V. O. Kapustyan

1. Introduction

The modern stage of functioning of railways is conditioned by the necessity of providing competitiveness with other types of transport. The decision of this problem supposes introduction of high-speed passenger transport as well as heavy freight trains. For this purpose on the railways measures on the increase of speed of movement are carried out, new electric locomotives of large power are created, different ways of strengthening traction power supply are applied (see, for example, [1]).

The traction power system of the electrified section of the railway (TPS) is a set of territorially dispersed and operating electric power stations. This

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system can include traction substations, sectionalizing stations, parallel connection points, contact network devices and power transmission lines between them, united by a common purpose and intended for processing and transmission of qualitative electric energy to electric rolling stock (ERS). The peculiarities of electric power transmission through the traction network is the change in the position of ERS and the change in their operating modes, the restrictions imposed by trains on each other depending on their relative location, as well as the restrictions associated with the technology of the transportation process as a whole. One of the main indicators of the quality of transmitted electrical energy to ERS is the level of voltage on the current collectors of electric locomotives, and the nature of the factors affecting on this level, which are nonlinear and non-stationary.

Ensuring the stable and reliable operation of any technical system is an important task that requires its solution. Voltage resilience is the ability of the power system to maintain stable and acceptable voltage levels on all bus systems (BS) in both normal and post-emergency and repair modes. The criterion for the stability of the power system in terms of voltage is that, in the current mode, the value of reactive power on the same BS should increase at each BS with increasing voltage. Dynamic voltage stability is associated with the evaluation and support of the voltage within $1 - 2$ seconds immediately after a large disturbance. Static voltage stability belongs to the form of stability, determined mainly by the static characteristics of the load and network parameters.

The existent system of traction electric supply of direct current not always is able to provide the transmission electric power of necessary capacity for speed trains. In this connection there can be the following limitations: lowering of voltage in the contact network is below than normal level 2.7 kV for ordinary motion (below than level 2.9 kV for high speed motion) and heatings of wires of contact network, that will promote in the loss of their mechanical durability. Lowering of voltage in the receiver of current diminishes the speed of movement of trains. Thus, the saving of level of consumable power results in the increase of current of electric locomotive and loss of electric power in the contact network.

Since the dynamical state of any technical system is described by a system of differential equations, the study of the problem of the stability of its motion reduces to the study of the stability of solutions of differential equations. To calculate the static stability of the power system, it is necessary to compile a system of differential equations of transient processes, linearize these equations, and obtain the characteristic equation of a system of linearized equations. Together, these equations constitute a mathematical model of the energy system [2], as a result of which solutions it is possible to obtain algebraic and other stability criteria for the system under consideration.

In this connection, there is an urgent need to consider the problem of determining the stability of TPS as an initially nonlinear problem. At present, an approach based on the analysis of signals produced by the system is widely used to study the properties of complex systems, including experimental studies. This is very important in cases when it is practically impossible to describe the process under study mathematically, but we have at our disposal certain observable quantities, which allow to build a wanted model [3].

Now there is a great number of methods of diminishment of losses of electric power in the contact network [2], [3]. In the present paper the diminishment of losses of electric power and increase of power efficiency of the systems of electric supply will be attained due to introduction of new dynamic models describing the behavior of current and voltage in these networks. Due to these models new methods of calculation of parameters of contact electric networks can be offered. Such methods allow to apply new organizational measures resulting in diminishment of losses of electric energy in contact networks.

Let

$$x_0 = x(t_0), x_1 = x(t_1), ..., x_n = x(t_n) \tag{1.1}$$

be a finite sequence of numerical values of some scalar dynamical variable $x(t)$ measured with the constant time step Δt in the moments $t_i = t_0 + i\Delta t$; $x_i = x(t_i)$; $i = 0, 1, ..., n$. Sequence (1.1) is called a time series [4] – [10].

In future the time series (1.1) will characterize a voltage (or current) in the contact network measured through the equal intervals of time.

A common practice in chaotic time series analysis has been to reconstruct the phase space by utilizing the delay-coordinate embedding technique, and then to compute the dynamical invariant magnitudes such as unstable periodic orbits, a fractal dimension of the underlying chaotic set, and its Lyapunov spectrum. As a large body of literature exists on applying of the technique of the time series to study chaotic attractors [10] – [15], a relatively unexplored issue is its applicability to dynamical systems of differential equations depending on parameters. Our focus will be concentrated on the analysis of influence of parameters of found dynamic system on the behavior of its solutions. These parameters are determined by the structure of the time series (1.1) and choice of approximating functions in right sides of the got system of differential equations.

To create a model by measuring the variables characterizing any dynamic process, it is necessary to solve the following four main problems.

It is known that any dynamic process depends on many variables. Most of these variables are functions of some small number of independent variables. Identifying these independent variables leads to the first problem.

Problem 1. Determine the dimension of phase space in which the explored process takes place.

Usually, a continuous dynamic process is described using a system of differential equations. It is important to specify the class of those functions that form the right-hand sides of the differential equations. In applied modeling problems, the right-hand sides of differential equations are given by polynomials. In this connection, the second problem arises.

Problem 2. Establish degrees of monomials and their composition that form the polynomial right-hand sides of the differential equations.

Problem 3. After the structure of the differential equations describing the dynamic process is established, it is necessary to determine the numerical value of coefficients of these equations.

Problem 4. Using the specifics of the problem, to establish analytical formulas of the coefficients of the found system of differential equations as functions of characteristics of individual elements of the direct current traction power supply system (it can be capacitors, resistances, inductances, and so on).

We must say that of all these problems, the fourth is the most difficult. Indeed, for such a complex system as the direct current traction power supply system we can not specify all electrical elements included in its composition. Therefore, in this paper we confine ourselves to solving only the first three problems.

Note that in great numbers papers on the problem of minimization of losses of electric power in contact networks, analytical simulation techniques are used. In these papers mathematical models are linear or, at the best, linearized. Their use gives positive results. However, losses of electric power in the contact network may be diminished only on 20 – 30 percents [1].

Therefore, construction of mathematical models allowing suggest practical methods to decrease the losses of electric power in the contact network is the primary purpose of the present paper. These models are constructed on the basis of the real information taken from the electric system of locomotive. As experimental information the results of measuring of currents and voltages taken on the segment of Nyzhne-Dneprovsk Knot – Pyatykhatky of Prydneprovskaya Railway (Ukraine) are used.

It should be said that we do not know publications in which the direct current traction power supply system was modeled only by measuring currents and voltages (see [16, 17]). The fact is that a large number of electrical elements forming such system practically exclude the possibility of constructing a detailed model through the combination of the equations of its individual elements. (Such model would represent a set of several hundred different equations.)

Therefore, the main achievements of this work are:

– creation of a new model of direct current traction power supply system, which describes this system not as a set of interconnected elements, but as a single dynamic element whose behavior is determined only by changes in voltages and currents flowing through it;

– development of a universal methodology for modeling of direct current traction power supply systems suitable for any segments of contact networks located anywhere in the world.

2. Embedding Method for Chaotic Time Series Analysis

The material of this section is well known (see [4]). The results of section are placed in the present article only for the convenience of readers.

Let sequence (1.1) be the time series. In principle, the measured time series comes from an underlying dynamical system that evolves the state variable in time according to a set of deterministic rules, which are generally represented by a set of differential equations, with or without the influence of noise. Mathematically, any such set of differential equations can be easily converted to a set of first-order, autonomous equations. The dynamical variables from all the first-order equations constitute the phase space, and the number of such variables is the dimension of the phase space, which we denote by $\mathbf{M}$. The phase space dimension can in general be quite large (in some cases it may be infinite) [10, 11, 16, 18].

However, it often occurs that the asymptotic evolution of the system lives on a dynamical invariant set of a finite dimension. The assumption here is that the details of the system equations in the phase space and of the asymptotic invariant set that determines what can be observed through experimental probes, are unknown. The task is to estimate, based solely on one or few time series, practically useful statistical quantities characterizing the invariant set, such as its dimension, its dynamical skeleton, and its degree of sensitivity on initial conditions. The delay-coordinate embedding technique established by Takens [9], in particular, his famous embedding theorem guarantees that a topological equivalence of the phase space of the intrinsic unknown dynamical system can be reconstructed from the time series, based on which characteristics of the dynamical invariant set can be estimated.

Let

$$\dot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}(t)), \mathbf{y} \in \mathbf{M} \subset \mathbb{R}^p \tag{2.1}$$

be the autonomous p-dimensional system of ordinary differential equations in the phase space $\mathbf{M}$.

We will consider that system (2.1) satisfies in the phase space $\mathbf{M}$ (an open region in $\mathbb{R}^p$) to the conditions of the known Cauchy Theorem about existence and uniqueness of solutions. Then for any initial condition $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbf{M}$ it is possible uniquely to define the solution $\mathbf{y}(t)$ systems (2.1) on the formula $\mathbf{y}(t) = \mathbf{W}^t(\mathbf{y}_0)$, where $\mathbf{W}^t$ is an evolution operator. (A domain $\mathbf{G} \subset \mathbf{M}$ of the phase space $\mathbf{M}$ under action of the evolution operator passes, generally speaking, in another domain $\mathbf{G}_t = \mathbf{W}^t(\mathbf{G}) \subset \mathbf{M}$. If $\mathbf{G}_t = \mathbf{W}^t(\mathbf{G}) = \mathbf{G}$, then the domain $\mathbf{G}$ is called an invariant subset of the phase space $\mathbf{M}$ with respect to the action of the evolution operator $\mathbf{W}^t$.)

The compact invariant with respect to the evolution operator set $\mathbf{H} \subset \mathbf{M}$ is called attracting if there exists an open set $\mathbf{U} \subset \mathbf{M}$ containing $\mathbf{H}$ such that for almost all $\mathbf{y} \in \mathbf{U}$ $\lim\limits_{t\to\infty} \mathbf{W}^t(\mathbf{y}) \in \mathbf{H}$. The indecomposable on two compact invariant subsets attracting set $\mathbf{H}$ is called an attractor.

It is known [12] that it is possible to get the attractor satisfactory image of a small dimension, if instead of the phase vector $\mathbf{y}(t)$ to use m-dimensional vectors derived from the time series (1.1) on the following formula:

$$\mathbf{x}_i = \begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+m-1} \end{pmatrix}; \quad i \to 0, 1l, 2l, ..., il, ..., \tag{2.2}$$

where l is a positive integer.

Consider the m-dimensional autonomous dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{Q}(\mathbf{x}(t)), \mathbf{x} \subset \mathbb{R}^m, \tag{2.3}$$

for which the following conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}(t_0 + \tau) = \mathbf{x}_1, ...,\ \mathbf{x}(t_i) = \mathbf{x}(t_0 + i\tau) = \mathbf{x}_i$$

are fulfilled. The magnitude $\mathbf{x}_i$ depends on $\mathbf{x}_0$ and τ, but it does not depend on t_0. We will especially emphasize that the map $\mathbf{Q} : \mathbb{R}^m \to \mathbb{R}^m$, determining the right side of systems (2.3), it is not known. In addition, it is clear that the role of the number l in (2.2) plays the number τ. The magnitude τ is called a delay parameter of the time series (1.1).

Introduce the evolution operator $\mathbf{P}^t : \mathbb{R}^m \to \mathbb{R}^m$ of system (2.3). For any vector $\mathbf{x} \in \mathbb{R}^m$ the action of this operator in a coordinate form looks like:

$$\mathbf{P}^t(\mathbf{x}) = (P_0^t(\mathbf{x}), P_1^t(\mathbf{x}), ..., P_{m-1}^t(\mathbf{x}))^T.$$

Let $t = \tau$. Consider the sequence of real numbers

$$h_k = P_0^\tau(\mathbf{x}_k), h_{k+1} = P_1^\tau(\mathbf{x}_k), h_{k+2} = P_2^\tau(\mathbf{x}_k), ..., h_{k+m-1} = P_{m-1}^\tau(\mathbf{x}_k). \tag{2.4}$$

Introduce the new vector $\mathbf{z}_k$ under the formula: $\mathbf{z}_k = (h_k, h_{k+1}, ..., h_{k+m-1})^T$. Then there must be an operator $\mathbf{\Delta} : \mathbb{R}^m \to \mathbb{R}^m$ depending only on $\mathbf{Q}$ and τ such that $\mathbf{z} = \mathbf{\Delta}(\mathbf{x})$, where $\mathbf{x} = (x_i, x_{i+1}, ..., x_{i+m-1})^T$ is one of vectors (2.2).

Theorem 2.1. *[9] Let d be a dimension of the attractor $\mathbf{\Sigma}$ generated by system (2.3). Then for almost all $\tau > 0$ and $m \geq 2d+1$ the mapping $\mathbf{\Delta}$ will be continuous and one-to-one.*

Theorem 2.1 means that if in the space $\mathbb{R}^m$ to select the set $\mathbf{H}_k$ such that $\forall \mathbf{x}_k \in \mathbf{H}_k$ we have $\mathbf{\Delta}(\mathbf{x}_k) \in \mathbf{H}_k$, then on this set the map $\mathbf{\Delta}$ is invertible and $\forall k$ $\mathbf{x}_k = \mathbf{\Delta}^{-1}(\mathbf{z}_k)$.

By $\mathbb{N}$ denote the set of natural numbers.

Theorem 2.2. *[19] Let $i_1, i_2, ..., i_l, ...$ be an infinite increasing sequence of positive integers. If system (2.3) is a dissipative then for any compact open subset $\mathbf{\Phi} \subset \mathbb{R}^m$, any $\tau > 0$, and almost all $\mathbf{x} \in \mathbf{\Phi}$ the inclusion $(\mathbf{P}^\tau)^{i_l}(\mathbf{x}) = \underbrace{\mathbf{P}^\tau(\mathbf{P}^\tau(...(\mathbf{P}^\tau(\mathbf{x}))...))}_{i_l} \in$ $\mathbf{\Phi}$; $l \in \mathbb{N}$ takes place.*

Thus, Theorem 2.2 (which is sometimes called the Poincare recurrence theorem) asserts that in the phase space of the dissipative system any trajectory beginning from the almost liked point A of this space in some finite time (even very large) will pass as much as close to A.

Theorems 2.1 and 2.2 allowed to create the necessary research instrument which is used presently in the theory of the dynamic systems. Indeed, as the time series (1.1) has only a finite number of terms and, consequently, it is bounded, there are no justified arguments in order to assert that at the further measurements we will derive very large values of terms of this series. Further, the time series (1.1) describes the behavior of some phase variable of the explored dynamic process. If we assume that the number of such phase variables is finite, then it is possible to consider that there exists the evolution operator, which controls by the behavior of this dynamic system in some finite-dimensional space. In addition, most systems describing the dynamics of one or another processes in our world are dissipative. Thus, the use of Theorems 2.1 and 2.2 for description of dynamics of the dissipative finite-dimensional systems becomes more than justified.

Eckmann [6] have introduced tools which visualize the recurrence of states $\mathbf{x}_i$ in the phase space. Usually, the phase state does not have a dimension (it is more than two or three) which allows it to be pictured. Higher dimensional phase spaces can only be visualized by projection into the two or three dimensional subspaces [10], [12].

Now by $\mathbf{x}(i)$ denote the point $\mathbf{x}_i = (x_i, x_{i+1}, ..., x_{i+m-1})^T$, which is built from the elements of the time series (1.1) describing the change of some scalar variable (or some coordinate of the vector variable, if a phase trajectory in m-dimensional space is considered); $i \to 0, 1l, 2l, ..., il$. If $il + m - 1 > n$, then number i must be replaced by the number $k \to kl = il - [n/l]l$, where $[n/l]$ is an integer part of the number n/l. We will consider that $l = \tau$.

Introduce in the first quadrant of the cartesian system of coordinates the graphic square matrix $T \in \mathbb{R}^{(n+1)\times(n+1)}$, which is built on the following algorithm: if point $\mathbf{x}(i)$ is close enough to the point $\mathbf{x}(j)$ (the concept of "closeness" will be defined below) then such points are called recurrence, and in the matrix T a black point with coordinates (i, j) are put. If point $\mathbf{x}(i)$ is not near to the point $\mathbf{x}(j)$, then in the matrix T no marks is done. The matrix T is called a recurrence plot of time series (1.1) [6], [10].

Let

$$\mathbf{R}_{ij} = \mathbf{\Theta}(\epsilon_i - \|\mathbf{x}_i - \mathbf{x}_j\|),\ \mathbf{x}_i, \mathbf{x}_j \in \mathbb{R}^m;\ i, j = 0, 1, ..., n,$$

be a real function accepting only two values: 0 and 1. (Here we have $\mathbf{\Theta}(\xi) = 1$, if $\xi \geq 0$ and $\mathbf{\Theta}(\xi) = 0$, if $\xi < 0$: it is the Heaviside function; $\|\mathbf{v}\| = \sqrt{v_1^2 + ... + v_m^2}$ is the Euclidian norm of the vector $\mathbf{v} \in \mathbb{R}^m$; ϵ_i is a radius of ball with a center in the point $\mathbf{x}_i$.)

In the future, it is possible to be restricted to the situation, when $\forall i, j\ \epsilon_i = \epsilon_j = \epsilon$. In this case positive number ϵ is called a recurrence threshold and we have symmetry of the recurrence plot with respect to the diagonal of the first quadrant. Indeed, if point $\mathbf{x}_i$ is near to the point $\mathbf{x}_j$, then the reverse statement must be right: the point $\mathbf{x}_j$ is near to the point $\mathbf{x}_i$.

In the present paper we want to apply the instruments of the recurrence analysis for research of periodic trajectories in the dynamic systems described by nD autonomous systems of differential equations. In order that such research was correct it is necessary to provide the boundedness of solutions of the explored systems.

3. Mathematical Statement of Problem and Its Discussion

We will assume that we can measure the voltage and current, and also if it is possible other dynamic characteristics of contact electric network. We also suppose that among these characteristics can be derivatives with respect to t from the voltage and current. (If the derivatives can not be measured, it is assumed that there exist smooth enough approximations of these derivatives.)

A choice of equations of model of describing the dynamics of one or another processes is a difficult task. The experiments show that the most logical approach describing dynamics of electrical engineering models is based on the use of the known physical laws. In particular there can be the conservation of energy laws.

By $U(t)$ and $I(t)$ denote respectively the voltage and current in contact electric network. The following laws are most known: the electric energy is accumulated in a capacitor according to the law $E_C = k_C U^2$; the electric energy is accumulated in an inductor according to the law $E_L = k_L I^2$; the electric energy is transformed into heat energy on a resistor according to the law $E_R = k_R UI$. (Here k_C, k_L, and k_R are constants.) In addition, a speed of change of energy $\dot{E}_C = 2k_C U\dot{U}$, $\dot{E}_L = 2k_L I\dot{I}$ or $\dot{E}_R = k_R(\dot{U}I + U\dot{I})$, and magnitudes $U, \dot{U}, I, \dot{I}$ also influences on the dynamics of electric network.

Thus, the vast class of electric networks can be described by quadratic differential equations depending on the linear $U, \dot{U}, I, \dot{I}$, and quadratic U^2, $U\dot{U}$, I^2, $I\dot{I}, UI$, $\dot{U}I, U\dot{I}$ terms.

We consider that there is n characteristics (measurements): $z_1(t_i), ..., z_n(t_i)$, $i = 1, 2, ..., N$. In addition, we also suppose that these measurements are noisy. Thus, we have multivariate time series

$$z_1(t_i) = x_1(t_i) + \theta_1(t_i), ..., \quad z_n(t_i) = x_n(t_i) + \theta_n(t_i), \tag{3.1}$$

which defined for $\forall t_i \in (t_1, t_N)$. Here $\forall i = 1, 2, ..., N$, we have $t_i = i\Delta t$ and $\Delta t = (t_N - t_1)/N$. In addition, we suppose that $\theta_1(t_i), ..., \theta_n(t_i)$ are Gaussian (white) noises, unable by definition to produce statistically systematical errors [9], [13], [14], [16], [18].

Finally, we assume that $x_1(t_i), ..., x_n(t_i)$ is a discrete approximation of some curve $\mathbf{x}(t) = (x_1(t), ..., x_n(t))^T \in \mathbb{R}^n$ [13], [14]. In the turn, it is assumed that the curve $\mathbf{x}(t)$ is a solution of some quadratic differential equations system. The necessity of such description is dictated by the considerations resulted higher.

Let $(c_1, ..., c_n)^T$ and $A = (a_{ij}), B_1, ..., B_n \in \mathbb{R}^{n\times n}$ be a real vector and real matrices, and let the matrices $B_1, ..., B_n$ be symmetrical.

Principal problem. *Construct the quadratic system of differential equations*

$$\begin{cases} \dot{x}_1(t) = \sum\limits_{j=1}^{n} a_{1j}x_j(t) + \mathbf{x}^T(t)B_1\mathbf{x}(t) + c_1 \equiv f_1(\mathbf{x}(t)), \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ \dot{x}_n(t) = \sum\limits_{j=1}^{n} a_{nj}x_j(t) + \mathbf{x}^T(t)B_n\mathbf{x}(t) + c_n \equiv f_n(\mathbf{x}(t)) \end{cases} \tag{3.2}$$

such that there exists bounded solution $\mathbf{x}(t)$ *(*$\lim\limits_{t\to\infty} \|\mathbf{x}(t)\| < \infty$*) of this system, which approximates time-variate series (3.1) with given accuracy in the set points* $t_1, ..., t_N$ *at the fixed choice of the vector of initial values* $\mathbf{x}(0) = (x_{10}, ..., x_{n0})^T$*.*

Further, we use the procedure for determining unknown quadratic right sides of the system of differential equations (3.2), which was suggested in [12] – [15]. This procedure is based on the least squares method and the fact that we know sufficient precision the components of $\mathbf{x}(t)$ and its derivative $\dot{\mathbf{x}}(t)$.

We will use the following designations: $\mathbf{x}(t_i) = (x_1(t_i), x_2(t_i), ..., x_n(t_i))^T = (x_{1i}, x_{2i}, ..., x_{ni})^T$, $\dot{\mathbf{x}}(t_i) = (\dot{x}_1(t_i), \dot{x}_2(t_i), ..., \dot{x}_n(t_i))^T = (\dot{x}_{1i}, \dot{x}_{2i}, ..., \dot{x}_{ni})^T$, where $\dot{x}_{ki} = (x_{k,i+1} - x_{ki})/\Delta t$; $k = 1, ..., n$; $i = 0, 1, ..., N$.

Introduce the matrix of unknown coefficients of system (3.2):

$$Y = \begin{pmatrix} c_1 & a_{11} & \cdots & a_{1n} & b_{11}^{(1)} & \cdots & b_{nn}^{(1)} & 2b_{12}^{(1)} & \cdots & 2b_{n-1,n}^{(1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ c_n & a_{n1} & \cdots & a_{nn} & b_{11}^{(n)} & \cdots & b_{nn}^{(n)} & 2b_{12}^{(n)} & \cdots & 2b_{n-1,n}^{(n)} \end{pmatrix} \in \mathbb{R}^{n\times m},$$

where $m = 1 + 2n + n(n-1)/2 = (n+1)(n+2)/2$.

Introduce also $(N \times m)$-matrix

$$X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{n1} & x_{11}^2 & \cdots & x_{n1}^2 & x_{11}x_{21} & \cdots & x_{n-1,1}x_{n,1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{1N} & \cdots & x_{nN} & x_{1N}^2 & \cdots & x_{nN}^2 & x_{1N}x_{2N} & \cdots & x_{n-1,N}x_{n,N} \end{pmatrix}$$

and $(N \times n)$-matrix

$$\dot{X}_D = \begin{pmatrix} \dot{x}_{11} & \cdots & \dot{x}_{n1} \\ \vdots & \cdots & \vdots \\ \dot{x}_{1N} & \cdots & \dot{x}_{nN} \end{pmatrix},$$

elements of which are known. Then by the least square method [12,20,21], we have $Y^T = (X^TX)^{-1}X^T\dot{X}_D$. Further, the following is said in work [13]. In view of the fact that number N may be chosen arbitrary large, a high precision reconstruction may be achieved. Thus, we can expect that the solution of reconstructed system will be near the purified solution $\mathbf{x}(t)$.

However, it should be said that one important circumstance, which can arise up at a reconstruction, remained outside the attention of the authors of article [14]. The point is that in [14] it is assumed that this interval (t_1, t_N) is finite. If the problem of long-term prediction is considered, it is necessary to assume that $t_N \to \infty$. In this case a reconstruction must be fulfilled so that system (3.2) had the bounded solutions [22]– [24].

Finally, we mark that the presence of all quadratic elements in the right side of system (3.2) do not always result in that the built model will adequately describe the explored process. Therefore, the choice of base of quadratic part of system (3.2) must be argued by the real information about the studied process, by which it is impossible to neglect.

4. Simulation of Dynamics of Current and Voltage in Contact Electric Network

We consider the following scheme of placing of traction substations on the segment Pyatykhatky – Nyzhne-Dneprovsk Knot [1]:

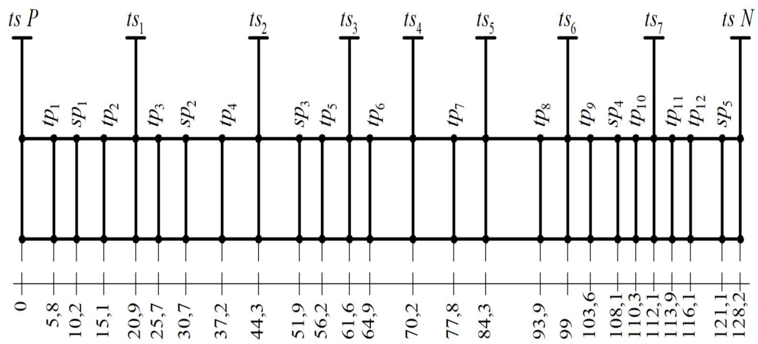


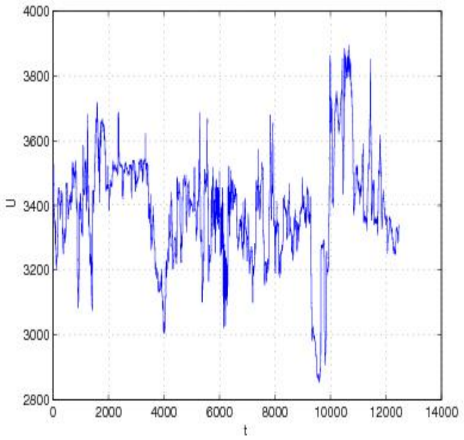
Fig. 4.1. The scheme of placing of traction substations

The length of this segment is 128.2 kM. Distances between the traction substations (in kM) are indicated on Fig. 4.1.

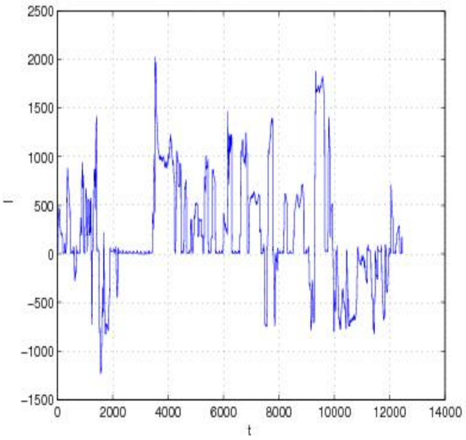
On Fig.4.1 the following designations are accepted: ts is a traction substation; tsP is the traction substation of Pyatykhatky; tsN is the traction substation of Nyzhne-Dneprovsk Knot; $tp_1, ..., tp_{12}$ are posts of parallel connection of contact suspension ways; $sp_1, ..., sp_5$ are sectioning posts.

Measurings of current and voltage will be realized by a measuring laboratory which moves on the segment together with a locomotive with a constant speed. Time of passing of all segment Pyatykhatky – Nyzhne-Dneprovsk Knot is 12470 sec.

On the segment Pyatykhatky – Nyzhne-Dneprovsk Knot the temporal dependences for current and voltage in the contact network were built. In all 12470 measurings with an interval of 1 second were done. The graphs of these dependences are represented on Fig. 4.2:



(a1)



(a2)

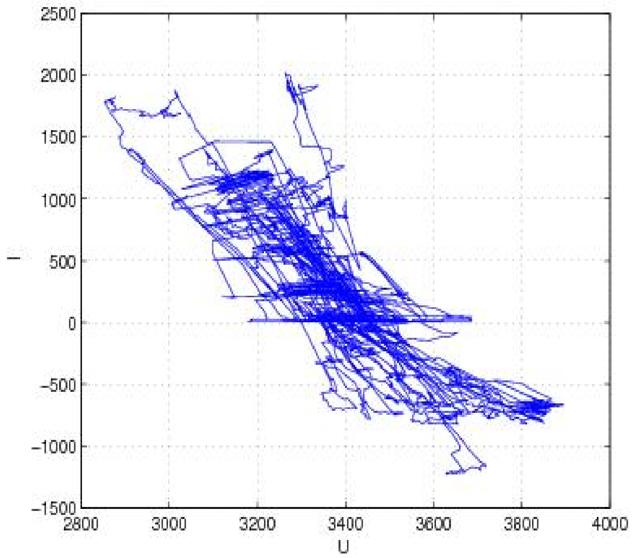


Fig. 4.2. The behavior of voltage $U(t)$(a1), current $I(t)$(a2), and $U-I$ characteristic of contact network (experimental data)

A standard procedure for modeling of the direct current traction power supply system consists of the following steps:

1. Construct a scheme replacing the direct current traction power supply system of the considered segment (no load; see Fig. 4.3).

2. Introduce in the model obtained at the first step, an element describing a motion of locomotive (it is a load; see Fig. 4.4).

3. Calculate using the Ohm and Kirchhoff laws, the voltage and current variations along in all the segment Pyatykhatky – Nyzhne-Dneprovsk Knot.

4. If the measured values of current and voltage significantly differ from

calculated in the step 3, then the topology of substitution scheme (see Fig. 4.4) and the characteristics of elements forming this scheme must be changed.

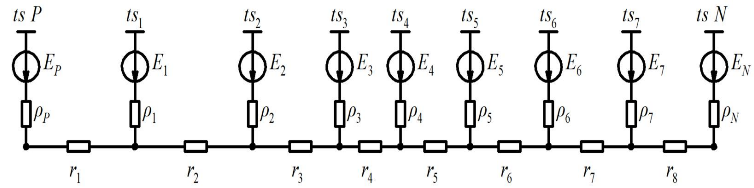


Fig. 4.3. The equivalent electric circuit for substitution of the segment Pyatykhatky – Nyzhne-Dneprovsk Knot (no load)

On Fig. 4.3 the following designations are accepted: $E_P, E_1, ..., E_N$ are voltages of idling of traction substations; $\rho_P, \rho_1, ..., \rho_N$ are internal resistances of traction substations; $r_1, r_2, ..., r_8$ are resistances between substations.

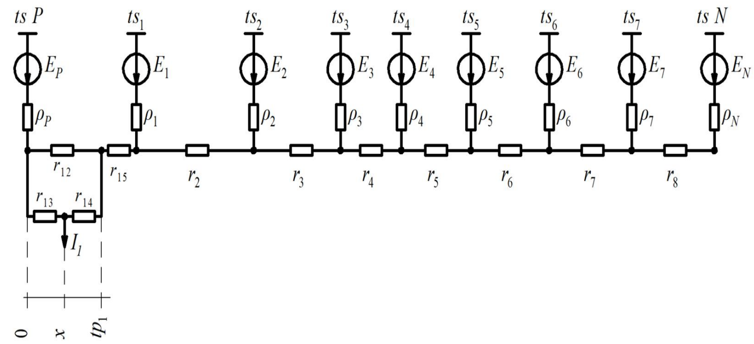


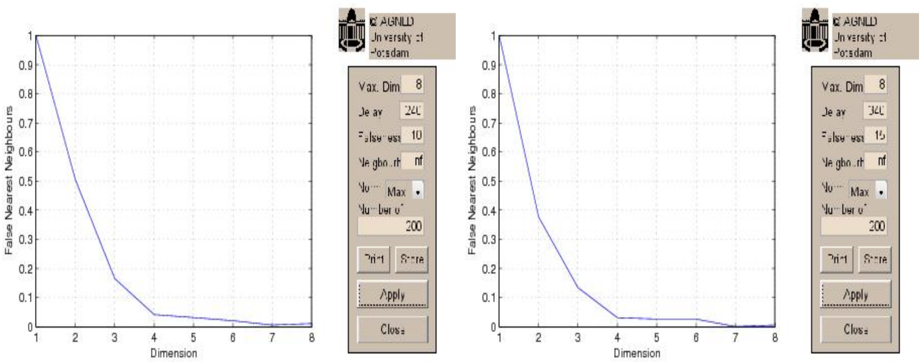
Fig. 4.4. The equivalent electric circuit for substitution of the segment Pyatykhatky – Nyzhne-Dneprovsk Knot (with load)

On Fig. 4.4 the following designations are accepted: x is a coordinate of locating an electric locomotive at a given time (in kM); r_{12} is the resistance of the traction network of the section of the first path between tsP and tp_1; r_{15} is the equivalent resistance of both paths section of the traction network between tp_1 and ts_1; r_{13} and r_{14} are the traction network resistances of the second path section between tsP and x, and x and tp_1, respectively. The values of the resistances r_{13} and r_{14} depend on the location of the train.

It is clear that such procedure leads to a very approximate model of the direct current traction power supply system (see, for example, [17]). Therefore, in this paper we propose another modeling method based on recurrence analysis.

A preliminary analysis of the obtained data shows that they are nonstationary. In this connection, the solution of problems of modeling and forecasting of nonstationary processes is of particular relevance. We point out that nonstationarity can manifest itself in the appearance of a deterministic or stochastic trend that varies in time with variance and covariance. There are two main purposes of analyzing time series: the determination of the nature time series and prediction (the prediction of future values of time series by present and past values). Both these goals require that the series model be identified and, more or less, formally described [25] – [28].

In order that to successfully fulfill the modeling of processes of represented on Fig. 4.2 it is necessary to verify the conditions of Theorems 2.1 and 2.2 (see Fig. 4.5):



(a1) (a2)

Fig. 4.5. The search optimal embedding dimension by means of false nearest neighbours

We take advantage of the least squares method. In accord to the done calculations an embedding dimension space n must be not less than 4. In future we will consider that $n = 4$. It should be noted here that dimension 5 can also be considered.

Thus, the dynamic system assumes existence of limit cycle of dimension $N = 1$ (in this case, we have $n = 4 > 2 \cdot N + 1 = 3$). Exactly bifurcations of limit cycles result in an appearance of chaotic dynamics.

On all following Fig. 4.6– 4.12 the voltage $U(t) = x(t)$ and current $I(t) = z(t)$ is measured in kilovolts (kV) and kiloamperes (kA).

1. The base of quadratic part of system (3.2) consists of two elements $\{xy, x^2\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.0193 - 0.0072x(t) + 0.0218y(t) - \epsilon_1 z(t) + 0.0057u(t) \\ \qquad -0.0039x(t)y(t) + 0.000422x^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.0294 - 0.0145x(t) - 0.8506y(t) - \epsilon_2 z(t) - 0.0095u(t) \\ \qquad +0.2380x(t)y(t) + 0.0017x^2(t). \end{cases} \tag{4.1}$$

2. The base of quadratic part of system (3.2) consists of two elements $\{zu, z^2\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = \epsilon_1 - 0.005634x(t) + 0.017369y(t) - 0.001177z(t) + 0.018429u(t) \\ \qquad -0.0225244z(t)u(t) + 0.00016z^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = \epsilon_2 - 0.003521x(t) - 0.083527y(t) - 0.002236z(t) - 0.057069u(t) \\ \qquad +0.0985z(t)u(t) - 0.00096z^2(t). \end{cases} \tag{4.2}$$

3. The base of quadratic part of system (3.2) consists of two elements $\{xz, xy + zu\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.0192 - 0.0056x(t) + 0.0083y(t) - 0.00044z(t) - 0.0396u(t) \\ \qquad -0.0004x(t)z(t) + 0.0139(x(t)u(t) + z(t)y(t)), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.0123 - 0.0035x(t) - 0.0336y(t) - 0.0006z(t) + 0.4143u(t) \\ \qquad -0.0006x(t)z(t) - 0.1279(x(t)u(t) + z(t)y(t)). \end{cases} \tag{4.3}$$

4. The base of quadratic part of system (3.2) consists of two elements $\{xz, xy + zu\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.019297079 - 0.0056x(t) + 0.0084y(t) - 0.0004z(t) - 0.0396u(t) \\ \qquad -0.0004x(t)z(t) + 0.0143(x(t)u(t) + z(t)y(t)), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.012284 - 0.0035x(t) - 0.03362y(t) - 0.0006z(t) + 0.4143u(t) \\ \qquad -0.0005x(t)z(t) - 0.1269(x(t)u(t) + z(t)y(t)). \end{cases} \tag{4.4}$$

5. The base of quadratic part of system (3.2) consists of two elements $\{xy, zu\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.0138 - 0.0040x(t) - 0.2526y(t) - 0.0008z(t) + 0.0162u(t) \\ \qquad +0.0791x(t)y(t) - 0.1782z(t)u(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.0095 - 0.0027x(t) + 0.1703y(t) - \epsilon z(t) - 0.0486u(t) \\ \qquad -0.0705x(t)y(t) + 0.147z(t)u(t). \end{cases} \tag{4.5}$$

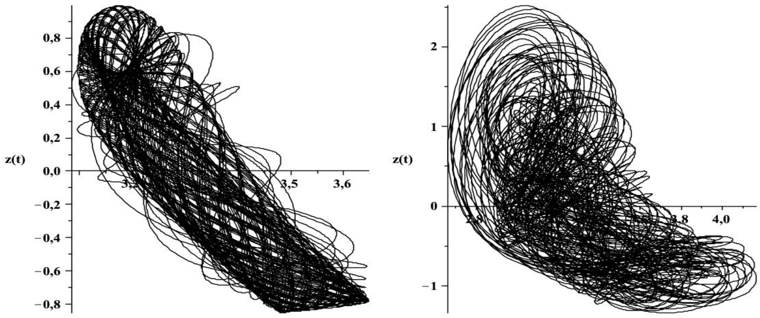
6. The base of quadratic part of system (3.2) consists of four elements $\{xy, xu, zy, zu\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.0137 - 0.0040x(t) - 0.2516y(t) - \epsilon_1 z(t) + 0.1418u(t) \\ \qquad +0.0776x(t)y(t) - 0.0370x(t)u(t) + 0.0128z(t)y(t) - 0.0286z(t)u(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.0096 - 0.0027x(t) + 1.1072y(t) - \epsilon_2 z(t) + 0.1237u(t) \\ \qquad -0.3401x(t)y(t) - 0.0496x(t)u(t) - 0.1029y(t)z(t) + 0.112z(t)u(t). \end{cases} \tag{4.6}$$

7. The base of quadratic part of system (3.2) consists of six elements $\{xy, xz, xu, x^2, y^2, z^2\}$:

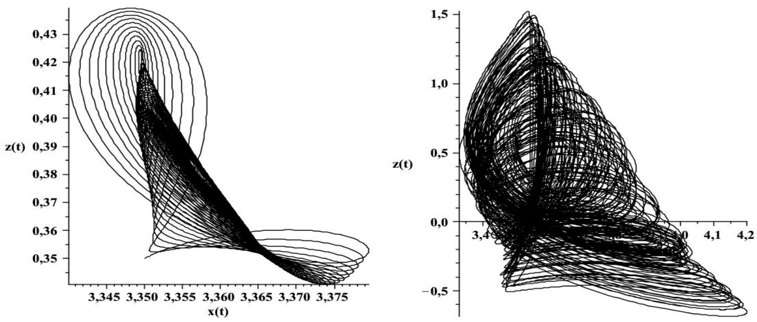
$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.0422 - 0.0206x(t) - 0.2367y(t) - 0.0042z(t) + 0.1494u(t) \\ \qquad +0.0732x(t)y(t) + 0.0010x(t)z(t) - 0.0206x(t)u(t) + 0.0024x^2(t) \\ \qquad +0.2794y^2(t) + 0.0037z^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.0882 - 0.049x(t) + 1.0144y(t) + \epsilon z(t) + 0.0394u(t) - 0.3144x(t)y(t) \\ \qquad +0.00073x(t)z(t) - 0.0240x(t)u(t) + 0.0068x^2(t) - 2.1513y^2(t). \end{cases} \tag{4.7}$$

The graphs of solutions of the corresponding systems of differential equations are below represented. (The starting point for integration was always chosen near an equilibrium point.)



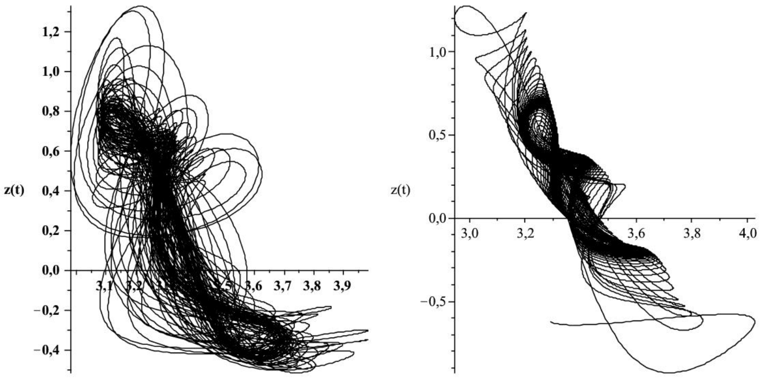
(c1) (c2)

Fig. 4.6. The behavior $U(t) - I(t)$ characteristic (c1) for system (4.1) at $\epsilon_2 = 0.0019$ and $\epsilon_1 = 0.0008$(quasiperiodic behavior), and the same characteristic (c2) at $\epsilon_2 = 0.0019$ and $\epsilon_1 = 0.00088$(chaotic behavior)



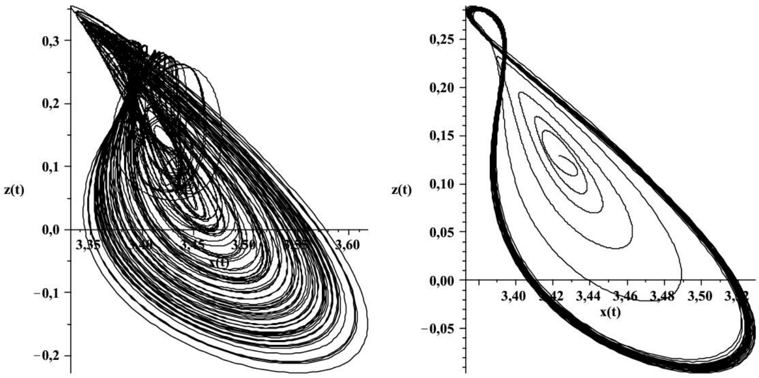
(c1) (c2)

Fig. 4.7. The behavior $U(t) - I(t)$ characteristic (c1) for system (4.2) at $\epsilon_1 = 0.01934$ and $\epsilon_2 = 0.01281$(quasiperiodic behavior), and the same characteristic (c2) at $\epsilon_1 = 0.0207$ and $\epsilon_2 = 0.01363$(chaotic behavior)



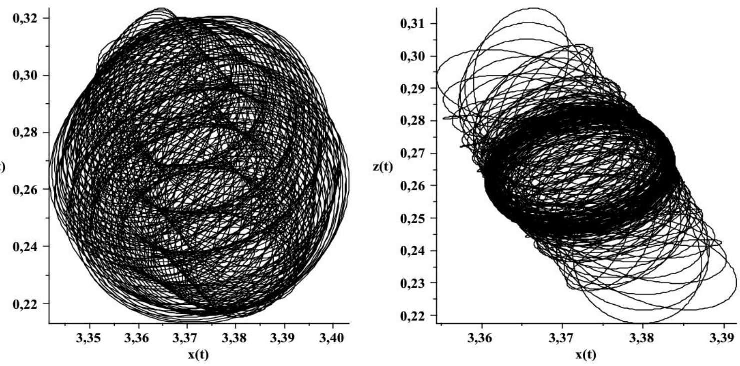
(c1) (c2)

Fig. 4.8. The behavior $U(t)-I(t)$ characteristic (c1) for system (4.3) and the same characteristic (c2) for system (4.4)



(c1) (c2)

Fig. 4.9. The behavior $U(t)-I(t)$ characteristic (c1) for system (4.5) at $\epsilon=-0.0019$(chaos), and the same characteristic (c2) at $\epsilon=-0.0020$



(c1) (c2)

Fig. 4.10. The behavior $U(t)-I(t)$ characteristic (c1) for system (4.6) at $\epsilon_1=-0.0008$, $\epsilon_2=-0.00185$(quasiperiodic behavior), and the same characteristic (c2) at at $\epsilon_1=-0.0008$, $\epsilon_2=-0.00187$(quasiperiodic behavior)

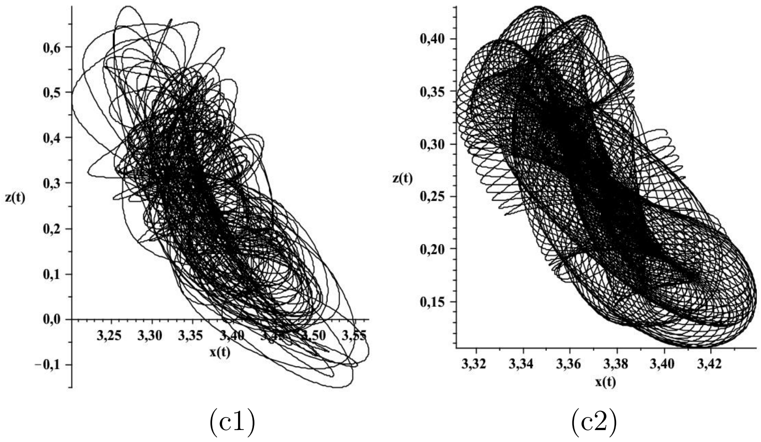


Fig. 4.11. The behavior $U(t)-I(t)$(c1) for system (4.6) at $\epsilon_1 = -0.00086$, $\epsilon_2 = -0.00187$(chaos), and the same characteristic (c2) at at $\epsilon_1 = -0.00082$, $\epsilon_2 = -0.00186$(quasiperiodic behavior)

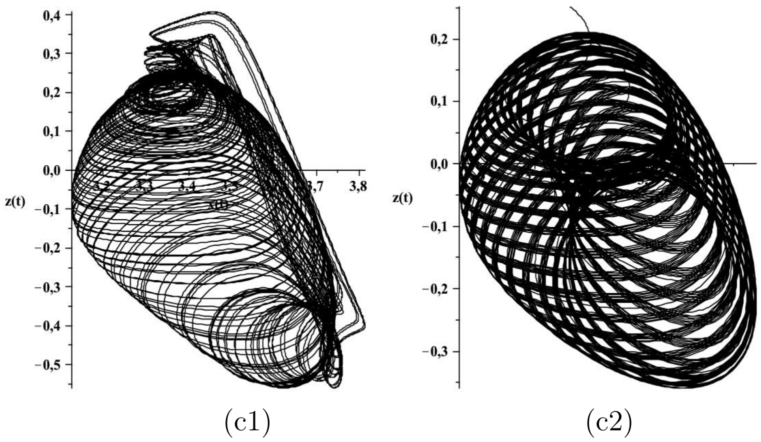


Fig. 4.12. The behavior $U(t) - I(t)$ characteristic (c1) for system (4.7) at $\epsilon = 0.0039$(chaos), and the same characteristic (c2) at $\epsilon = 0.0044$(quasiperiodic behavior)

It should be noted that all the considered models are chaotic: an arbitrarily small change in the parameters of the model leads to a radically different behavior of this model. (Nevertheless, one can always find such parameter values for which a limit cycle appears in the system. his fact can be used to construct stabilizing control laws. Another application of the obtained limit cycle can be the search for limit tori.)

Now there is a question: what from models (4.1)–(4.7) most adequately describes the behavior of process of represented on Fig. 4.2? Comparison of trajectories of the real chaotic system and its model does not enable to speak about their adequacy. Therefore, we decided to define the adequacy beginning from comparison of $U - I$ characteristics of model and real process. In this connection we build a parallelogram $ABCD$ in the coordinate system $U - I$ (see Fig. 4.13):

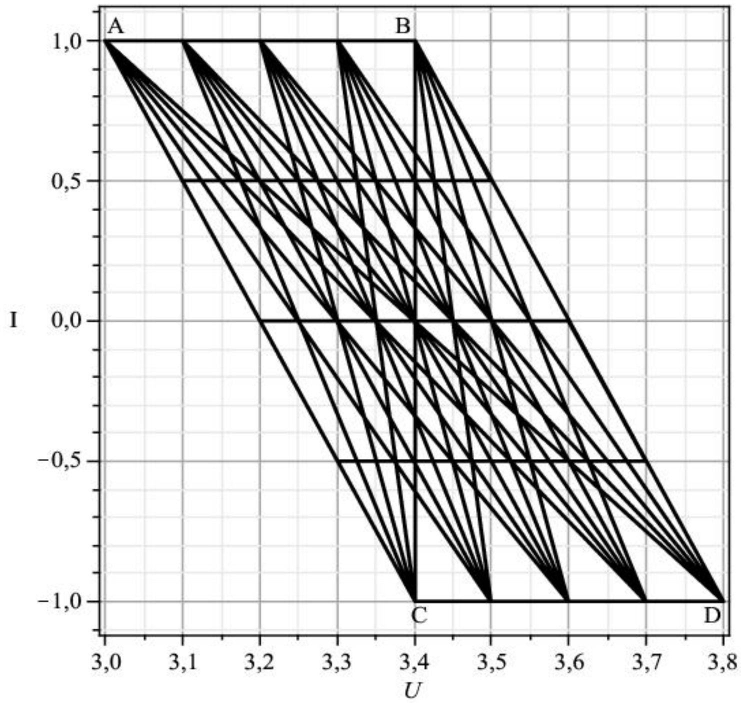


Fig. 4.13. The framework of $U - I$ characteristic for the direct current power supply system

This parallelogram must possess the following features:

a) taking into account that we deal with the direct current power supply system bases AB and CD of parallelogram $ABCD$ must be parallels to axis U;

b) the vertices of parallelogram must be disposed in points: $A(U_{\min}, I_{\max})$, $B(U_{\min+\delta}, I_{\max})$, $C(U_{\min}, I_{\min})$, $D(U_{\max}-\delta, I_{\min})$, where $I_{\min}(I_{max})$ is a minimum (maximum) current with the exception of some random fluctuations; $U_{\min}(U_{max})$ is a minimum (maximum) voltage with the exception of some random fluctuations; δ is a magnitude of change of voltage at the fixed current.

Definition 4.1. The parallelogram $ABCD$ is called a framework of $U-I$ characteristic for the direct current power supply system.

Let $\mathbb{S} \in \mathbb{R}^2$ be the set of all internal points of the parallelogram $ABCD$. Now we are ready to answer on the question about adequacy of model and real process.

By $\mathbb{M}_1, ..., \mathbb{M}_k \in \mathbb{R}^2$ denote $U-I$ characteristics of models, which it is possible to build beginning from the real process of represented on Fig. 4.2).

Introduce the Hausdorff distance $d_H(\mathbb{S}, \mathbb{M}_i)$ between the sets $\mathbb{S}$ and $\mathbb{M}_i$, $i \in \{1, ..., k\}$ [29].

Definition 4.2. The set $\mathbb{S}$ is called adequate to the set $\mathbb{M}_m$, if $d_H(\mathbb{S}, \mathbb{M}_m) \leq d_H(\mathbb{S}, \mathbb{M}_i), i = 1, ..., k; 1 \leq m \leq k$.

We notice that exact calculation of the Hausdorff distance $d_H(\mathbb{S},\mathbb{M})$ for the complex sets $\mathbb{S}$ and $\mathbb{M}$ is practically impossible. Therefore in the present work we will be limited to rough enough estimations of the magnitude $d_H(\mathbb{S},\mathbb{M})$.

Thus, the analysis of models (4.1)–(4.7) shows that only the behavior of model (4.1) is adequate to the real process of represented on Fig. 4.2.

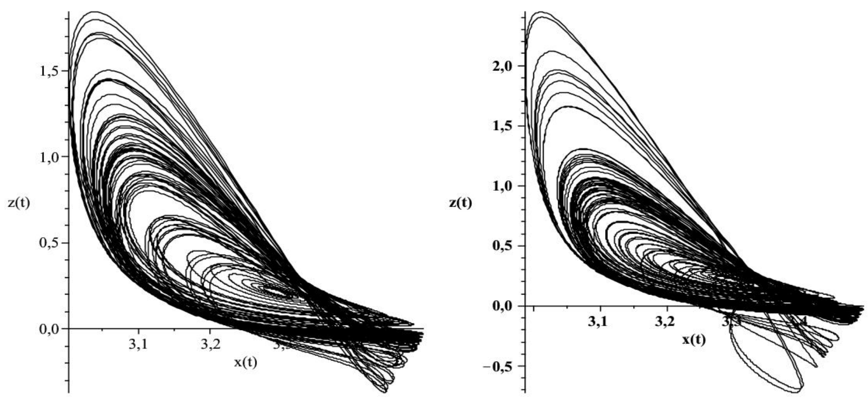
5. Modeling of Other Contact Networks

All the above equations modeled the experimental data of voltage and current changes (they are shown in Fig. 4.2) in the contact network. In order to dwell on specific equations modeling the dynamics, it is necessary to use other data representing the dynamics of processes in other contact networks. These data, describing changes in voltage and current on other railway lines, are presented in Fig. 5.17. (Note that the above data, generally speaking, is non-stationary. Therefore, for good modeling it is necessary to have such data as much as possible.)

Now we use the methodology of Sections 3 and 4. Then, we get the following equations and the corresponding behavior of their chaotic solutions (see Fig. 5.14–Fig. 5.16):

1. The base of quadratic part of system (3.2) consists of two elements $\{xz, xu+zy\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.00393 - 0.0012x(t) + 0.0059y(t) - 0.00154z(t) - 0.13146u(t) \\ \qquad +0.0005x(t)z(t) + \epsilon_1(x(t)u(t) + z(t)y(t)), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.01016 - 0.0029x(t) + 0.0006y(t) + 0.00228z(t) + 1.14108u(t) \\ \qquad -0.001481x(t)z(t) - \epsilon_2(x(t)u(t) + z(t)y(t)). \end{cases} \tag{5.1}$$

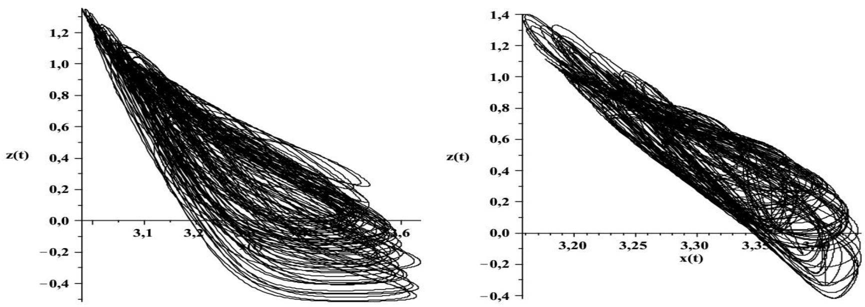


(a1) (b1)

Fig. 5.14. The behavior of $U - I$ characteristic for $\epsilon_1 = 0.04339, \epsilon_2 = 0.3710$ (a1) and $\epsilon_1 = 0.04335, \epsilon_2 = 0.3709(a1)$(b1)

2. The base of quadratic part of system (3.2) consists of two elements $\{z^2, zu\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.00623 - 0.00118696x(t) + 0.00957y(t) - 0.00012z(t) + 0.0114u(t) \\ \qquad +\epsilon_1 z^2(t) - 0.01674z(t)u(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.001951 - 0.00035x(t) + 0.06579y(t) - 0.000878z(t) - 0.0871u(t) \\ \qquad -\epsilon_1 z^2(t) + 0.145988z(t)u(t). \end{cases} \tag{5.2}$$

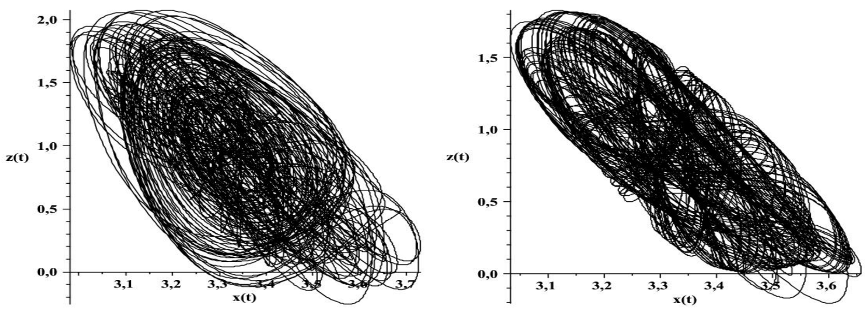


(a1) (b1)

Fig. 5.15. The behavior of $U - I$ characteristic for $\epsilon_1 = 0.0001, \epsilon_2 = 0.0019$ (a1) and $\epsilon_1 = 0.000091, \epsilon_2 = 0.0011$ (b1)

3. The base of quadratic part of system (3.2) consists of two elements $\{x^2, xy\}$:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.029438 - 0.015765x(t) + 0.48059y(t) - 0.000055z(t) + 0.002253u(t) \\ \qquad +0.002082x^2(t) - 0.1423665x(t)y(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = -0.05098 + 0.03155x(t) - 3.3642y(t) - 0.0001783z(t) - 0.006276u(t) \\ \qquad -\epsilon_1 x^2(t) + 0.99914x(t)y(t). \end{cases} \tag{5.3}$$



(a1) (b1)

Fig. 5.16. The behavior of $U - I$ characteristic for $\epsilon_1 = 0.004724$(a1) and $\epsilon_1 = 0.004739$(b1)

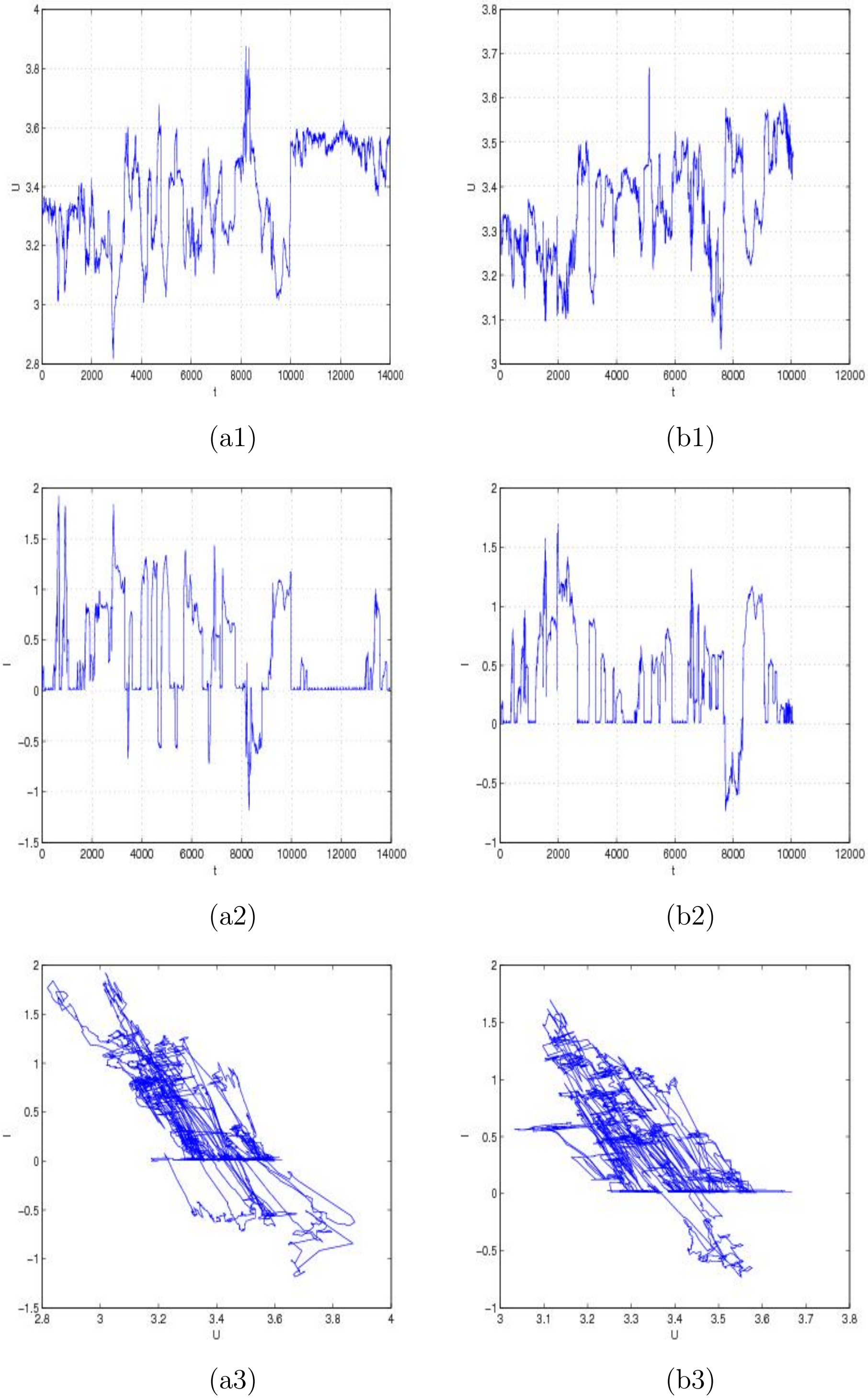


Fig. 5.17. The behavior of voltage $U(t)$(a1,b1), current $I(t)$(a2,b2), and $U-I$ characteristic (a3,b3) of different contact networks in Ukraine (experimental data)

Using the methodology of Sections 3 and 4, as well as Fig. 4.13, we arrive at the following conclusion: the most adequate description of processes in the contact network is achieved when pairs of quadratic nonlinearities $\{x^2, xy\}$ or $\{z^2, zu\}$ or $\{xz, zy + ux\}$ are used as nonlinear terms in system (3.2).

6. Research of System (4.1)

For verification of conditions Theorem 2.2 we will carry beginning of coordinates of system (4.1) in the point $(3.3657, 0, -0.0388, 0)$. In the total we get such system

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -0.0043x(t) + 0.0086y(t) - 0.00088z(t) + 0.0057u(t) \\ \qquad -0.0039x(t)y(t) + 0.000422x^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = -0.0030x(t) - 0.0495y(t) - 0.0019z(t) - 0.0095u(t) \\ \qquad +0.2380x(t)y(t) + 0.0017x^2(t) \end{cases} \tag{6.1}$$

The condition of dissipativity for system (6.1)

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{u}}{\partial u} = 0 + 0.0086 + 0 - 0.0095 = -0.0009 < 0$$

(and for system (4.1)) is fulfilled. On Fig. 6.18 possible solutions of system (4.1) are shown.

Fig. 4.2 and 5.17 show that processes in contact networks are chaotic. It is known that any chaotic processes begin from bifurcations of limit cycles. Therefore, we show the existence of such cycles using the example of system (4.1).

Thus, system (4.1) most corresponds to experimental information represented on Fig. 4.2.

6.1. Existence of limit cycles

Definition 6.1. A real cubic form $f(x, y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3$ is called trilinear, if the real factorization

$$f(x, y) = (b_0x+b_1y)(c_0x+c_1y)(d_0x+d_1y) \not\equiv (h_0x+h_1y)^2(r_0x+r_1y) \not\equiv (p_0x+p_1y)^3$$

takes place.

Consider the following real quadratic system

$$\begin{cases} \dot{x}(t) = a_{11}x(t) + a_{12}y(t) + b_{11}x^2(t) + b_{12}x(t)y(t) + b_{22}y^2(t), \\ \dot{y}(t) = a_{21}x(t) + a_{22}y(t) + c_{11}x^2(t) + c_{12}x(t)y(t) + c_{22}y^2(t). \end{cases} \tag{6.2}$$

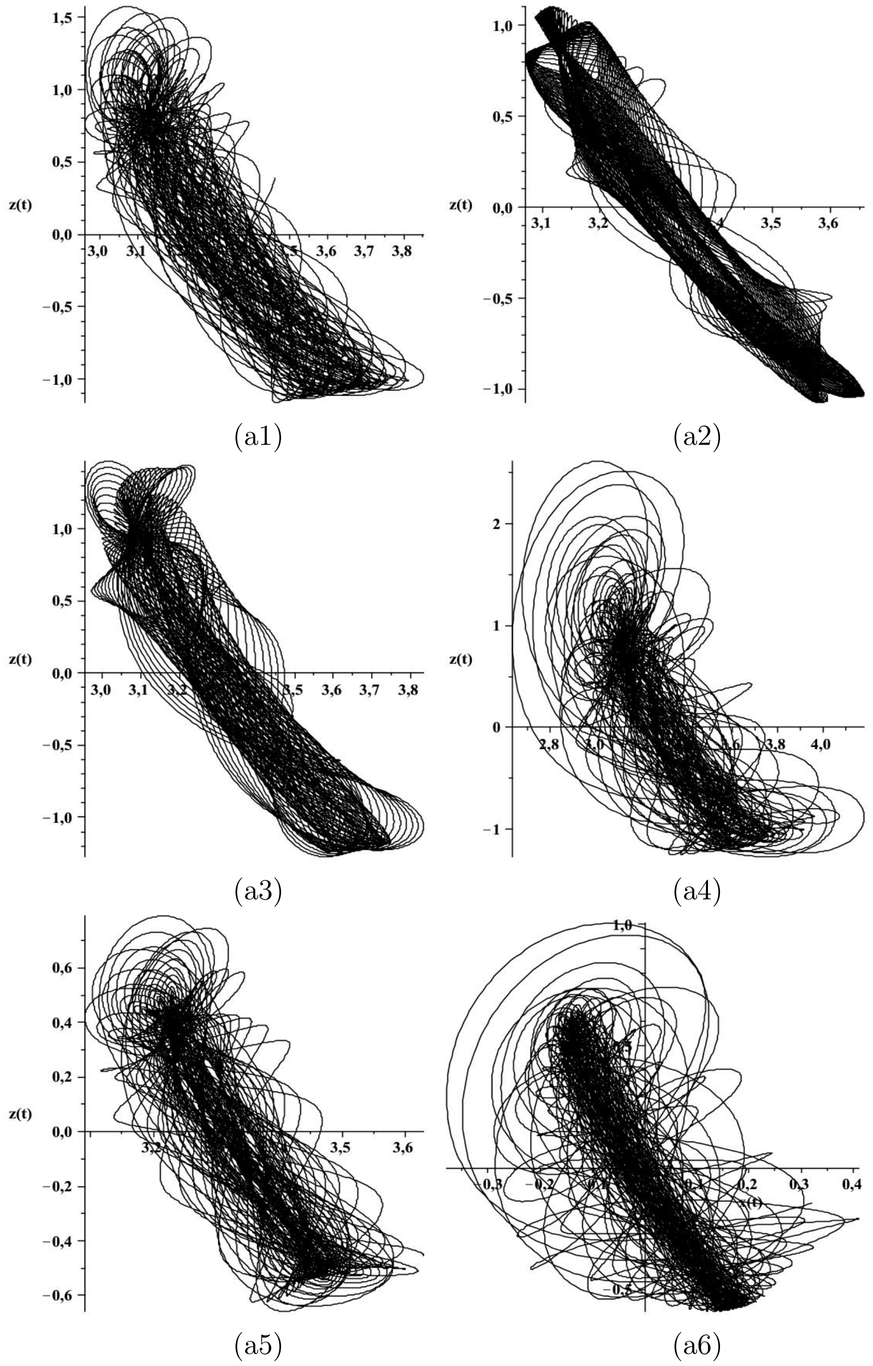


Fig. 6.18. The phase portraits of system (4.1) at the parameters: $\epsilon_1 = -0.00084$, $\epsilon_2 = -0.0018$ (a1); $\epsilon_1 = -0.00084$, $\epsilon_2 = -0.0017$ (a2); $\epsilon_1 = -0.00086$, $\epsilon_2 = -0.0017$ (a3); $\epsilon_1 = -0.00085$, $\epsilon_2 = -0.0018$ (a4); $\epsilon_1 = -0.00086$, $\epsilon_2 = -0.0019$ (a5); and system (6.1)

Assume that the equilibrium $(0, 0)^T$ of system (6.2) be an unstable focus. By suitable real linear replacements of variables x and y, we reduce system (6.2) to the following form:

$$\begin{cases} \dot{x}(t) = \alpha x(t) + \beta y(t) + d_{11}x^2(t) + d_{12}x(t)y(t) + d_{22}y^2(t), \\ \dot{y}(t) = -\beta x(t) + \alpha y(t) + e_{11}x^2(t) + e_{12}x(t)y(t) + e_{22}y^2(t). \end{cases} \tag{6.3}$$

Now we construct the following cubic form: $g(x,y) \equiv x \cdot (d_{11}x^2 + d_{12}xy + d_{22}y^2) + y \cdot (e_{11}x^2 + e_{12}xy + e_{22}y^2)$.

Theorem 6.1. *Let for system (6.3) the form $g(x,y)$ be trilinear. Suppose also that in the same system there are two equilibriums, one of which is an unstable focus or center (the number $\alpha \geq 0$ is small enough, $\beta \neq 0$) and the other is a saddle. Then in this system there exists a stable limit cycle.*

Proof. Using system (6.3), we introduce the following continuously differentiable function $V(t) = 0.5(x^2(t) + y^2(t))$. Thus, we have:

$$\dot{V}(t) \equiv x(t)\dot{x}(t) + y(t)\dot{y}(t) = \alpha \cdot (x^2(t) + y^2(t)) + g(x,y)$$

$$\equiv \alpha \cdot (x^2(t) + y^2(t)) + (q_1x(t) + q_2y(t))(s_{11}x(t) + s_{12}y(t))(s_{21}x(t) + s_{22}y(t)).$$

Without loss of generality, we can consider that $q_1s_{11}s_{21} > 0$. Otherwise, by replacement $x(t) \to -x(t)$ (or $y(t) \to -y(t)$), we obtain implementation of the condition $q_1s_{11}s_{21} > 0$. (If $q_1s_{11}s_{21} \neq 0$, then it is always possible.)

In addition, we can also consider that $\alpha = 0$.

By $\lambda_1 = -q_2/q_1, \lambda_2 = -s_{12}/s_{11}, \lambda_3 = -s_{22}/s_{21}$ denote roots of polynomial $(q_1\lambda + q_2)(s_{11}\lambda + s_{12})(s_{21}\lambda + s_{22})$. Without loss of generality, we can consider that $\lambda_1 < \lambda_2 < \lambda_3$.

Rewrite the cubic function $H(x,y) \equiv (q_1x + q_2y)(s_{11}x + s_{12}y)(s_{21}x + s_{22}y)$ in the following way:

$$H(\cos\phi, \sin\phi) = (q_1\cos\phi + q_2\sin\phi)(s_{11}\cos\phi + s_{12}\sin\phi)(s_{21}\cos\phi + s_{22}\sin\phi),$$

where $\cos\phi = x/\sqrt{2V}, \sin\phi = y/\sqrt{2V}$. From here it follows that

$$\delta_{min} \leq H(\cos\phi, \sin\phi) \leq \delta_{max},$$

where $\delta_{min} < 0 (\delta_{max} > 0)$ is a minimum (maximum) of function $H(\cos\phi, \sin\phi)$.

Consider the trigonometric polynomial

$$H_1(\cos\phi, \sin\phi, V)$$

$$= \frac{\alpha}{\sqrt{2V}} + (q_1\cos\phi + q_2\sin\phi)(s_{11}\cos\phi + s_{12}\sin\phi)(s_{21}\cos\phi + s_{22}\sin\phi),$$

where a number $V > 0$ must be chosen so that the real trigonometric periodic function $H_1(\cos\phi, \sin\phi, V)$ has three real distinct roots on a period. (The number V must be large enough.)

By V^* denote the minimal value $V > 0$ at which the periodic function $H_1(\cos\phi, \sin\phi, V)$ has three real roots on the period.

Let $\lambda_1^* < \lambda_2^* < \lambda_3^*$ be three real distinct roots of function $H_1(\cos\phi, \sin\phi, V^*)$ following in succession. It is obviously that $\lambda_1^* < \lambda_1 < \lambda_2 < \lambda_2^*$ and $\lambda_2 < \lambda_2^* < \lambda_3^* < \lambda_3$. In addition it is possible to consider that function $H_1(\cos\phi, \sin\phi, V^*)$ on the interval $(\lambda_1^*, \lambda_2^*)$ is positive, and on the interval $(\lambda_2^*, \lambda_3^*)$ is negative.

In addition, for function $H_1(\cos\phi, \sin\phi, V^*)$ we have

$$\delta_{min}^* \le H_1(\cos\phi, \sin\phi, V^*) \le \delta_{max}^*; \quad \delta_{min} < \delta_{min}^* < 0 < \delta_{max} < \delta_{max}^*,$$

where $\delta_{min}^* < 0(\delta_{max}^* > 0)$ is a minimum (maximum) of the real periodic function $H_1(\cos\phi, \sin\phi, V^*)$.

Further, we have that if $\lambda_1^* < \tan\phi < \lambda_2^*$, then $H_1(\cos\phi, \sin\phi, V^*) > 0$, and if $\lambda_2^* < \tan\phi < \lambda_3^*$, then $H_1(\cos\phi, \sin\phi, V^*) < 0$.

Thus, if $\lambda_1^* < \tan\phi < \lambda_3^*$, then we have

$$0.5V(t) \cdot (\alpha + \sqrt{2V(t)} \cdot \delta_{min}^*) \le \dot{V}(t) \le 0.5V(t) \cdot (\alpha + \sqrt{2V(t)} \cdot \delta_{max}^*).$$

Since $(0,0)^T$ there is a repellent point, then from here it follows that the maximal value of function V must be bounded.

To investigate system (6.3), we will use the iterative Euler method:

$$\begin{cases} x_{j+1} = x_j + (\alpha x_j + \beta y_j + d_{11}x_j^2 + d_{12}x_jy_j + d_{22}y_j^2)\Delta t, \\ y_{j+1} = y_j + (-\beta x_j + \alpha y_j + e_{11}x_j^2 + e_{12}x_jy_j + e_{22}y_j^2)\Delta t. \end{cases} \tag{6.4}$$

where $x_0 = x(0), y_0 = y(0)$ and $\Delta t > 0$ is a integration step; $j = 0, ..., m \to \infty$.

In addition, we construct the iterated procedure

$$V_{j+1} = V_j + V_j \cdot (\alpha + \sqrt{2V_j} \cdot H(\cos\phi_j, \sin\phi_j))\Delta t; \tag{6.5}$$

where $V_0 = x_0^2 + y_0^2$, $\cos\phi_j = x_j/\sqrt{2V_j}$, $\sin\phi_j = x_j/\sqrt{2V_j}$, $j = 0, ..., m \to \infty$. (Here we did replacement $V(t) \to x^2(t) + y^2(t)$.)

Assume that the initial pair values (x_0, y_0) such that $y_0/x_0 = (\cos\phi_0/\sin\phi_0) \in (\lambda_1^*, \lambda_2^*)$.

Construct the iterative process $V_0 \ge V_1 \ge ... \ge V_m \ge ...$, which is defined by formulas (6.5). It is obviously that we have a positive monotone decreasing bonded sequence. It means that for small enough Δt the iterative process (6.5) is convergence. Therefore, $\lim\limits_{j\to\infty} V_j = C(x_l, y_l) = const$. Here (x_l, y_l) is a solution of equation $\alpha + \sqrt{x^2 + y^2}H(\cos\phi, \sin\phi) = 0$ $(\alpha \approx 0)$.

Now we again assume that the initial pair values (x_0, y_0) such that $y_0/x_0 = (\cos\phi_0/\sin\phi_0) \in (\lambda_2^*, \lambda_3^*)$.

Construct the iterative process $V_0 \le V_1 \le ... \le V_m \le ...$, which is defined by formulas (6.5). Now we already have a positive monotone increasing bonded sequence. Therefore, we again have $\lim\limits_{j\to\infty} V_j = C(x_l, y_l) = const$.

Let S be a bounded set containing the point $(0,0)$. (It can be a circle of small enough radius.) Now we will organize the iterative process (6.5) from any point of the set S. In this case, in acording to LaSall's Theorem [30] all limit points of the process (6.5) form a positively invariant set C.

Since system (6.3) has also the saddle equilibrium, then her separatrix is a restriction for any trajectory of the system of beginning in the small neighbouring of the point $(0,0)^T$. It means that sequence (6.4) (it is $(x_0, y_0)^T$, $(x_1, y_1)^T, ...$) must converge to the set C. Thus, this is a stable limit cycle. $\square$

Consider the following system:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = a_{10} + a_{11}x(t) + a_{12}y(t) + a_{13}z(t) + a_{14}u(t) \\ \qquad\quad + b_{12}x(t)y(t) + b_{11}x^2(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = a_{20} + a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + a_{24}u(t) \\ \qquad\quad + b_{21}x(t)y(t) + b_{22}x^2(t), \end{cases} \tag{6.6}$$

In system (6.6) it is possible to select two subsystems:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = a_{10} + a_{11}x(t) + a_{12}y(t) + a_{13}z(t) + a_{14}u(t) \\ \qquad\quad + b_{12}x(t)y(t) + b_{11}x^2(t). \end{cases} \tag{6.7}$$

and

$$\begin{cases} \dot{z}(t) = u(t), \\ \dot{u}(t) = a_{20} + a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + a_{24}u(t) \\ \qquad\quad + b_{21}x(t)y(t) + b_{22}x^2(t). \end{cases} \tag{6.8}$$

At system (6.7) it is possible to look as on the system with external perturbation $F(t) \equiv a_{13}z(t) + a_{14}u(t)$. Therefore, it is importantly to research the proper behavior of the unperturbed system

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = a_{10} + a_{11}x(t) + a_{12}y(t) + b_{12}x(t)y(t) + b_{11}x^2(t), \end{cases} \tag{6.9}$$

where $b_{11}b_{12} \neq 0$.

Assume that system (6.9) has real equilibriums: $(\lambda_1, 0)$ and $(\lambda_2, 0)$, where $\lambda_{1,2}$ are real roots of the equation $a_{10} + a_{11}\lambda + b_{11}\lambda^2 = 0$. Introduce the new variable x_1 in system (6.9) under the formula $x_1 = x - \lambda_1$ (or $x_1 = x - \lambda_2$).

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = h_{11}x(t) + h_{12}y(t) + b_{12}x(t)y(t) + b_{11}x^2(t). \end{cases} \tag{6.10}$$

(For simplicity we have left the former designation of variables x.)

Corollary 6.1. *Assume that for the system (6.10) conditions $h_{11} < 0, h_{12} = 0, b_{12} < 0, b_{11} > 0$ are fulfilled. Then in system (6.10) there exists the stable limit cycle.*

Proof. The quadratic form $b_{12}xy+b_{11}x^2$ it is possible to represent as $x(b_{11}x+b_{12}y)$. Now we replace the variable y under the formula: $y_1 = b_{11}x + b_{12}y$. In this case system (6.10) can be represented in the following form:

$$\begin{cases} \dot{x}(t) = h_{11}x(t) + h_{12}y(t) + d_1x(t)y(t), \\ \dot{y}(t) = h_{21}x(t) + h_{22}y(t) + d_2x(t)y(t). \end{cases} \tag{6.11}$$

For simplicity we have left the former designation of variables y.

Suppose that eigenvalues of matrix $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ are complex: $\mu + i\nu, \mu - i\nu$. (Here $\mu > 0, \nu \neq 0$.) By suitable linear replacements of variables $x \to s_{11}x + s_{12}y$, $x \to s_{21}x + s_{22}y$, we reduce system (6.11) to such aspect:

$$\begin{cases} \dot{x}(t) = \mu x(t) + \nu y(t) + q_1 \cdot (s_{11}x(t) + s_{12}y(t))(s_{21}x(t) + s_{22}y(t)), \\ \dot{y}(t) = -\nu x(t) + \mu y(t) + q_2 \cdot (s_{11}x(t) + s_{12}y(t))(s_{21}x(t) + s_{22}y(t)). \end{cases} \tag{6.12}$$

In addition, by replacement $x(t) \to -x(t)$ (or $y(t) \to -y(t)$, we must obtain implementation of the condition $q_1s_{11}s_{21} > 0$. In this case the conditions of Theorem 6.1 are fulfilled. □

Now we can apply Theorem 6.1 to system (6.12) (and (6.9)). To do this, we show the sequence of bifurcations leading to the appearance of chaotic dynamics from the limit cycle (see Fig. 6.19).

Let A, ω be an amplitude and frequency of external perturbation $A\sin(\omega t)$. (If $A = 0$, then the perturbation there doesn't exist.)

For system (6.7) the perturbed system is

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = 0.0193 - 0.0072x(t) + 0.0218y(t) - 0.0039x(t)y(t) \\ \qquad +0.000422x^2(t) + A\sin(\omega t). \end{cases} \tag{6.13}$$

All conditions of Theorem 6.1 for system (6.12) are satisfied. (It can be confirmed directly.) Appearance in system (6.13) of a stable limit cycle at the corresponding external perturbations is shown on Fig. 6.19.

Further, on system (6.8) it is also possible to look as on a linear system with external perturbation $G(t) \equiv a_{21}x(t) + a_{22}y(t) + b_{21}x(t)y(t) + b_{22}x^2(t)$. In order that the solution of this linear system was bounded it is necessary that the equilibrium $(-a_{20}/a_{23}, 0)$ unperturbed system was stable and the function $G(t)$ was bounded [30]. For system

$$\begin{cases} \dot{z}(t) = u(t), \\ \dot{u}(t) = 0.0294 - 0.0019z(t) - 0.0095u(t) + G(t) \end{cases} \tag{6.14}$$

both these conditions are fulfilled. The equilibrium ($z^* = -15.4737, u^* = 0$) (at $G(t) \equiv 0$) is a stable focus. In system (6.14) the boundedness of function $G(t)$ is guaranteed by the boundedness of solutions of system (6.13) [23, 24].

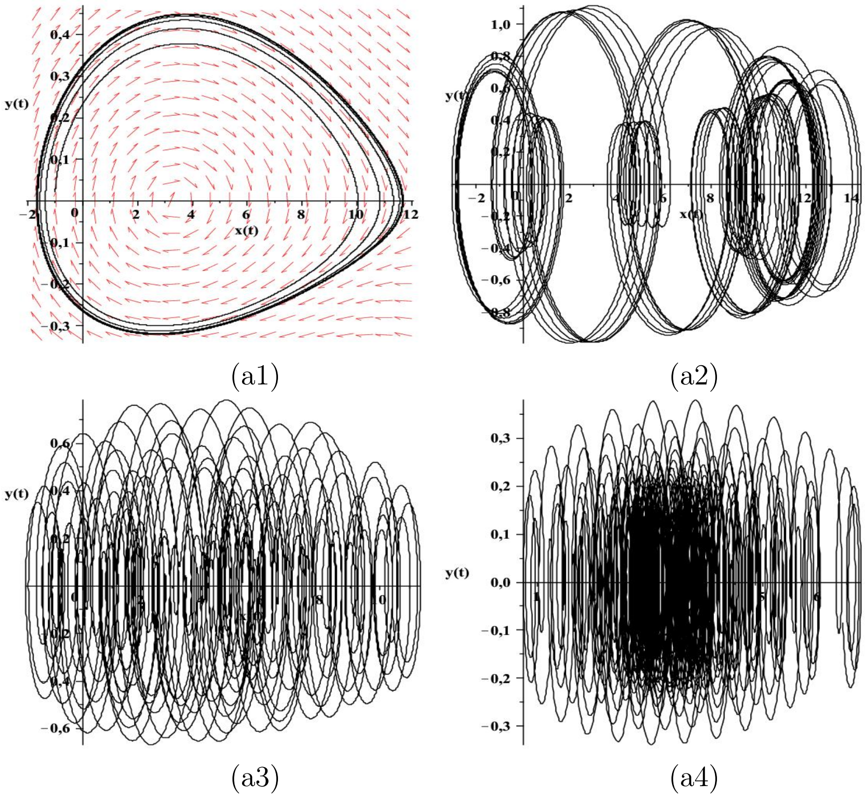


Fig. 6.19. The bifurcations of limit cycle for system (6.13) at $A = 0$ (a1); $A = 0.3, \omega = 0.45$ (a2); $A = 0.3, \omega = 0.85$ (a3); $A = 0.3, \omega = 1.45$ (a4)

Thus, on system (4.1) it is also possible to look as on a system of two connected 2D circuits of describing oscillations of current and voltage in the contact electric network.

7. Remarks on Design of Voltage Regulator

At certain values of parameters system (6.6) describes the dynamics of changes in voltage and current in a contact network. Note that voltage $U(t) = x(t)$ and current $I(t) = z(t)$ are measured using a mobile laboratory that moves at constant velocity v along a contact network [31]. Note that in order to study processes in the contact network, it is more convenient to employ a dynamic model in which voltage $U(t) = x(t)$ and current $I(t) = z(t)$ are represented as functions $U(s) = x(s)$, $I(s) = z(s)$ of distance s from some starting point. Such representation is very convenient when the stabilization of voltage in contact network is implemented from some fixed points along the route of the train, for example, at traction substations or gain points. Furthermore, we shall assume that voltage control is executed using the regulator $Uinput(s) = f(U(s), \dot{U}(s), I(s), \dot{I}(s))$, where $f(...)$ is the real function of its arguments. Thus, the transition from model

(6.6), where $x(t)$ and $z(t)$ are represented as a function of time t, has been achieved through the replacement of independent variable t with independent variable s, according to formula $s = v_s t$. In this case, $y(t) \to v_s y(s)$, $u(t) \to v_s u(s)$, and system (6.6) transforms to the following system:

$$\begin{cases} \dot{x}(s) = y(s), \\ \dot{y}(s) = (a_{10} + a_{11}x(s) + a_{12}v_s y(s) + a_{13}z(s) + a_{14}v_s u(s) \\ \qquad +b_{12}v_s x(s)y(s) + b_{11}x^2(s))/v_s^2, \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = (a_{20} + a_{21}x(s) + a_{22}v_s y(s) + a_{23}z(s) + a_{24}v_s u(s) \\ \qquad +b_{21}v_s x(s)y(s) + b_{22}x^2(s))/v_s^2, \end{cases} \tag{7.1}$$

where velocity $v_s = const$ is measured in m/s and $U(s) = x(s)$, $I(s) = z(s)$ are some functions of distance. (For simplicity, we kept the former designations for dependent variables x and z in the newly derived system. The variables $x(t)$ and $z(t)$ are replaced with variables $x(s)$ and $z(s)$.

Now we will introduce a control law.

Model (7.1) is designed to study the stability of the voltage in the contact network. This model was obtained by observing the chaotic behavior of voltage and current. In future, for simplicity we will consider that $v_s = 1$.

Introduce in system (6.6) a new variable $v = x^2$. Then we get the following system

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = a_{10} + a_{11}x(t) + a_{12}y(t) + a_{13}z(t) + a_{14}u(t) \\ \qquad +b_{12}x(t)y(t) + b_{11}v(t), \\ \dot{z}(t) = u(t), \\ \dot{u}(t) = a_{20} + a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + a_{24}u(t) \\ \qquad +b_{21}x(t)y(t) + b_{22}v(t), \\ \dot{v}(t) = 2x(t)y(t). \end{cases} \tag{7.2}$$

Hence it is already possible to establish parameters under which the voltage in system (7.2) (or (6.6)) can be stabilized [30, 32]. (The corresponding characteristics of real behavior of the voltage and current are given in Fig. 5.17.)

Now we will do an attempt yet to simplify system (7.2) and to do this system of more suitable for further researches.

We will consider that system (6.6) has an equilibrium $P = (x^*, y^*, z^*, u^*)$. Then we can transfer the origin of coordinates in the point P. In this case in system (7.2) we have $a_{10} = a_{20} = 0$. Therefore, we can consider that the condition $a_{10} = a_{20} = 0$ is satisfied.

Introduce the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & b_{11} \\ 0 & 0 & 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_{22} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ b_{12} \\ 0 \\ b_{21} \\ 2 \end{pmatrix}.$$

Then by the algorithm of indicated in [33], the following result can be got:

Theorem 7.1. *[33] If the conditions $a_{10} = a_{20} = 0$ and*

$$\det(B, AB, A^2B, A^3B, A^4B) \neq 0$$

are hold, then by linear replacements of variables $(x, y, z, u, v) \to (v_1, v_2, v_3, v_4, v_5)$ system (7.2) can be reduced to the following canonical form:

$$\begin{pmatrix} \dot v_1(t) \\ \dot v_2(t) \\ \dot v_3(t) \\ \dot v_4(t) \\ \dot v_5(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -d_4 & -d_3 & -d_2 & -d_1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (\sum\limits_{i=1}^{5} p_i v_i) \sum\limits_{i=1}^{5} q_i v_i \end{pmatrix}, \tag{7.3}$$

where numbers $d_1, ..., d_4, p_i, q_i \in \mathbb{R}; i = 1, ..., 5$.

The canonical form (7.3) is generalization of the known *Bezout's states column model* [33]. (The system (7.3) is interesting to those that unlike the system (7.2), this system has only one nonlinearity.)

The last equation can be generalized in the following way. We will assume that the model of direct current power supply system is association of several oscillatory circuits for description of the voltage, current, electromagnetic induction, and so on. Then model (7.3) may be generalized in the following form:

$$\dot{\mathbf{v}}(t) = A\mathbf{v}(t) + B \cdot \left(\sum_{i=1}^{n+1} p_i v_i(t)\right)\left(\sum_{i=1}^{n+1} q_i v_i(t)\right). \tag{7.4}$$

Here,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & -d_n & -d_{n-1} & \dots & -d_1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1},$$

$\mathbf{v} = (v_1, ..., v_{n+1})^T$, $d_1, ..., d_n, p_i, q_i \in \mathbb{R}; i = 1, ..., n+1$. This equation can be useful at the further study of the model of direct current power supply system.

Now we show a simple method of construction of stabilizing linear feedback for system (6.6).

Consider the following control system:

$$\begin{cases} \dot x(t) = y(t), \\ \dot y(t) = a_{10} + a_{11}x(t) + a_{12}y(t) + a_{13}z(t) + a_{14}u(t) + p_U F(t) \\ \qquad +b_{12}x(t)y(t) + b_{11}x^2(t), \\ \dot z(t) = u(t), \\ \dot u(t) = a_{20} + a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + a_{24}u(t) + p_I F(t) \\ \qquad +b_{21}x(t)y(t) + b_{22}x^2(t), \end{cases} \tag{7.5}$$

where a pair of coefficients $(p_U, p_I) \neq 0$ can take any real values and $F(t)$ is a control.

Introduce the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, B = \begin{pmatrix} 0 \\ p_U \\ 0 \\ p_I \end{pmatrix}.$$

Let $a_{10} = a_{20} = 0$ and $\det(B, AB, A^2B, A^3B) \neq 0$. Then by the algorithm of indicated in [30], [33], we can reduced system (6.6) to such aspect:

$$\begin{pmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{v}_3(t) \\ \dot{v}_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d_4 & -d_3 & -d_2 & -d_1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} G_1(v_1, ..., v_4) \\ G_2(v_1, ..., v_4) \\ G_3(v_1, ..., v_4) \\ F(t) + G_4(v_1, ..., v_4) \end{pmatrix}, \tag{7.6}$$

where $G_1(v_1, ..., v_4), ..., G_4(v_1, ..., v_4)$ are quadratic forms and $d_1, ..., d_4 \in \mathbb{R}$.

Introduce in system (7.6) a linear feedback by the formula

$$F(t) = k_1 v_1(t) + k_2 v_2(t) + k_3 v_3(t) + k_4 v_4(t),$$

where $k_1, ..., k_4$ are indeterminate coefficients. Then a linear part of the closed by the feedback system has a characteristic polynomial $h(\lambda) = \lambda^4 + (k_1 - d_1)\lambda^3 + (k_2 - d_2)\lambda^2 + (k_3 - d_3)\lambda + k_4 - d_4$.

Choose the coefficients $(k_i - d_i)$ of polynomial $h(\lambda); i = 1, ..., 4$, so that this polynomial became the Hurwitz polynomial [34]. In this case the origin of the closed by feedback system will be stable (see Fig. 7.20):

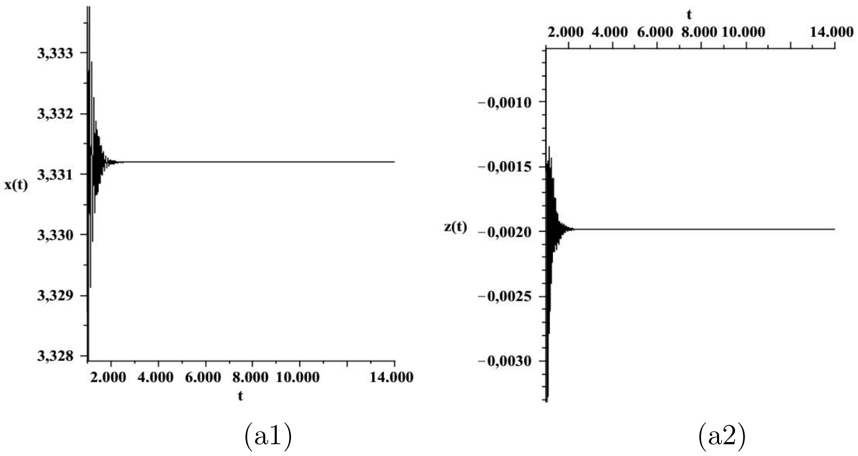


Fig. 7.20. Plots for voltage (a1) and current (a2) for system (6.6) at $b_{12} = -0.01$. (Other values of parameters the same as in system (4.1))

Let S be a linear transformation reducing system (7.5) to form (7.4):

$$(x(t), y(t), z(t), u(t))^T = S \cdot (v_1(t), v_2(t), v_3(t), v_4(t))^T; \quad \det S \neq 0.$$

Then in order that the origin of system (7.5) ($a_{10} = a_{20} = 0$) would be stable it is enough to give the control law by the formula

$$F(t) = (k_1, k_2, k_3, k_4) \cdot S^{-1}(x(t), y(t), z(t), u(t))^T.$$

An area of stability of power system is the set of its modes, in which static stability is provided for a certain composition of the generators and a fixed circuit of the electric network. A surface bounding a set of stable regimes is called a boundary of region of static stability [32]. The stability regions are constructed in the coordinates of the parameters that affect the stability of the regime. The calculated and experimentally determined areas of stability are used to set the dispatch restrictions on the regime of the power system (in the form of dispatch instructions) and to configure automation facilities to prevent possible violations of static stability. Obviously, reliable and stable operation of the power system in modes directly adjacent to the boundary of the stability region is impossible. In these modes, any, even weak disturbances in the power system or spontaneous minor weighting of the regime will lead to a violation of stability. Changes in the regime of the power system (active and reactive overflows, voltage and frequency) are primarily associated with load fluctuations in load nodes: the inclusion and disconnection of individual electrical installations, start up and shutdown of enterprises, changes in their operating mode according to technology conditions, etc. Part of these changes are regular character, due to daily, weekly, seasonal regimes. Such changes are described by the corresponding load schedules and predictable enough.

Thus, the estimate of stability regions is another problem for future research. The basis for such studies is system (7.3).

The problem of voltage regulation is the topic of future work. For this regulation, external controls can be introduced into system (7.2). Then the system (7.2) can be taken in one of the following forms: either

$$\begin{pmatrix} \ddot{U}(t) \\ \ddot{I}(t) \end{pmatrix} = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{pmatrix} \cdot \begin{pmatrix} \dot{U}(t) \\ \dot{I}(t) \end{pmatrix} + \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} U(t) \\ I(t) \end{pmatrix}$$
$$+ \begin{pmatrix} b_{11}U^2(t) + b_{12}U(t) \cdot \dot{U}(t) \\ b_{22}U^2(t) + b_{21}U(t) \cdot \dot{U}(t) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \tag{7.7}$$

or

$$\begin{pmatrix} \ddot{U}(t) \\ \ddot{I}(t) \end{pmatrix} = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{pmatrix} \cdot \begin{pmatrix} \dot{U}(t) \\ \dot{I}(t) \end{pmatrix} + \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} U(t) \\ I(t) \end{pmatrix}$$
$$+ \begin{pmatrix} b_{11}I^2(t) + b_{12}I(t) \cdot \dot{I}(t) \\ b_{22}I^2(t) + b_{21}I(t) \cdot \dot{I}(t) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \tag{7.8}$$

where $u_1(t), u_2(t)$ are external controls.

Note that if system (6.6) is a system without equilibria, then $\forall n > 2$ by affine replacements of variables it can be reduced to the following form

$$\dot{\mathbf{v}}(t) = A_0 + A\mathbf{v}(t) + B \cdot \left(\sum_{i=1}^{n+1} p_i v_i(t) \right) \left(\sum_{i=1}^{n+1} q_i v_i(t) \right), \tag{7.9}$$

where the matrices A and B are the same as in (7.4); $A_0 = (0, ..., 0, r)^T \in \mathbb{R}^{n+1}$, $r \neq 0$, and $rp_1q_1 > 0$.

8. Conclusion and Analysis of Results

One of the main problems arising in modeling any dynamic process is the problem of determination of dimension of phase space in which this process occurs. In article [35], which is devoted to the study of chaotic processes in the self-exciting homopolar disc dynamo, for modeling of the dynamics of this system three and five dimensional systems of differential equations were used.

It should be said that researches fulfilled in [35] are based on the known models, for which Problems 1 – 4 were already solved (see Section 1). A purpose of these publications it is the search of hidden attractors and establishment of their properties.

Note that the problems considered in [35] can be raised and for system (6.6). However, it is possible only when the results of verifications and tests fully will confirm adequacy of system (6.6) and the direct current traction power supply system. This adequacy can be set by the recurrence analysis methods [10].

At the recurrence analysis of recurrence plots an important role play lengths of diagonal lines (we will emphasize that on recurrence plots the length of line characterizes a response time of trajectory in some region of phase space) [5], [10], [13], [18]. We performed such analysis of diagonal lines, but its results were rather rough.

It should be said that in the general case it is impossible to achieve a good correspondence between the model and the complex process that this model describes. Therefore, in this work, the adequacy of the model and the process was evaluated by the deviations of the current and voltage obtained in the simulation and experiment (a current-voltage characteristic $U - I$).

Comparison of the experimental information on Fig. 4.2 and solutions of system equations (4.1) shows that there is satisfactory description of dynamics of the direct current power supply system on the interval 4000 seconds – 10000 seconds. (However, it is necessary to notice that among all systems of equations (4.1) – (4.7) only the phase portrait of system (4.1) most adequate to the real phase portrait on Fig. 4.2. Thus, there is good quality coincidence of the experimental $U - I$ characteristic with $U - I$ characteristic of system (4.1).)

In order to attain a greater accuracy it is necessary to specify the coefficients of system (4.1) (or (6.6)). It is possible to do by the use of artificial neural networks. In the future authors hope to get back to this problem.

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Received 12.3.2019

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **27**, Issue **1**, June **2019**, pp. 56–66, DOI 10.15421/141903

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

PROJECTION-ITERATION REALIZATION OF A NEWTON-LIKE METHOD FOR SOLVING NONLINEAR OPERATOR EQUATIONS

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Abstract. We consider the problem of existence and location of a solution of a nonlinear operator equation with a Fréchet differentiable operator in a Banach space and present the convergence results for a projection-iteration method based on a Newton-like method under the Cauchy's conditions, which generalize the results for the projection-iteration realization of the Newton-Kantorovich method. The proposed method unlike the traditional interpretation is based on the idea of whatever approximation of the original equation by a sequence of approximate operator equations defined on subspaces of the basic space with the subsequent application of the Newton-like method to their approximate solution. We prove the convergence theorem, obtain the error estimate and discuss the advantages of the proposed approach and some of its modifications.

Key words: nonlinear equation, Fréchet differentiable operator, Newton-like method, projection-iteration method, approximation, convergence, error estimate.

2010 Mathematics Subject Classification: 65J15, 65B99, 47A58.

Communicated by Prof. V. V. Semenov

1. Introduction

The fundamental tool in numerical analysis, operations research, optimization and control is Newton's method originally intended to solve algebraic equations. The basic ideas of the method, the main theoretical results of convergence, the latest developments in this area, the most up-to-date versions of the method, as well as its various applications can be found, for instance, in papers [1, 4, 5, 12–18]. Newton's method has been studied in more detail under the so-called Kantorovich conditions (the derivative of the equation operator is invertible at the initial point and satisfies the Lipschitz condition in the considered domain), under the Vertgeim conditions (the operator derivative is invertible at the initial point but satisfies only Hölder condition) and under the Mysovskih conditions (the derivative is invertible at all points in the considered domain and its inverse operator is bounded).

To solve nonlinear functional equations, other iterative methods as well as projection (approximation) type methods are also used; a survey of the relevant literature is contained, for instance, in [8]. In the same source, to solve operator

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equations of the first kind, studies have been performed for methods called the projection-iteration ones based on the following idea. An equation of the form

$$Au = f \tag{1.1}$$

with a nonlinear operator A acting on a Banach space X ($f \in X$ is a known element), is approximated by a sequence of approximate equations

$$A_n u_n = f_n, \quad n = 1,\ 2, \dots, \tag{1.2}$$

where A_n is a nonlinear operator acting on a subspace X_n of the original space ($X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X, \quad X_1 \neq \varnothing$). To solve approximate equations (1.2), some iterative method is used, at that for each of these equations only a few approximations $u_n^{(k)}$ ($k = 1,\ 2, \dots,\ k_n$) are found and the last of them $u_n^{(k_n)}$ is assumed to be equal to the initial approximation $u_{n+1}^{(0)}$ in the iterative process for the next, $(n+1)$-th approximate equation. The sequence $\{u_n^{(k_n)}\}_{n=1}^{\infty} \subset X$ is considered as a sequence of approximations to a solution u^* of equation (1.1). This approach to finding an approximate solution of the original equation naturally eliminates the difficulties that arise when solving the same equation using the conventional projection method and also makes it easier to choose the suitable initial approximation compared to solving the original equation using the iterative method.

In this paper, to solve nonlinear operator equation (1.1), the projection-iteration implementation of the Newton-like method [6] is studied under generalized Cauchy's conditions, which instead of the inverse operator to the derivative in the considered domain imply the existence of some linear operator close to it. The problems of substantiation of the projection-iteration schemes of both the basic Newton-Kantorovich method under such conditions and some of its modifications are considered.

2. Preliminaries

Let us consider equation (1.1) $Au = f$ with a nonlinear operator A which acts on a Banach space X and is Fréchet differentiable on some ball $S(u_N^{(0)}, R) = \{u \in X : \|u - u_N^{(0)}\| \le R\}$ of this space. We approximate equation (1.1) by the sequence of approximate equations (1.2) $A_n u_n = f_n, \quad n = 1,\ 2, \dots$ with nonlinear operators A_n, each of which acts on the respective subspace $X_n \subset X$ and is Fréchet differentiable on the set $\Omega_n = X_n \cap S(u_N^{(0)}, R)$ beginning with some number $n = N \ge 1$; $f_n = P_n f$, P_n is a linear projector which maps X onto X_n $(P_n \colon X \to X_n, \quad P_n u_n = u_n$ for $u_n \in X_n)$.

Assume that for each $n \ge N$ the following proximity conditions hold:

$$\|A_n u_n - P_n A u_n\| \le \alpha_n, \quad \|A_n'(u_n) - P_n A'(u_n)\|_{X_n \to X_n} \le \alpha_n', \quad \forall u_n \in \Omega_n; \tag{2.1}$$

$$\|P_n A u - A u\| \le \beta_n, \quad \|P_n A'(u) - A'(u)\|_{X \to X} \le \beta_n', \quad \forall u \in S(u_N^{(0)}, R); \tag{2.2}$$

$$\|P_n f - f\| \le \gamma_n, \quad \forall f \in X, \tag{2.3}$$

where α_n, α_n', β_n, β_n', $\gamma_n \to 0$ when $n \to \infty$. We will also assume that the derivative $A_n'(u_n)$ on the set Ω_n satisfies the Lipschitz condition

$$\|A_n'(u_n) - A_n'(v_n)\|_{X_n \to X_n} \le L' \, \|u_n - v_n\|, \quad \forall u_n, \, v_n \in \Omega_n, \quad n \ge N, \tag{2.4}$$

where $L' > 0$ is a Lipschitz constant.

If there exists a continuous linear operator $\Gamma_n(u_n) = [A_n'(u_n)]^{-1}$ for all $u_n \in \Omega_n \, (n \ge N)$ then one can apply the Newton-Kantorovich method [14] to each of equations (1.2) beginning from the number $n = N$, and construct a sequence of approximations to the solution u^* of equation (1.1) by the formulas

$$u_n^{(k+1)} = u_n^{(k)} - [A_n'(u_n^{(k)})]^{-1}(A_n u_n^{(k)} - f_n), \quad k = 0, \, 1, \ldots, \, k_n - 1; \tag{2.5}$$
$$u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \ge N; \quad u_N^{(0)} \in \Omega_N \subset X.$$

In paper [3] the theorem is given on the existence of a solution u^* to equation (1.1), on the domain of its location, as well as on the convergence of projection-iteration process (2.5) under the Cauchy-type conditions. The following theorem is a generalization of the mentioned theorem, when instead of operators

$$\Gamma(u) = [A'(u)]^{-1}, \quad u \in S(u_N^{(0)}, R)$$

and $\Gamma_n(u_n) = [A_n'(u_n)]^{-1}$, $u_n \in \Omega_n \;\; (n \ge N)$, it is required the existence only of an operator $D(u)$, $u \in S(u_N^{(0)}, R)$ in X and an operator $D_n(u_n)$, $u_n \in \Omega_n$ in X_n, which are close to $\Gamma(u)$ and $\Gamma_n(u_n)$ respectively.

Theorem 2.1. *Let the operator A be Fréchet differentiable on some ball $S(u_N^{(0)}, R) \subset X$ and let for all $n \ge N$ the operator A_n be Fréchet differentiable on the set $\Omega_n = X_n \cap S(u_N^{(0)}, R)$, at that let its derivative $A_n'(u_n)$ satisfy on Ω_n the Lipschitz condition* (2.4). *Assume that the proximity conditions* (2.1)–(2.3) *hold true and there exist a linear operator $D(u)$ on X and linear operators $D_n(u_n)$ on X_n such that*

$$\|D(u)\|_{X \to X} \le b, \quad \|E - D(u)A'(u)\|_{X \to X} \le \delta < \; 1, \quad \forall u \in S(u_N^{(0)}, R); \tag{2.6}$$

$$\|E - D_n(u_n)A_n'(u_n)\|_{X_n \to X_n} \le \delta_n < \; 1, \quad \forall u_n \in \Omega_n, \quad n \ge N, \tag{2.7}$$

where $b > 0$, $\delta > 0$, $\delta_n > 0$; E is an identity operator on X. If the initial approximation $u_N^{(0)} \in \Omega_N$ satisfies the conditions

$$\|A_N u_N^{(0)} - f_N)\| \le \eta_N^{(0)}, \quad h_N^{(0)} = b_N^2 L' \eta_N^{(0)} < 2, \quad r_N = b_N \eta_N^{(0)} G_N \le R, \tag{2.8}$$

where

$$\begin{aligned} b_N &= b/\left(1 - b(\alpha'_N + \beta'_N) - \delta\right), \\ G_N &= H_N + \sum_{m=N}^{\infty} (h_N^{(0)}/2)^{2^{S_m}-1} < 2H_N, \\ s_m &= \sum_{i=N}^{m} (k_i - 1), \\ H_N &= \sum_{m=0}^{\infty} (h_N^{(0)}/2)^{2^{S_m}-1}, \end{aligned}$$

then equation (1.1) *has in the ball* $S(u_N^{(0)}, r_N)$ *a solution* u^* *to which the based on Newton's method process* (2.5) *converges with the error estimate*

$$\|u_n^{(k_n)} - u^*\| \le b_N \eta_N^{(0)} V_n (h_N^{(0)}/2)^{2^{S_n}-1}, \quad n \ge N, \tag{2.9}$$

where

$$V_n = \sum_{m=0}^{\infty} (h_N^{(0)}/2)^{2^{S_n}(2^m-1)} + \sum_{m=n+1}^{\infty} (h_N^{(0)}/2)^{2^{S_m}-2^{S_n}} < 2H_N.$$

The proof of Theorem 2.1 can be found in [3].

3. Proving the convergence theorem

Let us consider, to solve the operator equation (1.1), a projection-iteration process, like (2.5) with the replacement of the operator $\Gamma_n(u_n^{(k)}) = [A'_n(u_n^{(k)})]^{-1}$ by an operator $D_n(u_n^{(k)})$ close to it:

$$\begin{gathered} u_n^{(k+1)} = u_n^{(k)} - D_n(u_n^{(k)})(A_n u_n^{(k)} - f_n), \quad k = 0,\ 1, \ldots,\ k_n - 1; \\ u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \ge N; \quad u_N^{(0)} \in \Omega_N \subset X. \end{gathered} \tag{3.1}$$

The following theorem establishes the sufficient conditions of feasibility and convergence in the ball $S(u_N^{(0)}, R)$ of the approximations sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty} \subset X$ determined by formulas (3.1) to a solution u^* of equation (1.1).

Theorem 3.1. *Let all the conditions of Theorem 2.1 hold true and let, moreover, the derivative* $A'(u)$ *satisfy on* $S(u_N^{(0)}, R)$ *the Lipschitz condition*

$$\|A'(u) - A'(v)\|_{X \to X} \le L\ \|u - v\|, \quad \forall u, v \in S(u_N^{(0)}, R); \quad L > 0. \tag{3.2}$$

Assume that $bL\delta/(1-\delta) < 1$*, where* $b > 0$*,* $\delta > 0$ *are defined in* (2.6)*, and that* $\delta_n \to 0$ *in condition* (2.7) *when* $n \to \infty$*. If the initial approximation* $u_N^{(0)} \in \Omega_N$ *satisfies the first condition* (2.8)*,*

$$h_N^{(0)} = b_N^2 L' \eta_N^{(0)} + \frac{2 b_N L' \delta_N}{1 - \delta_N} < 2, \quad r_N = b_N \eta_N^{(0)} G_N \le R, \tag{3.3}$$

where

$$\begin{aligned} b_N &= b/(1-b\rho_N), \\ \rho_N &= \alpha'_N + \beta'_N + L'\delta_N/(1-\delta_N) + L\delta/(1-\delta), \\ G_N &= H_N + \sum_{m=N}^{\infty} (h_N^{(0)}/2)^{S_m} < 2H_N, \\ s_m &= \sum_{i=N}^{m} (k_i - 1), \end{aligned}$$

$$H_N = 1/(1-h_N^{(0)}/2),$$

then equation (1.1) *has in the ball* $S(u_N^{(0)}, r_N) \subset X$ *a solution* u^* *to which the projection-iteration process of approximations* (3.1) *converges with the error estimate*

$$\|u_n^{(k_n)} - u^*\| \le b_N \eta_N^{(0)} V_n (h_N^{(0)}/2)^{S_n}, \quad n \ge N, \tag{3.4}$$

where $V_n = H_N + \sum\limits_{m=n+1}^{\infty} (h_N^{(0)}/2)^{S_m - S_n} < 2H_N$.

Proof. First of all, we note that the second condition in (2.6) implies the existence of bounded inverse operator $[D(u)]^{-1}$, $u \in S(u_N^{(0)}, R)$; while taking into account (3.2) the estimate $\|[D(u)]^{-1}\|_{X\to X} \le L/(1-\delta)$ holds for all $u \in S(u_N^{(0)}, R)$. Similarly, from the conditions (2.7) and (2.4) there follows the existence of bounded inverse operators $[D_n(u_n)]^{-1}$, $u_n \in \Omega_n$ with the norm $\|[D_n(u_n)]^{-1}\|_{X_n\to X_n} \le L'/(1-\delta_n)$, $n \ge N$. Further, based on the first condition (2.6) and the proximity conditions (2.1), (2.2) the existence of operators $D_n(u_n)$ implies their boundedness, beginning with some $n = N_1 \ge N$. Indeed, since for $u_n \in \Omega_n$, $z_n \in X_n$

$$\begin{aligned} \|[D_n(u_n)]^{-1} z_n - [D(u_n)]^{-1} z_n\| \le \big(&\|[D_n(u_n)]^{-1} - A'_n(u_n)\|_{X_n\to X_n} \\ &+ \|A'_n(u_n) - P_n A'(u_n)\|_{X_n\to X_n} + \|P_n A'(u_n) - A'(u_n)\|_{X\to X} \\ &+ \|A'(u_n) - [D(u_n)]^{-1}\|_{X\to X} \big) \|z_n\| \le \rho_n \|z_n\|, \end{aligned}$$

where $\rho_n = L'\delta_n/(1-\delta_n) + \alpha'_n + \beta'_n + L\delta/(1-\delta)$, then

$$\begin{aligned} \|[D_n(u_n)]^{-1} z_n\| &\ge \|[D(u_n)]^{-1} z_n\| - \|[D_n(u_n)]^{-1} z_n - [D(u_n)]^{-1} z_n\| \\ &\ge (1-b\rho_n)/b\, \|z_n\|, \end{aligned}$$

and since under the conditions of the theorem $b\rho_n < 1$ for $n \ge N_1$, then for these numbers n we will have

$$\|D_n(u_n)\|_{X_n\to X_n} \le b_n = b/(1-b\rho_n), \quad u_n \in \Omega_n. \tag{3.5}$$

Let us prove the feasibility of process (3.1). Note that the possibility of replacing equations (1.2) by linearized equations

$$A_n u_n^{(k)} + [D_n(u_n^{(k)})]^{-1}(u_n - u_n^{(k)}) = f_n, \quad k = 0, 1, \dots; \quad n \ge N$$

respectively follows from the existence of continuous operators $[D_n(u_n)]^{-1}$ close to $A'_n(u_n)$, $u_n \in \Omega_n$ for the specified n. We establish (by mathematical induction) that all subsequent approximations $u_n^{(0)}$ for $n > N$ have the same properties (2.8), (3.3) and that they belong to the ball $S(u_N^{(0)}, r_N) \subset X$. Based on the theorem conditions, it can be shown that for $n = N,\ N+1, \dots,\ m$

$$\|A_n u_n^{(0)} - f_n\| \le \eta_n^{(0)}, \quad h_n^{(0)} = b_n^2 L' \eta_n^{(0)} + \frac{2b_n L' \delta_n}{1 - \delta_n} < 2. \tag{3.6}$$

In addition, as it follows from the proof of Theorem 2 of [6], at any fixed n $(N \le n \le m)$ the conditions

$$\|A_n u_n^{(k)} - f_n\| \le \eta_n^{(k)}, \quad h_n^{(k)} = b_n^2 L' \eta_n^{(k)} + \frac{2b_n L' \delta_n}{1 - \delta_n} < 2 \tag{3.7}$$

hold for each number $k = 1,\ 2, \dots,\ k_n$. We show the feasibility of (3.6) for $n = m+1$. Insofar as

$$\|A_{m+1} u_{m+1}^{(0)} - f_{m+1}\| \le \|A_{m+1} u_{m+1}^{(0)} - A_m u_{m+1}^{(0)}\| + \|A_m u_m^{(k_m)} - f_m\| + \|f_m - f_{m+1}\|,$$

then based on the proximity conditions (2.1)–(2.3) from the relations

$$\begin{aligned}
\|A_{m+1} u_{m+1}^{(0)} - A_m u_{m+1}^{(0)}\| &\le \|A_{m+1} u_{m+1}^{(0)} - P_{m+1} A u_{m+1}^{(0)}\| \\
&+ \|P_{m+1} A u_{m+1}^{(0)} - A u_{m+1}^{(0)}\| + \|A u_m^{(k_m)} - P_m A u_m^{(k_m)}\| \\
&+ \|P_m A u_m^{(k_m)} - A_m u_m^{(k_m)}\| \le \alpha_{m+1} + \beta_{m+1} + \beta_m + \alpha_m;
\end{aligned}$$

$$\|f_m - f_{m+1}\| \le \|P_m f - f\| + \|f - P_{m+1} f\| \le \gamma_m + \gamma_{m+1}$$

and from the first of the conditions (3.7) we obtain:

$$\|A_{m+1} u_{m+1}^{(0)} - f_{m+1}\| \le \theta_m + \eta_m^{(k_m)} = \eta_{m+1}^{(0)}, \tag{3.8}$$

where $\theta_m = \alpha_m + \alpha_{m+1} + \beta_m + \beta_{m+1} + \gamma_m + \gamma_{m+1}$, that is, the first of the conditions (3.6) for $n = m+1$ holds true. Let us show the fulfillment of the second one.

Proof of the Theorem 2 of [6] implies that for any $k = 0,\ 1, \dots,\ k_m - 1$

$$\begin{aligned}
\|A_m u_m^{(k+1)} - f_m\| &= \|A_m u_m^{(k+1)} - A_m u_m^{(k)} - [D_m(u_m^{(k)})]^{-1}(u_m^{(k+1)} - u_m^{(k)})\| \\
&\le \frac{L'}{2} \|u_m^{(k+1)} - u_m^{(k)}\|^2 + \frac{L'\delta_m}{1 - \delta_m} \|u_m^{(k+1)} - u_m^{(k)}\| \\
&\le \frac{L'}{2} b_m^2 \eta_m^{(k)2} + \frac{L'\delta_m}{1 - \delta_m} b_m \eta_m^{(k)} = \frac{h_m^{(k)}}{2} \eta_m^{(k)} = \eta_m^{(k+1)},
\end{aligned} \tag{3.9}$$

so

$$\eta_m^{(k_m)} = \frac{h_m^{(k_m-1)}}{2} \eta_m^{(k_m-1)} = \dots = \frac{1}{2^{k_m}} h_m^{(k_m-1)} h_m^{(k_m-2)} \dots h_m^{(0)}\, \eta_m^{(0)}.$$

Because in (3.8) $\theta_m \to 0$ when $m \to \infty$ and because by virtue of (3.7) $h_m^{(k_m-1)} < 2$, there exists a number $m = N_2 \geq N$ beginning with which

$$\eta_{m+1}^{(0)} \leq \frac{1}{2^{k_m-1}} h_m^{(k_m-2)} h_m^{(k_m-3)} \ldots h_m^{(0)} \eta_m^{(0)}. \tag{3.10}$$

Since, obviously $b_{m+1} \leq b_m$, $\delta_{m+1} \leq \delta_m$, then taking into account (3.10) and (3.7) we have for all $m \geq N_2$:

$$\begin{aligned} h_{m+1}^{(0)} &= b_{m+1}^2 L' \eta_{m+1}^{(0)} + \frac{2b_{m+1} L' \delta_{m+1}}{1 - \delta_{m+1}} \\ &\leq b_m^2 L' \frac{1}{2^{k_m-1}} h_m^{(k_m-2)} h_m^{(k_m-3)} \ldots h_m^{(0)} \eta_m^{(0)} + \frac{2b_m L' \delta_m}{1 - \delta_m} \\ &= b_m^2 L' \eta_m^{(k_m-1)} + \frac{2b_m L' \delta_m}{1 - \delta_m} = h_m^{(k_m-1)} < 2, \end{aligned} \tag{3.11}$$

that is, the second of the conditions (3.6) for $n = m + 1$ also holds true.

Let number $N := \max\{N_1, N_2\}$ be the initial one in formulas (3.1).

Let's show that the approximations $u_{n+1}^{(0)}$ belong to the ball $S(u_N^{(0)}, r_N) \subset X$ for all $n \geq N$. It's obvious that

$$\|u_{n+1}^{(0)} - u_N^{(0)}\| \leq \sum_{m=N}^{n} \|u_{m+1}^{(0)} - u_m^{(0)}\|, \quad n \geq N;$$

in turn, for each $m = N,\ N+1, \ldots,\ n$

$$\|u_{m+1}^{(0)} - u_m^{(0)}\| = \|u_m^{(k_m)} - u_m^{(0)}\| \leq \sum_{k=0}^{k_m-1} \|u_m^{(k+1)} - u_m^{(k)}\|.$$

Based on formulas (3.1), (3.5), (3.7), (3.9) for any numbers $m = N,\ N+1, \ldots,\ n$ and $k = 0,\ 1, \ldots,\ k_m - 1$ we obtain:

$$\begin{aligned} \|u_m^{(k+1)} - u_m^{(k)}\| &\leq \|D_m(u_m^{(k)})\|_{X_m \to X_m} \|A_m u_m^{(k)} - f_m\| \leq b_m \eta_m^{(k)} \\ &= b_m \frac{1}{2^k} h_m^{(k-1)} h_m^{(k-2)} \ldots h_m^{(0)} \eta_m^{(0)}, \end{aligned}$$

and because of

$$\begin{aligned} h_m^{(k+1)} &= b_m^2 L' \eta_m^{(k+1)} + \frac{2b_m L' \delta_m}{1 - \delta_m} \\ &= b_m^2 L' \frac{h_m^{(k)}}{2} \eta_m^{(k)} + \frac{2b_m L' \delta_m}{1 - \delta_m} \\ &< b_m^2 L' \eta_m^{(k)} + \frac{2b_m L' \delta_m}{1 - \delta_m} \\ &= h_m^{(k)} < 2, \quad k = 0,\ 1, \ldots,\ k_m - 1, \end{aligned} \tag{3.12}$$

we have

$$\|u_m^{(k+1)} - u_m^{(k)}\| \le b_m\big(h_m^{(0)}/2\big)^k \eta_m^{(0)}, \quad k = 0,\ 1, \dots,\ k_m - 1.$$

Let's evaluate here $\eta_m^{(0)}$ and $h_m^{(0)}$ $(N+1 \le m \le n)$ through $\eta_N^{(0)}$ and $h_N^{(0)}$. Applying (3.12) in formulas (3.10) and (3.11), we obtain the relations

$$\eta_{m+1}^{(0)} < \big(h_m^{(0)}/2\big)^{k_m-1}\eta_m^{(0)};$$
$$h_{m+1}^{(0)} \le h_m^{(k_m-1)} < h_m^{(0)} \le h_{m-1}^{(k_{m-1}-1)} < h_{m-1}^{(0)} \le \dots < h_N^{(0)}, \quad m \ge N,$$

which implies that

$$\begin{aligned}\eta_m^{(0)} &< \big(h_{m-1}^{(0)}/2\big)^{k_{m-1}-1}\eta_{m-1}^{(0)} \\ &< \big(h_{m-1}^{(0)}/2\big)^{k_{m-1}-1}\big(h_{m-2}^{(0)}/2\big)^{k_{m-2}-1}\eta_{m-2}^{(0)} < \dots \\ &< \big(h_N^{(0)}/2\big)^{S_{m-1}}\eta_N^{(0)},\end{aligned}$$

where $s_{m-1} = \sum\limits_{i=N}^{m-1}(k_i - 1),\ m = N+1,\ N+2, \dots,\ n$. With this in mind

$$\begin{gathered}\|u_m^{(k+1)} - u_m^{(k)}\| \le b_N\big(h_N^{(0)}/2\big)^{S_{m-1}+k}\eta_N^{(0)}, \\ k = 0,\ 1, \dots,\ k_m - 1; \quad m = N+1,\ N+2, \dots,\ n; \\ \|u_N^{(k+1)} - u_N^{(k)}\| \le b_N\big(h_N^{(0)}/2\big)^k\eta_N^{(0)}, \quad k = 0,\ 1, \dots,\ k_N - 1,\end{gathered} \tag{3.13}$$

so

$$\begin{aligned}\|u_{n+1}^{(0)} - u_N^{(0)}\| &\le \sum_{m=N}^{n}\sum_{k=0}^{k_m-1}\|u_m^{(k+1)} - u_m^{(k)}\| \\ &\le b_N\eta_N^{(0)}\left[\sum_{k=0}^{k_N-1}\big(h_N^{(0)}/2\big)^k + \sum_{m=N+1}^{n}\sum_{k=0}^{k_m-1}\big(h_N^{(0)}/2\big)^{S_{m-1}+k}\right] \\ &= b_N\eta_N^{(0)}\left[\sum_{k=0}^{S_n}\big(h_N^{(0)}/2\big)^k + \sum_{m=N}^{n-1}\big(h_N^{(0)}/2\big)^{S_m}\right] \\ &< b_N\eta_N^{(0)}G_N = r_N, \quad n \ge N,\end{aligned}$$

that is, each $u_{n+1}^{(0)}$ where $n \ge N$ (and also all $u_n^{(k)}$ $(k = 1,\ 2, \dots,\ k_n)$ by virtue of the Theorem 2 from [6]) belong to the ball $S(u_N^{(0)}, r_N)$. Thus, the feasibility of process (3.1) is proved.

Let's now show that the sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty}$, which is determined by formulas (3.1), converges in $S(u_N^{(0)}, r_N)$. Using (3.13) for any numbers $n \ge N$ and $p \in \mathbb{N}$

we write:

$$\begin{aligned}
\|u_{n+p}^{(k_{n+p})} - u_n^{(k_n)}\| &\le \sum_{m=n+1}^{n+p} \|u_m^{(k_m)} - u_{m-1}^{(k_{m-1})}\| = \sum_{m=n+1}^{n+p} \|u_m^{(k_m)} - u_m^{(0)}\| \\
&\le \sum_{m=n+1}^{n+p} \sum_{k=0}^{k_m-1} \|u_m^{(k+1)} - u_m^{(k)}\| \\
&\le b_N \eta_N^{(0)} \sum_{m=n+1}^{n+p} \sum_{k=0}^{k_m-1} \big(h_N^{(0)}/2\big)^{S_{m-1}+k} \\
&= b_N \eta_N^{(0)} \left[\sum_{k=0}^{k_{n+1}-1} \big(h_N^{(0)}/2\big)^{S_n+k} + \sum_{m=n+2}^{n+p} \sum_{k=0}^{k_m-1} \big(h_N^{(0)}/2\big)^{S_{m-1}+k} \right] \\
&= b_N \eta_N^{(0)} \big(h_N^{(0)}/2\big)^{S_n} \left[\sum_{k=0}^{S_{n+p}-S_n} \big(h_N^{(0)}/2\big)^{k} + \sum_{m=n+1}^{n+p-1} \big(h_N^{(0)}/2\big)^{S_m-S_n} \right] \\
&< b_N \eta_N^{(0)} \big(h_N^{(0)}/2\big)^{S_n} 2H_N. \qquad (3.14)
\end{aligned}$$

Since $h_N^{(0)} < 2$, then $\|u_{n+p}^{(k_{n+p})} - u_n^{(k_n)}\| \to 0$ when $n \to \infty$, that means the fundamentality of the sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty} \subset S(u_N^{(0)}, r_N)$. By virtue of the completeness of the space X, there exists an element $u^* \in S(u_N^{(0)}, r_N)$ such that $u^* = \lim\limits_{n\to\infty} u_n^{(k_n)}$. Passing to the limit at $p \to \infty$ in (3.14) and denoting

$$\begin{aligned}
V_n &= \lim_{p\to\infty} \left[\sum_{k=0}^{S_{n+p}-S_n} \big(h_N^{(0)}/2\big)^{k} + \sum_{m=n+1}^{n+p-1} \big(h_N^{(0)}/2\big)^{S_m-S_n} \right] \\
&= \sum_{k=0}^{\infty} \big(h_N^{(0)}/2\big)^{k} + \sum_{m=n+1}^{\infty} \big(h_N^{(0)}/2\big)^{S_m-S_n}, \quad n \ge N,
\end{aligned}$$

we obtain the error estimate (3.4).

To prove that the limit u^* of the sequence $\{u_n^{(k_n)}\}_{n=N}^{\infty}$ is a solution of equation (1.1), we consider the residual of method (3.1) on the n-th step ($n \ge N$):

$$\begin{aligned}
\|Au_n^{(k_n)} - f\| &\le \|Au_{n+1}^{(0)} - P_{n+1}Au_{n+1}^{(0)}\| + \|P_{n+1}Au_{n+1}^{(0)} - A_{n+1}u_{n+1}^{(0)}\| \\
&\quad + \|A_{n+1}u_{n+1}^{(0)} - f_{n+1}\| + \|f_{n+1} - f\| \le \beta_{n+1} + \alpha_{n+1} + \eta_{n+1}^{(0)} + \gamma_{n+1}.
\end{aligned}$$

Since, α_{n+1}, β_{n+1}, γ_{n+1}, $\eta_{n+1}^{(0)} \to 0$ when $n \to \infty$, and since the operator A is continuous due to Fréchet differentiability, then by tending $n \to \infty$ in the last inequality, we obtain that $Au^* = f$. The theorem is proved. □

Note that the projection-iteration implementation (3.1) of the Newton-like method generally converges more slowly than the projection-iteration process (2.5) based on the classical Newton's method. An exception is the case, when $\delta = 0$,

$\delta_n = 0$ $(n \geq N)$ in formulas (2.6), (2.7), that leads to the transformation of method (3.1) into (2.5); in such a situation, the error estimate (3.4) for method (3.1) (or, equivalently, method (2.5)) is significantly overestimated, and for this case the more appropriate result is contained in Theorem 2.1.

For equation (1.1) under the Theorem 2.1 conditions, along with the projection-iteration method (2.5) based on the Newton's method, one can consider the approximation process based on the modified Newton's method:

$$u_n^{(k+1)} = u_n^{(k)} - [A_n'(u_n^{(0)})]^{-1}(A_n u_n^{(k)} - f_n), \quad k = 0,\ 1, \ldots,\ k_n - 1;$$
$$u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \geq N; \quad u_N^{(0)} \in \Omega_N \subset X,$$

and under the Theorem 3.1 conditions, along with the projection-iteration method (3.1), one can consider the approximation process based on the modified Newton-like method:

$$u_n^{(k+1)} = u_n^{(k)} - D_n(u_n^{(0)})(A_n u_n^{(k)} - f_n), \quad k = 0,\ 1, \ldots,\ k_n - 1;$$
$$u_{n+1}^{(0)} = u_n^{(k_n)}, \quad n \geq N; \quad u_N^{(0)} \in \Omega_N \subset X.$$

Such the projection-iteration processes (although they converge more slowly than the process (2.5) based on the Newton's method) are less laborious, since for each $n \geq N$ they use operators $[A_n'(u_n^{(0)})]^{-1}$ or $D_n(u_n^{(0)})$ which correspond only to the initial point $u_n^{(0)} \in \Omega_n$, and this obviously leads to a computational overhead reduction in numerical implementation.

We note, finally, that while solving nonlinear operator equations of the form (1.1), as follows from the proofs of Theorems 2.1, 3.1 on the convergence of projection-iteration methods based on the Newton's method and the Newton-like one respectively, the convergence of corresponding sequences $\{u_n^{(k_n)}\}_{n=N}^{\infty}$ (when $n \to \infty$) towards an exact solution u^* in X occurs under an arbitrary choice of numbers k_n. However, to prevent a sharp increase with increasing n of amount of computations needed to find the next approximation, we have to consider a problem of the appropriate choice of numbers k_n at each $n \geq N$. Some recommendations on this issue have been given in [2]. In particular, there has been considered a way to choose numbers k_n so that the element $u_n^{(k_n)}$ would be a good initial approximation for the $(n+1)$-th approximate equation of the form (1.2), that is, that the residual $A_{n+1}u_{n+1}^{(0)} - f_{n+1}$ would have, if possible, a small value. The idea underlying this way to choose numbers k_n also makes it possible to determine the most acceptable number $n+p$ $(p \geq 1)$ of the approximate equation following the n-th one in the sequence of equations (1.2). Some other ways to choose numbers k_n in projection-iteration methods of solving nonlinear equations as well as their application in solving specific problems, can be found in [3,7,9–11].

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Received 12.03.2019

JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **27**, Issue **1**, June **2019**, pp. 67–88, DOI 10.15421/141904

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

AN EXPLICIT SOLVER TO THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION IN A DISK IN POLYNOMIALS

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Communicated by Prof. V. Kapustyan

Abstract. Explicit formulas for the solution to the Dirichlet problem for the Laplace equation in a disk in polynomials are derived and discussed.

Key words: the Dirichlet problem, explicit solution in polynomials, degenerate elliptic equation.

2010 Mathematics Subject Classification: 31A25, 31B20, 35A09, 35G15, 35J25.

1. Introduction to the solver

Following [1, 2], we consider the well known Dirichlet problem for the Laplace equation in a disk of radius c centered at point $\boldsymbol{x}_0$ in the plane $\mathbb{R}^2$ parameterized by cartesian orthogonal coordinates $\boldsymbol{x} = (x_1, x_2)$

$$\begin{cases} \Delta_{\boldsymbol{x}} u(\boldsymbol{x}) = 0\,, & \boldsymbol{x} \in \mathcal{B}^2_c(\boldsymbol{x}_0) := \left\{\boldsymbol{x} \colon |\boldsymbol{x} - \boldsymbol{x}_0|^2 < c^2\right\}, \\ u(\boldsymbol{x}) = Q'_m(\boldsymbol{x})\,, & \boldsymbol{x} \in \mathcal{S}^2_c(\boldsymbol{x}_0) := \left\{\boldsymbol{x} \colon |\boldsymbol{x} - \boldsymbol{x}_0|^2 = c^2\right\}, \end{cases} \tag{1.1}$$

where the boundary function is a polynomial of degree m

$$Q_m(\boldsymbol{x}) = \sum_{p+q=0}^{m} a_{p,q}\, x_1^p\, x_2^q\,, \tag{1.2}$$

$p, q \in \mathbb{Z}\backslash\mathbb{Z}_-$, $a_{p,q} \in \mathbb{R}$, and prime means that the domain of definition of $Q_m(\boldsymbol{x})$ is restricted to $\mathcal{S}^2_c(\boldsymbol{x}_0)$.

The above problem is known [3] to have a unique solution, and the solution is evidently to be a polynomial $U_m(\boldsymbol{x})$ of degree m.

In [1] we proved the following

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Proposition 1.1. The solution to the Dirichlet problem (1.1) admits the following representation

$$u(\boldsymbol{x}) = U_m(\boldsymbol{x}) = F_2(\boldsymbol{x})\, P_{m-2}(\boldsymbol{x}) + Q_m(\boldsymbol{x})\,, \tag{1.3}$$

where $F_2(\boldsymbol{x})$ is the polynomial of degree 2 specifying the boundary of the disk: $F_2(\boldsymbol{x}) = c^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2 = 0$, and $P_{m-2}(\boldsymbol{x})$ is a uniquely determined polynomial of degree $m-2$.

The way of proving proposition 1.1 was based on the Fourier method. Then, in [2] we gave an other proof based on the Poisson integral formula for the solution. Representation (1.3) was illustrated in [1] by numerous examples where c, $\boldsymbol{x}_0$ and $P_m(\boldsymbol{x})$ were changed: first, we obtained solutions to (1.1) using the Fourier method and, second, showed representation (1.3) to hold by dividing polynomial solution $U_m(\boldsymbol{x})$ by polynomial $F_2(\boldsymbol{x})$ with remainder $Q_m(\boldsymbol{x})$. From this some shrewd readers of [1] concluded that the method we used to prove representation (1.3) gives no explicit formula for polynomial $P_{m-2}(\boldsymbol{x})$. Hence, in the current study we derive the proper explicit formulas for polynomial $P_{m-2}(\boldsymbol{x})$ literally following [1].

The article is arranged as follows.

As in [1], we apply the direct transformation of independent variables $\boldsymbol{x} \to \boldsymbol{y}$: $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{x}_0$, and replace the original Dirichlet problem (1.1) with the following derived one

$$\begin{cases} \Delta_{\boldsymbol{y}} w(\boldsymbol{y}) = 0\,, & \boldsymbol{y} \in \mathcal{B}_c^2(\boldsymbol{0})\,, \\ \quad w(\boldsymbol{y}) = R_m'(\boldsymbol{y})\,, & \boldsymbol{y} \in \mathcal{S}_c^2(\boldsymbol{0})\,, \end{cases} \tag{1.4}$$

where the disk is centered at the origin of cartesian coordinates $\boldsymbol{y} = (y_1, y_2)$, $w(\boldsymbol{y}) := u(\boldsymbol{y} + \boldsymbol{x}_0)$, and

$$R_m(\boldsymbol{y}) := Q_m(\boldsymbol{y} + \boldsymbol{x}_0) = \sum_{p+q=0}^{m} b_{p,q}\, y_1^p\, y_2^q = \sum_{p+q=0}^{m} b_{p,q}\, R_{p,q}(\boldsymbol{y})\,. \tag{1.5}$$

Representation (1.3) for the solution to the derived Dirichlet problem reads

$$w(\boldsymbol{y}) = W_m(\boldsymbol{y}) = G_2(\boldsymbol{y})\, S_{m-2}(\boldsymbol{y}) + R_m(\boldsymbol{y})\,, \tag{1.6}$$

where $G_2(\boldsymbol{y}) := F_2(\boldsymbol{y} + \boldsymbol{x}_0) = c^2 - |\boldsymbol{y}|^2$, $S_{m-2}(\boldsymbol{y}) := P_{m-2}(\boldsymbol{y} + \boldsymbol{x}_0)$.

Then, in Section 2 we consider the contributions

$$W_{p,q}(\boldsymbol{y}) = G_2(\boldsymbol{y})\, S_{p,q}(\boldsymbol{y}) + R_{p,q}(\boldsymbol{y}) \tag{1.7}$$

of the monomials $R_{p,q}(\boldsymbol{y}) = y_1^p y_2^q$ of the boundary polynomial $R_m(\boldsymbol{y})$ (1.5) to the solution $W_m(\boldsymbol{y})$ (1.6), where

$$W_m(\boldsymbol{y}) = \sum_{p+q=0}^{m} W_{p,q}(\boldsymbol{y}) \,, \tag{1.8}$$

$$S_{m-2}(\boldsymbol{y}) = \sum_{p+q=0}^{m} b_{p,q}\, S_{p,q}(\boldsymbol{y}) \,, \tag{1.9}$$

$W_{p,q}(\boldsymbol{y})$ and $S_{p,q}(\boldsymbol{y})$ are polynomials of degree $p+q$ and $p+q-2$ respectively. To find the polynomials $S_{p,q}(\boldsymbol{y})$ we use the Fourier method. For representation (1.3) for the solution to the original Dirichlet problem to be obtained one should apply the inverse transformation of independent variables $\boldsymbol{y} \to \boldsymbol{x} \colon \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$.

In Section 3 we illustrate the resulted explicit formulas of Section 2 for $S_{p,q}(\boldsymbol{y})$ by numerous examples.

In Section 4 we show how to simplify the resulted explicit formulas for $S_{p,q}(\boldsymbol{y})$ and eventually how to find polynomial $S_{m-2}(\boldsymbol{y})$ not treating the monomials $R_{p,q}(\boldsymbol{y})$ separately.

In Section 5 we give supplementary data for readers to check propositions of Section 4.

In Section 6 we discuss briefly some other methods to find polynomials $S_{p,q}(\boldsymbol{y})$.

In Section 7 we show in what way other methods could help us obtain polynomials $S_{p,q}(\boldsymbol{y})$.

2. The Fourier method to find $S_{p,q}(\boldsymbol{y})$

Before applying the well known Fourier method [3] to derive explicit formulas for representation (1.6), we give a brief description of the method to clarify the main idea we utilized in [1].

Let a boundary monomial $R'_{p,q}$ ($p+q \geqslant 3$ and is odd) to have the following Fourier series (possible cases are presented in Tbl. 1 at p. 83)

$$\mathring{R}'_{p,q}(\varphi) = c^{\,p+q} \sum_{\mu=0}^{\frac{p+q-1}{2}} a_{2\mu+1} \cos\left[(2\mu+1)\,\varphi\right],$$

then extension of the boundary monomial to disk $\overline{\mathcal{B}^2_c(\mathbf{0})}$ is evidently to be

$$\mathring{R}'_{p,q}(\varphi) = r^{\,p+q} \sum_{\mu=0}^{\frac{p+q-1}{2}} a_{2\mu+1} \cos\left[(2\mu+1)\,\varphi\right],$$

and the solution to the Dirichlet problem is as follows

$$\mathring{W}_{p,q}(r,\varphi) = c^{p+q} \sum_{\mu=0}^{\frac{p+q-1}{2}} \left(\frac{r}{c}\right)^{2\mu+1} a_{2\mu+1} \cos\left[(2\mu+1)\,\varphi\right],$$

where the circle over the function name indicates changing cartesian coordinates to polar ones: $y_1 = r\cos\varphi$, $y_2 = r\sin\varphi$.

Then we transform the solution identically as

$$\begin{aligned}\mathring{W}_{p,q}(r,\varphi) &= \mathring{W}_{p,q}(r,\varphi) - \mathring{R}_{p,q}(r,\varphi) + \mathring{R}_{p,q}(r,\varphi)\\ &= \sum_{\mu=1}^{\frac{p+q-1}{2}} \left[c^{p+q}\left(\frac{r}{c}\right)^{2\mu+1} - r^{p+q}\right] a_{2\mu+1}\cos\left[(2\mu+1)\,\varphi\right] + \mathring{R}_{p,q}(r,\varphi)\end{aligned}$$

and manipulate the expression in brackets algebraically (provided $2\mu+1 < p+q$ and $5 \leqslant p+q$)

$$\begin{aligned}.c^{p+q}\left(\frac{r}{c}\right)^{2\mu+1} - r^{p+q} &= c^{p+q-2\mu-1}\, r^{2\mu+1} - r^{p+q}\\ &= c^{p+q-2\mu-1}\, r^{2\mu+1} - r^{p+q-2\mu-1}\, r^{2\mu+1}\\ &= r^{2\mu+1}\left(c^{p+q-2\mu-1} - r^{p+q-2\mu-1}\right)\\ &= r^{2\mu+1}\left(c^2 - r^2\right) A_{p+q-2\mu-3}(c,r),\end{aligned}$$

where homogeneous in c and r polynomials are defined as

$$A_{2k}(c,r) = \begin{cases} 1, & k=0,\\ c^{2k} + c^{2k-2}r^2 + \ldots + c^2 r^{2k-2} + r^{2k}, & k>1. \end{cases} \tag{2.1}$$

Eventually we obtain the required representation as

$$\begin{aligned}\mathring{W}_{p,q}(r,\varphi) &= \left(c^2 - r^2\right) \sum_{\mu=1}^{\frac{p+q-1}{2}-1} a_{2\mu+1}\, A_{p+q-2\mu-3}(c,r)\, r^{2\mu+1}\cos\left[(2\mu+1)\,\varphi\right] + \mathring{R}_{p,q}(r,\varphi)\\ &= \left(c^2 - r^2\right) \sum_{\mu=1}^{\frac{p+q-1}{2}-1} a_{2\mu+1}\, A_{p+q-2\mu-3}(c,r)\, H_{2\mu+1,1}(r,\varphi) + \mathring{R}_{p,q}(r,\varphi),\end{aligned}$$

where first there appear harmonic polynomials [8]

$$\mathring{H}_{k,1}(r,\varphi) = r^k \cos k\varphi, \qquad \mathring{H}_{k,2}(r,\varphi) = r^k \sin k\varphi. \tag{2.2}$$

To obtain the Fourier series of boundary monomials $R_{p,q}$ we use well known formulas [4] for powers of trigonometric functions cos and sin in terms of these functions of multiples of the argument

$$\begin{cases} \cos^p \varphi = \dfrac{1}{2^{p-1}} \displaystyle\sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \, \cos \varphi_\mu \,, \\ \sin^q \varphi = \dfrac{1}{2^{q-1}} \displaystyle\sum_{\gamma=0}^{\frac{q-1}{2}} C_q^\gamma \, \sin \varphi_\gamma \, (-1)^{\frac{q-1}{2}+\gamma} \,, \end{cases} \tag{2.3}$$

when p and q are odd integers, and

$$\begin{cases} \cos^p \varphi = \dfrac{1}{2^p} \, C_p^{\frac{p}{2}} + \dfrac{1}{2^{p-1}} \displaystyle\sum_{\mu=0}^{\frac{p}{2}-1} C_p^\mu \, \cos \varphi_\mu \,, \\ \sin^q \varphi = \dfrac{1}{2^q} \, C_q^{\frac{q}{2}} + \dfrac{1}{2^{q-1}} \displaystyle\sum_{\gamma=0}^{\frac{q}{2}-1} C_q^\gamma \, \cos \varphi_\gamma \, (-1)^{\frac{q}{2}+\gamma} \,, \end{cases} \tag{2.4}$$

when p and q are even integers; and $\varphi_\mu := p_\mu \, \varphi = (p - 2\mu) \, \varphi$, $\varphi_\gamma := q_\gamma \, \varphi = (q - 2\gamma) \, \varphi$, and formulas

$$\begin{cases} 2 \, \cos \varphi_1 \, \cos \varphi_2 = \cos (\varphi_1 - \varphi_2) + \cos (\varphi_1 + \varphi_2) \,, \\ 2 \, \sin \varphi_1 \, \cos \varphi_2 = \sin (\varphi_1 - \varphi_2) + \sin (\varphi_1 + \varphi_2) \,. \end{cases} \tag{2.5}$$

To transform harmonic polynomials (2.2) to cartesian variables we use well known formulas [4] for trigonometric functions cos and sin of multiples of the argument in terms of powers of these functions (for p, q being odd and even respectively)

$$\begin{cases} \cos p\varphi = \displaystyle\sum_{\mu=0}^{\frac{p-1}{2}} (-1)^\mu \, C_p^{2\mu} \, \cos^{p-2\mu} \varphi \, \sin^{2\mu} \varphi \,, \\ \sin q\varphi = \displaystyle\sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^\gamma \, C_q^{2\gamma+1} \, \cos^{q-2\gamma-1} \varphi \, \sin^{2\gamma+1} \varphi \,, \end{cases} \tag{2.6}$$

$$\begin{cases} \cos p\varphi = \displaystyle\sum_{\mu=0}^{\frac{p}{2}} (-1)^{\mu}\, C_p^{2\mu}\, \cos^{p-2\mu}\varphi\, \sin^{2\mu}\varphi\,, \\ \sin q\varphi = \displaystyle\sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\gamma}\, C_q^{2\gamma+1}\, \cos^{q-2\gamma-1}\varphi\, \sin^{2\gamma+1}\varphi\,. \end{cases} \tag{2.7}$$

2.1. Trivial cases

We call trivial those cases when the boundary monomials are harmonic ones of degree 1 and 2: a) $p = 1$, $q = 0$; b) $p = 0$, $q = 1$; c) $p = 1$, $q = 1$. Then polynomials $S_{1,0}$, $S_{0,1}$ and $S_{1,1}$ are evidently to equal zero identically.

2.2. Other missing cases

Other missing cases are as follows: a) $p \geqslant 3$ is odd and $q = 0$; b) $p = 0$ and $q > 1$ is odd; c) $p \geqslant 2$ is even and $q = 0$; d) $p = 0$ and $q \geqslant 2$ is even. They are thoroughly studied or discussed in [1].

2.3. $p \geqslant 1$ is odd and $q \geqslant 1$ is odd, $p + q \geqslant 4$

First we find the Fourier series for the given boundary monomial restricted to $\mathcal{S}_c^2(\mathbf{0})$

$$\begin{aligned} \mathring{R}'_{p,q}(\varphi) &= c^{\,p} \cos^p\varphi\; c^{\,q} \sin^q\varphi \\ &= \frac{c^{\,p+q}}{2^{p+q-2}} \left(\sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu} \cos\varphi_{\mu} \right) \left(\sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_q^{\gamma} \sin\varphi_{\gamma} \right) \\ &= \frac{c^{\,p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \sin(\varphi_{\gamma} + \varphi_{\mu}) \\ &+ \frac{c^{\,p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \sin(\varphi_{\gamma} - \varphi_{\mu})\,. \end{aligned}$$

Then we extend the boundary monomial to $\overline{\mathcal{B}_c^2(\mathbf{0})}$

$$\begin{aligned} \mathring{R}_{p,q}(r,\varphi) = \frac{r^{\,p+q}}{2^{p+q-1}} &\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \sin(\varphi_{\gamma} + \varphi_{\mu}) \\ + \frac{r^{\,p+q}}{2^{p+q-1}} &\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \sin(\varphi_{\gamma} - \varphi_{\mu})\,, \end{aligned}$$

set up the solution to the Dirichlet problem following the Fourier method and apply the identical transformation discussed in Section 2

$$\begin{aligned}
\mathring{W}_{p,q}(r,\varphi) &= \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{\mu+\gamma>0} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} \left[c^{p+q} \left(\frac{r}{c}\right)^{q_\gamma + p_\mu} - r^{p+q} \right] \sin(\varphi_{\gamma+\mu}) \\
&+ \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu<q_\gamma} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} \left[c^{p+q} \left(\frac{r}{c}\right)^{q_\gamma - p_\mu} - r^{p+q} \right] \sin(\varphi_{\gamma-\mu}) \\
&- \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu>q_\gamma} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} \left[c^{p+q} \left(\frac{r}{c}\right)^{q_\gamma - p_\mu} - r^{p+q} \right] \sin(\varphi_{\mu-\gamma}) \\
&+ \mathring{R}_{p,q}(r,\varphi)\,,
\end{aligned}$$

where the following auxiliary notation is used: $\varphi_{\gamma+\mu} = \varphi_\gamma + \varphi_\mu$, $\varphi_{\gamma-\mu} = \varphi_\gamma - \varphi_\mu$, $\varphi_{\mu-\gamma} = \varphi_\mu - \varphi_\gamma$.

After some algebraic manipulations discussed and explained in Section 2 we present the polynomial $S_{p,q}$ in polar variables separately

$$\begin{aligned}
2^{p+q-1}\, \mathring{S}_{p,q}(r,\varphi) &= \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{\mu+\gamma>0} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} A_{2(\mu+\gamma)-2}(c,r)\, r^{p_\mu+q_\gamma} \sin\left[(q_\gamma + p_\mu)\,\varphi\right] \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu<q_\gamma} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} A_{2(p+\gamma-\mu)-2}(c,r)\, r^{q_\gamma-p_\mu} \sin\left[(q_\gamma - p_\mu)\,\varphi\right] \\
&- \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu>q_\gamma} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} A_{2(q+\mu-\gamma)-2}(c,r)\, r^{p_\mu-q_\gamma} \sin\left[(p_\mu - q_\gamma)\,\varphi\right].
\end{aligned}$$

Transforming independent variables from cartesian to polar ones we eventually find polynomial $S_{p,q}$

$$
\begin{aligned}
2^{p+q-1}\, S_{p,q}(\boldsymbol{y}) = {} & \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{\mu+\gamma>0} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(\mu+\gamma)-2}(c,|\boldsymbol{y}|)\, H_{p+q-2(\mu_\gamma),2}(\boldsymbol{y}) \\
& + \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu<q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(p+\gamma-\mu)-2}(c,|\boldsymbol{y}|)\, H_{q-p-2(\gamma-\mu),2}(\boldsymbol{y}) \\
& - \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu>q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(q+\mu-\gamma)-2}(c,|\boldsymbol{y}|)\, H_{p-q-2(\mu-\gamma),2}(\boldsymbol{y})
\end{aligned}
\tag{2.8}
$$

2.4. $p \geqslant 1$ is odd and $q \geqslant 2$ is even, $p+q \geqslant 3$

We again start considering new case with finding the Fourier series for the given boundary monomial restricted to $\mathcal{S}_c^2(\mathbf{0})$

$$
\begin{aligned}
\mathring{R}'_{p,q}(\varphi) &= c^{p}\cos^{p}\varphi\; c^{q}\sin^{q}\varphi \\
&= c^{p+q}\left(\frac{1}{2^{p-1}}\sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu}\cos\varphi_\mu\right)\left(\frac{1}{2^{q}}\, C_q^{\frac{q}{2}} + \frac{1}{2^{q-1}}\,\sum_{\gamma=0}^{\frac{q}{2}-1}(-1)^{\frac{q}{2}+\gamma}\, C_q^{\gamma}\sin\varphi_\gamma\right) \\
&= \frac{c^{p+q}}{2^{p+q-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu}\cos\varphi_\mu \\
&+ \frac{c^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\cos(\varphi_\mu+\varphi_\gamma) \\
&+ \frac{c^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\cos(\varphi_\mu-\varphi_\gamma)\,.
\end{aligned}
$$

Then as usually we extend the boundary monomial to the disk

$$
\begin{aligned}
\mathring{R}'_{p,q}(\varphi) &= \frac{r^{\,p+q}}{2^{p+q-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu} \cos\varphi_\mu \\
&\quad + \frac{r^{\,p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \cos(\varphi_\mu+\varphi_\gamma) \\
&\quad + \frac{r^{\,p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \cos(\varphi_\mu-\varphi_\gamma)\,,
\end{aligned}
$$

set up the solution to the problem and implement the identical transformation as follows

$$
\begin{aligned}
\mathring{W}_{p,q}(r,\varphi) &= \frac{1}{2^{p+q-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{p_\mu} - r^{\,p+q} \right] \cos\varphi_\mu \\
&\quad + \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{\mu+\gamma>0} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{q_\gamma+p_\mu} - r^{\,p+q} \right] \cos(\varphi_{\gamma+\mu}) \\
&\quad + \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu<q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{q_\gamma-p_\mu} - r^{\,p+q} \right] \cos(\varphi_{\gamma-\mu}) \\
&\quad + \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu>q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{q_\gamma-p_\mu} - r^{\,p+q} \right] \cos(\varphi_{\mu-\gamma}) \\
&\quad + \mathring{R}_{p,q}(r,\varphi)\,.
\end{aligned}
$$

Now the polynomial $S_{p,q}$ has been ready to be presented in polar variables

$$
\begin{aligned}
2^{p+q-1}\,\mathring{S}_{p,q}(r,\varphi) &= C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu}\, A_{q+2\mu-2}(c,r)\, r^{\,p_\mu} \cos p_\mu \varphi \\
&= \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{\mu+\gamma>0} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(\mu+\gamma)-2}(c,r)\, r^{\,p_\mu+q_\gamma} \cos\left[(p_\mu+q_\gamma)\,\varphi\right] \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu<q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(p+\gamma-\mu)-2}(c,r)\, r^{\,q_\gamma-p_\mu} \cos\left[(q_\gamma-p_\mu)\,\varphi\right] \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu>q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(q+\mu-\gamma)-2}(c,r)\, r^{\,p_\mu-q_\gamma} \cos\left[(p_\mu-q_\gamma)\,\varphi\right]
\end{aligned}
$$

and finally in cartesian ones

$$
\begin{aligned}
2^{p+q-1}\,S_{p,q}(\boldsymbol{y}) &= C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu}\, A_{q+2\mu-2}(c,|\boldsymbol{y}|)\, H_{p-2\mu,1}(\boldsymbol{y}) \\
&= \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{\mu+\gamma>0} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(\mu+\gamma)-2}(c,|\boldsymbol{y}|)\, H_{p+q-2(\mu+\gamma),1}(\boldsymbol{y}) \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu<q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(p+\gamma-\mu)-2}(c,|\boldsymbol{y}|)\, H_{q-p-2(\gamma-\mu),1}(\boldsymbol{y}) \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu>q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(q+\mu-\gamma)-2}(c,|\boldsymbol{y}|)\, H_{p-q-2(\mu-\gamma),1}(\boldsymbol{y})\,.
\end{aligned}
\tag{2.9}
$$

2.5. $p \geqslant 2$ is even and $q \geqslant 1$ is odd, $p+q \geqslant 3$

The Fourier series for the boundary monomial

$$\begin{aligned}
\mathring{R}'_{p,q}(\varphi) &= c^p \cos^p\varphi \; c^q \sin^q\varphi \\
&= c^{p+q} \left(\frac{1}{2^p} C_p^{\frac{p}{2}} + \frac{1}{2^{p-1}} \sum_{\mu=0}^{\frac{p}{2}-1} C_p^{\mu} \cos\varphi_\mu \right) \left(\frac{1}{2^{q-1}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma} C_q^{\gamma} \sin\varphi_\gamma \right) \\
&= \frac{c^{p+q}}{2^{p+q-1}} C_p^{\frac{p}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma} C_q^{\gamma} \sin\varphi_\gamma \\
&+ \frac{c^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} \sin(\varphi_\gamma + \varphi_\mu) \\
&+ \frac{c^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} \sin(\varphi_\gamma - \varphi_\mu)
\end{aligned}$$

is uniquely extended to the disk as the following function of polar variables

$$\begin{aligned}
\mathring{R}_{p,q}(r,\varphi) &= \frac{r^{p+q}}{2^{p+q-1}} C_p^{\frac{p}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma} C_q^{\gamma} \sin\varphi_\gamma \\
&+ \frac{r^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} \sin(\varphi_\gamma + \varphi_\mu) \\
&+ \frac{r^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma} C_p^{\mu} C_q^{\gamma} \sin(\varphi_\gamma - \varphi_\mu)\,.
\end{aligned}$$

Then the solution to the problem is set up as follows

$$\begin{aligned}
\mathring{W}_{p,q}(r,\varphi) &= \frac{1}{2^{p+q-1}}\, C_p^{\frac{p}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_q^{\gamma} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{q_\gamma} - r^{\,p+q} \right] \sin\varphi_\gamma \\
&+ \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{\mu+\gamma>0} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{q_\gamma+p_\mu} - r^{\,p+q} \right] \sin(\varphi_{\gamma+\mu}) \\
&+ \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu<q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{q_\gamma-p_\mu} - r^{\,p+q} \right] \sin(\varphi_{\gamma-\mu}) \\
&- \frac{1}{2^{p+q-1}} \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu>q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left[c^{\,p+q} \left(\frac{r}{c}\right)^{p_\mu-q_\gamma} - r^{\,p+q} \right] \sin(\varphi_{\mu-\gamma}) \\
&+ \mathring{R}_{p,q}(r,\varphi)\,,
\end{aligned}$$

from where the required polynomial is obtained in polar

$$\begin{aligned}
2^{p+q-1}\, \mathring{S}_{p,q}(r,\varphi) &= C_p^{\frac{p}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_q^{\gamma}\, A_{p+2\gamma-2}(c,r)\, r^{\,q_\gamma} \sin q_\gamma\varphi \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{\mu+\gamma>0} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(\mu+\gamma)-2}(c,r)\, r^{\,p_\mu+q_\gamma} \sin\left[(p_\mu+q_\gamma)\,\varphi\right] \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu<q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(p+\gamma-\mu)-2}(c,r)\, r^{\,q_\gamma-p_\mu} \sin\left[(q_\gamma-p_\mu)\,\varphi\right] \\
&- \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu>q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(q+\mu-\gamma)-2}(c,r)\, r^{\,p_\mu-q_\gamma} \sin\left[(p_\mu-q_\gamma)\,\varphi\right]
\end{aligned}$$

and then in cartesian variables

$$
\begin{aligned}
2^{p+q-1}\, S_{p,q}(\boldsymbol{y}) &= C_p^{\frac{p}{2}} \sum_{\gamma=0}^{\frac{q-1}{2}} (-1)^{\frac{q-1}{2}+\gamma}\, C_q^{\gamma}\, A_{p+2\gamma-2}(c,|\boldsymbol{y}|)\, H_{q-2\gamma,1}(\boldsymbol{y}) \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{\mu+\gamma>0} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(\mu+\gamma)-2}(c,|\boldsymbol{y}|)\, H_{p+q-2(\mu+\gamma),2}(\boldsymbol{y}) \\
&+ \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu<q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(p+\gamma-\mu)-2}(c,|\boldsymbol{y}|)\, H_{q-p-2(\gamma-\mu),2}(\boldsymbol{y}) \\
&- \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q-1}{2}}}_{p_\mu>q_\gamma} (-1)^{\frac{q-1}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(q+\mu-\gamma)-2}(c,|\boldsymbol{y}|)\, H_{p-q-2(\mu-\gamma),2}(\boldsymbol{y})\,.
\end{aligned}
\tag{2.10}
$$

2.6. $p \geqslant 2$ is even and $q \geqslant 2$ is even, $p+q \geqslant 4$

Using the Fourier series for the boundary monomial

$$
\begin{aligned}
\mathring{R}'_{p,q}(\varphi) &= c^p \cos^p\varphi\; c^q \sin^q\varphi \\
&= c^{p+q} \left(\frac{C_p^{\frac{p}{2}}}{2^p} + \frac{1}{2^{p-1}} \sum_{\mu=0}^{\frac{p}{2}-1} C_p^{\mu} \cos\varphi_\mu \right) \left(\frac{C_q^{\frac{q}{2}}}{2^q} + \frac{1}{2^{q-1}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_q^{\gamma} \sin\varphi_\gamma \right) \\
&+ \frac{c^{p+q}}{2^{p+q}}\, C_p^{\frac{p}{2}}\, C_q^{\frac{q}{2}} \\
&+ \frac{c^{p+q}}{2^{p+q-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p}{2}-1} C_p^{\mu} \cos(\varphi_\mu) + \frac{c^{p+q}}{2^{p+q-1}}\, C_p^{\frac{p}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_q^{\gamma} \cos(\varphi_\gamma) \\
&+ \frac{c^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \cos(\varphi_\gamma + \varphi_\mu) \\
&+ \frac{c^{p+q}}{2^{p+q-1}} \sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \cos(\varphi_\gamma - \varphi_\mu)
\end{aligned}
$$

and not extending it explicitly to the disk set up the solution to the problem in polar variables and implement the identical transformation

$$
\begin{aligned}
\mathring{W}_{p,q}(r,\varphi) = {} & \frac{1}{2^{p+q-1}}\left(\frac{1}{2}\,C_p^{\frac{p}{2}}\,C_q^{\frac{q}{2}} + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1}\sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu=q_\gamma}(-1)^{\frac{q}{2}+\gamma}\,C_p^{\mu}\,C_q^{\gamma}\right)\left[c^{\,p+q} - r^{\,p+q}\right] \\
& + \frac{1}{2^{p+q-1}}\sum_{\mu=0}^{\frac{p}{2}-1} C_p^{\mu}\left[c^{\,p+q}\left(\frac{r}{c}\right)^{p_\mu} - r^{\,p+q}\right]\cos\varphi_\mu \\
& + \frac{1}{2^{p+q-1}}\sum_{\gamma=0}^{\frac{q}{2}-1}(-1)^{\frac{q}{2}+\gamma}\,C_q^{\gamma}\left[c^{\,p+q}\left(\frac{r}{c}\right)^{q_\gamma} - r^{\,p+q}\right]\cos\varphi_\gamma \\
& + \frac{1}{2^{p+q-1}}\underbrace{\sum_{\mu=0}^{\frac{p}{2}-1}\sum_{\gamma=0}^{\frac{q}{2}-1}}_{\mu+\gamma>0}(-1)^{\frac{q}{2}+\gamma}\,C_p^{\mu}\,C_q^{\gamma}\left[c^{\,p+q}\left(\frac{r}{c}\right)^{q_\gamma+p_\mu} - r^{\,p+q}\right]\cos\varphi_{\gamma+\mu} \\
& + \frac{1}{2^{p+q-1}}\underbrace{\sum_{\mu=0}^{\frac{p}{2}-1}\sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu<q_\gamma}(-1)^{\frac{q}{2}+\gamma}\,C_p^{\mu}\,C_q^{\gamma}\left[c^{\,p+q}\left(\frac{r}{c}\right)^{q_\gamma-p_\mu} - r^{\,p+q}\right]\cos\varphi_{\gamma-\mu} \\
& + \frac{1}{2^{p+q-1}}\underbrace{\sum_{\mu=0}^{\frac{p}{2}-1}\sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu>q_\gamma}(-1)^{\frac{q}{2}+\gamma}\,C_p^{\mu}\,C_q^{\gamma}\left[c^{\,p+q}\left(\frac{r}{c}\right)^{p_\mu-q_\gamma} - r^{\,p+q}\right]\cos\varphi_{\mu-\gamma} \\
& + \mathring{R}_{p,q}(r,\varphi)\,.
\end{aligned}
$$

This case needs much more work compared to previous cases, and eventually we obtain the polynomial $S_{p,q}$ in polar variables

$$
\begin{aligned}
2^{p+q-1}\,\mathring{S}_{p,q}(r,\varphi) = & \left(\frac{1}{2}\, C_p^{\frac{p}{2}}\, C_q^{\frac{q}{2}} + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu = q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \right) A_{p+q-2}(r) \\
& + \sum_{\mu=0}^{\frac{p}{2}-1} C_p^{\mu}\, A_{q+2\mu-2}(r)\, r^{\,p_\mu} \cos\left(p_\mu \varphi\right) \\
& + \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_q^{\gamma}\, A_{p+2\gamma-2}(r)\, r^{\,q_\gamma} \cos\left(q_\gamma \varphi\right) \\
& + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{\mu+\gamma>0} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(\mu+\gamma)-2}(r)\, r^{\,q_\gamma+p_\mu} \cos\left[\left(p_\mu + q_\gamma\right)\varphi\right] \\
& + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu<q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(p+\gamma-\mu)-2}(r)\, r^{\,q_\gamma-p_\mu} \cos\left[\left(q_\gamma - p_\mu\right)\varphi\right] \\
& + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu>q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(q+\mu-\gamma)-2}(r)\, r^{\,p_\mu-q_\gamma} \cos\left[\left(p_\mu - q_\gamma\right)\varphi\right].
\end{aligned}
$$

Transforming the polynomial $S_{p,q}$ to cartesian variables is quite routine to obtain

$$
\begin{aligned}
2^{p+q-1}\, S_{p,q}(\boldsymbol{y}) = {} & \left(\frac{1}{2}\, C_p^{\frac{p}{2}}\, C_q^{\frac{q}{2}} + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu = q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma} \right) A_{p+q-2}(|\boldsymbol{y}|) \\
& + \sum_{\mu=0}^{\frac{p}{2}-1} C_p^{\mu}\, A_{q+2\mu-2}(|\boldsymbol{y}|)\, H_{p-2\mu,1}(\boldsymbol{y}) \\
& + \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}+\gamma}\, C_q^{\gamma}\, A_{p+2\gamma-2}(|\boldsymbol{y}|)\, H_{q-2\gamma,1}(\boldsymbol{y}) \\
& + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{\mu+\gamma>0} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(\mu+\gamma)-2}(|\boldsymbol{y}|)\; H_{p+q-2(\mu+\gamma),1}(\boldsymbol{y}) \\
& + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu < q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(p+\gamma-\mu)-2}(|\boldsymbol{y}|)\, H_{q-p-2(\gamma-\mu),1}(\boldsymbol{y}) \\
& + \underbrace{\sum_{\mu=0}^{\frac{p}{2}-1} \sum_{\gamma=0}^{\frac{q}{2}-1}}_{p_\mu > q_\gamma} (-1)^{\frac{q}{2}+\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, A_{2(q+\mu-\gamma)-2}(|\boldsymbol{y}|)\, H_{p-q-2(\mu-\gamma),1}(\boldsymbol{y})\,.
\end{aligned}
\tag{2.11}
$$

3. Examples on the Fourier method to find $S_{p,q}(\boldsymbol{y})$

Here and below we consider the Dirichlet problem (1.1), where $\boldsymbol{x}_0 = \boldsymbol{0}$, hence, all the resulted formulas, namely (2.8) – (2.11), are valid when formally replacing variables $\boldsymbol{y}$ with $\boldsymbol{x}$. For readers interested in deriving the explicit formulas for the coefficients of polynomials $S_{p,q}$ we place in Table 1 the Fourier series of all the boundary monomials $\mathring{R}'_{p,q}$. The resulted polynomials $S_{p,q}$ are placed in Table 2 and could attract attention of shrewd readers by very high level of packing the independent variables.

For example, the polynomial $S_{6,6}$ is written (in Table 2) using 3 non-zero coefficients (essential, or primary), but admits the following partial expanding

Table 1. The Fourier series of boundary monomials $\mathring{R}'_{p,q}(\varphi)$

No	The Fourier series
1	$\mathring{R}'_{7,0} = +\frac{35}{64}\, c^7 \cos\varphi + \frac{21}{84}\, c^7 \cos 3\varphi + \frac{7}{64}\, c^7 \cos 5\varphi + \frac{1}{64}\, c^7 \cos 7\varphi$
2	$\mathring{R}'_{0,7} = +\frac{35}{64}\, c^7 \sin\varphi - \frac{21}{84}\, c^7 \sin 3\varphi + \frac{7}{64}\, c^7 \sin 5\varphi + \frac{1}{64}\, c^7 \sin 7\varphi$
3	$\mathring{R}'_{8,0} = +\frac{35}{128}\, c^8 + \frac{7}{16}\, c^8 \cos 2\varphi + \frac{7}{32}\, c^8 \cos 4\varphi + \frac{1}{16}\, c^8 \cos 6\varphi + \frac{1}{128}\, c^8 \cos 8\varphi$
4	$\mathring{R}'_{8,0} = +\frac{35}{128}\, c^8 - \frac{7}{16}\, c^8 \cos 2\varphi + \frac{7}{32}\, c^8 \cos 4\varphi - \frac{1}{16}\, c^8 \cos 6\varphi + \frac{1}{128}\, c^8 \cos 8\varphi$
5	$\mathring{R}'_{3,3} = +\frac{3}{32}\, c^6 \sin 2\varphi - \frac{1}{32}\, c^6 \sin 6\varphi$
6	$\mathring{R}'_{3,4} = +\frac{3}{64}\, c^7 \cos\varphi - \frac{3}{64}\, c^7 \cos 3\varphi - \frac{1}{64}\, c^7 \cos 5\varphi + \frac{1}{64}\, c^7 \cos 7\varphi$
7	$\mathring{R}'_{4,3} = +\frac{3}{64}\, c^7 \sin\varphi + \frac{3}{64}\, c^7 \sin 3\varphi - \frac{1}{64}\, c^7 \sin 5\varphi - \frac{1}{64}\, c^7 \sin 7\varphi$
8	$\mathring{R}'_{4,4} = +\frac{3}{128}\, c^8 - \frac{1}{32}\, c^8 \cos 4\varphi + \frac{1}{128}\, c^8 \cos 8\varphi$
9	$\mathring{R}'_{3,5} = +\frac{3}{64}\, c^8 \sin 2\varphi - \frac{1}{64}\, c^8 \sin 4\varphi - \frac{1}{64}\, c^8 \sin 6\varphi + \frac{1}{128}\, c^8 \sin 8\varphi$
10	$\mathring{R}'_{5,3} = +\frac{3}{64}\, c^8 \sin 2\varphi + \frac{1}{64}\, c^8 \sin 4\varphi - \frac{1}{64}\, c^8 \sin 6\varphi - \frac{1}{128}\, c^8 \sin 8\varphi$
11	$\mathring{R}'_{3,6} = +\frac{3}{128}\, c^9 \cos\varphi - \frac{1}{32}\, c^9 \cos 3\varphi + \frac{3}{256}\, c^9 \cos 7\varphi - \frac{1}{256}\, c^9 \cos 9\varphi$
12	$\mathring{R}'_{6,3} = +\frac{3}{128}\, c^9 \sin\varphi + \frac{1}{32}\, c^9 \sin 3\varphi - \frac{3}{256}\, c^9 \sin 7\varphi - \frac{1}{256}\, c^9 \sin 9\varphi$
13	$\mathring{R}'_{5,5} = +\frac{5}{256}\, c^{10} \sin 2\varphi - \frac{5}{512}\, c^{10} \sin 6\varphi + \frac{1}{512}\, c^{10} \sin 10\varphi$
14	$\mathring{R}'_{6,6} = +\frac{5}{1024}\, c^{12} - \frac{15}{2048}\, c^{12} \cos 4\varphi + \frac{3}{1024}\, c^{12} \cos 8\varphi - \frac{1}{2048}\, c^{12} \cos 12\varphi$

$$\begin{aligned} 2^{11}\, S_{6,6} = {} & 10\left(c^{10} + c^8|\boldsymbol{x}|^2 + c^6|\boldsymbol{x}|^4 + c^4|\boldsymbol{x}|^6 + c^2|\boldsymbol{x}|^8 + |\boldsymbol{x}|^{10}\right) \\ & - 15\left(c^6 + c^4|\boldsymbol{x}|^2 + c^2|\boldsymbol{x}|^4 + \boldsymbol{x}|^6\right)\left(x_1^4 - 6\, x_1^2 x_2^2 + x_2^4\right) \\ & + \ 6\left(c^2 + |\boldsymbol{x}|^2\right)\left(x_1^8 - 28\, x_1^6 x_2^2 + 70\, x_1^4 x_2^4 - 28\, x_1^2 x_2^6 + x_2^8\right). \end{aligned} \tag{3.1}$$

Fully expanded polynomial $S_{6,6}$ is given in Table 4.

Table 2. Polynomials $S_{p,q}(\boldsymbol{x})$ (arguments $(c, |\boldsymbol{x}|)$ of the homogeneous polynomials A and argument $(\boldsymbol{x})$ of the harmonic polynomials H are not shown for brevity; '# of terms' in the last column refers to fully expanded polynomials, see Table 4 in Section 7)

No	$S_{p,q}(\boldsymbol{x})$	# of terms
1	$2^6\, S_{7,0} = +35A_4H_{1,1} + 21A_2H_{3,1} + 7A_0H_{5,1}$	6
2	$2^6\, S_{0,7} = +35A_4H_{1,2} - 21A_2H_{3,2} + 7A_0H_{5,2}$	6
3	$2^7\, S_{8,0} = +35A_6H_{0,1} + 56A_4H_{2,1} + 28A_2H_{4,1} + 8A_0H_{6,1}$	10
4	$2^7\, S_{0,8} = +35A_6H_{0,1} - 56A_4H_{2,1} + 28A_2H_{4,1} - 8A_0H_{6,1}$	10
5	$2^5\, S_{3,3} = +3A_2H_{2,2}$	3
6	$2^6\, S_{3,4} = +3A_4H_{1,1} - 3A_2H_{3,1} - A_0H_{5,1}$	5
7	$2^6\, S_{4,3} = +3A_4H_{1,2} + 3A_2H_{3,2} - A_0H_{5,2}$	5
8	$2^7\, S_{4,4} = +3A_6H_{0,1} - 4A_2H_{4,1}$	10
9	$2^7\, S_{3,5} = +6A_4H_{2,2} - 2A_2H_{4,2} - 2A_0H_{6,2}$	6
10	$2^7\, S_{5,3} = +6A_4H_{2,2} + 2A_2H_{4,2} - 2A_0H_{6,2}$	6
11	$2^8\, S_{3,6} = +6A_6H_{1,1} - 8A_4H_{3,1} + 3A_0H_{7,1}$	10
12	$2^8\, S_{6,3} = +6A_6H_{1,2} + 8A_4H_{3,2} - 3A_0H_{7,2}$	10
13	$2^9\, S_{5,5} = +10A_6H_{2,2} - 5A_2H_{6,2}$	10
14	$2^{11}\, S_{6,6} = +10A_{10}H_{0,1} - 15A_6H_{4,1} + 6A_2H_{8,1}$	21

4. Some improvements to find $S_{p,q}(\boldsymbol{y})$

From the explicit formulas (2.8) – (2.11) derived in Section 2 and the examples considered in Section 3 one could deduce that the contribution of a monomial $R_{p,q}(\boldsymbol{y})$ to the resulted polynomial $S_{m-2}(\boldsymbol{y})$ (1.9) is given as

$$2^{p+q-1}\, S_{p,q}(\boldsymbol{y}) = \sum_{\rho=0}^{\frac{p+q-1}{2}} A_{2\rho}(c, |\boldsymbol{y}|)\Big(\alpha_{2\rho}\, H_{p+q-2-2\rho,1}(\boldsymbol{y}) + \sigma_{2\rho}\, H_{p+q-2-2\rho,2}(\boldsymbol{y})\Big), \quad (4.1)$$

when $p+q$ is odd, and as

$$2^{p+q-1}\, S_{p,q}(\boldsymbol{y}) = \sum_{\rho=0}^{\frac{p+q}{2}-1} A_{2\rho}(c, |\boldsymbol{y}|)\Big(\alpha_{2\rho}\, H_{p+q-2-2\rho,1}(\boldsymbol{y}) + \sigma_{2\rho}\, H_{p+q-2-2\rho,2}(\boldsymbol{y})\Big), \quad (4.2)$$

when $p+q$ is even, where the coefficients $\alpha_{2\rho}$, $\sigma_{2\rho}$ are fully determined by the Fourier series of the monomial $R'_{p,q}(\boldsymbol{y})$ through formulas (2.8) – (2.11).

We show here how to simplify the computation of the coefficients $\alpha_{2\rho}$, $\sigma_{2\rho}$.

Proposition 4.1. The Fourier series of the Laplacian of a monomial $R_{p,q}(\boldsymbol{y})$, i. e. function $\Delta_{\boldsymbol{y}} R_{p,q}(\boldsymbol{y})$ restricted to $\mathcal{S}_c^2(\mathbf{0})$, fully determines (the coefficients $\alpha_{2\rho}, \sigma_{2\rho}$ of) the polynomial $S_{p,q}(\boldsymbol{y})$.

Proof. First, we derive a partial differential equation the polynomial $S_{p,q}(\boldsymbol{y})$ satisfies. For this we substitute representation (1.7) into the differential equation of the problem (1.4)

$$\Delta_{\boldsymbol{y}} \Big(\big(c^2 - |\boldsymbol{y}|^2\big)\, S_{p,q} \Big) = -\Delta_{\boldsymbol{y}} R_{p,q} \tag{4.3}$$

to obtain

$$\big(c^2 - |\boldsymbol{y}|^2\big)\, \Delta_{\boldsymbol{y}} S_{p,q} - 4 \left(y_1 \, \frac{\partial S_{p,q}}{\partial y_1} + y_2 \, \frac{\partial S_{p,q}}{\partial y_2} + S_{p,q} \right) = -\Delta_{\boldsymbol{y}} R_{p,q}\,, \tag{4.4}$$

a degenerate elliptic linear partial differential equation with variable coefficients.

Second, when $|\boldsymbol{y}| \to c$, the higher order terms of (4.4) vanish, and we obtain the following Robin boundary condition

$$4 \left(r \, \frac{\partial \mathring{S}_{p,q}(r,\varphi)}{\partial r} + \mathring{S}_{p,q}(r,\varphi) \right) \Bigg|_{r=c} = \Delta_{(r,\varphi)}\, \mathring{R}_{p,q}(r,\varphi) \big|_{r=c}\,, \tag{4.5}$$

written in polar variables.

Third, substituting representation (4.1) or (4.2) for the polynomial $S_{p,q}(\boldsymbol{y})$ on the left-hand side of the boundary condition (4.5) we obtain a trigonometric polynomial, whereas the right-hand side of (4.5) is easily reduced due to formulas (..) to the Fourier series of function $\Delta_{\boldsymbol{y}} R_{p,q}(\boldsymbol{y})$ restricted to $\mathcal{S}_c^2(\mathbf{0})$.

Comparing the respective coefficients of both trigonometric polynomials we easily find simple algebraic relations between the unknown coefficients of the polynomial $S_{p,q}(\boldsymbol{y})$ and the Fourier series of function $\Delta_{\boldsymbol{y}} R_{p,q}(\boldsymbol{y})$ restricted to $\mathcal{S}_c^2(\mathbf{0})$. For the sake of brevity we do not derive here the proper relations and leave this derivation to shrewd readers. $\square$

Since the Dirichlet problem (1.4) is linear it immediately follows from proposition 4.1 the following

Proposition 4.2. The Fourier series of the Laplacian of the boundary polynomial $R_m(\boldsymbol{y})$, i. e. function $\Delta_{\boldsymbol{y}} R_m(\boldsymbol{y})$ restricted to $\mathcal{S}_c^2(\mathbf{0})$, fully determines the polynomial $S_{m-2}(\boldsymbol{y})$.

5. Examples on improvements to find $S_{p,q}(\boldsymbol{y})$

For readers interested in deriving the explicit formulas for the coefficients of polynomials $S_{p,q}(\boldsymbol{y})$ we place in Table 3 the Fourier series of the Laplacians of monomials $R_{p,q}$, where the monomials $R_{p,q}$ are the same as in Table 1. The Fourier series in Table 3 supplemented with the boundary condition (4.5) fully determine the polynomials $S_{p,q}$ in Table 2.

Table 3. The Fourier series of functions $\Delta_{\boldsymbol{x}} R_{p,q}(\boldsymbol{x})$ restricted to $\mathcal{S}_c^2(\mathbf{0})$

No	The Fourier series
1	$\big(\Delta_{\boldsymbol{x}} R_{7,0}\big)' = +\dfrac{105}{4}\, c^5 \cos\varphi + \dfrac{105}{8}\, c^5 \cos 3\varphi + \dfrac{21}{8}\, c^5 \cos 5\varphi$
2	$\big(\Delta_{\boldsymbol{x}} R_{0,7}\big)' = +\dfrac{105}{4}\, c^5 \sin\varphi - \dfrac{105}{8}\, c^5 \sin 3\varphi + \dfrac{21}{8}\, c^5 \sin 5\varphi$
3	$\big(\Delta_{\boldsymbol{x}} R_{8,0}\big)' = +\dfrac{35}{2}\, c^6 + \dfrac{105}{4}\, c^6 \cos 2\varphi + \dfrac{21}{2}\, c^8 \cos 4\varphi + \dfrac{7}{4}\, c^6 \cos 6\varphi$
4	$\big(\Delta_{\boldsymbol{x}} R_{0,8}\big)' = +\dfrac{35}{2}\, c^6 - \dfrac{105}{4}\, c^6 \cos 2\varphi + \dfrac{21}{2}\, c^8 \cos 4\varphi - \dfrac{7}{4}\, c^6 \cos 6\varphi$
5	$\big(\Delta_{\boldsymbol{x}} R_{3,3}\big)' = +3\, c^4 \sin 2\varphi$
6	$\big(\Delta_{\boldsymbol{x}} R_{3,4}\big)' = +\dfrac{9}{4}\, c^5 \cos\varphi - \dfrac{15}{8}\, c^5 \cos 3\varphi - \dfrac{3}{8}\, c^5 \cos 5\varphi$
7	$\big(\Delta_{\boldsymbol{x}} R_{4,3}\big)' = +\dfrac{9}{4}\, c^5 \sin\varphi + \dfrac{15}{8}\, c^5 \sin 3\varphi - \dfrac{3}{8}\, c^5 \sin 5\varphi$
8	$\big(\Delta_{\boldsymbol{x}} R_{4,4}\big)' = +\dfrac{3}{2}\, c^6 - \dfrac{3}{2}\, c^6 \cos 4\varphi$
9	$\big(\Delta_{\boldsymbol{x}} R_{3,5}\big)' = +\dfrac{45}{16}\, c^6 \sin 2\varphi - \dfrac{3}{4}\, c^8 \sin 4\varphi - \dfrac{7}{16}\, c^6 \sin 6\varphi$
10	$\big(\Delta_{\boldsymbol{x}} R_{5,3}\big)' = +\dfrac{45}{16}\, c^6 \sin 2\varphi + \dfrac{3}{4}\, c^8 \sin 4\varphi - \dfrac{7}{16}\, c^6 \sin 6\varphi$
11	$\big(\Delta_{\boldsymbol{x}} R_{3,6}\big)' = +\dfrac{15}{8}\, c^7 \cos\varphi - \dfrac{9}{4}\, c^7 \cos 3\varphi + \dfrac{3}{8}\, c^7 \cos 7\varphi$
12	$\big(\Delta_{\boldsymbol{x}} R_{6,3}\big)' = +\dfrac{15}{8}\, c^7 \sin\varphi + \dfrac{9}{4}\, c^7 \sin 3\varphi - \dfrac{3}{8}\, c^7 \sin 7\varphi$
13	$\big(\Delta_{\boldsymbol{x}} R_{5,5}\big)' = +\dfrac{15}{8}\, c^8 \sin 2\varphi - \dfrac{5}{8}\, c^8 \sin 6\varphi$
14	$\big(\Delta_{\boldsymbol{x}} R_{6,6}\big)' = +\dfrac{45}{64}\, c^{10} - \dfrac{15}{16}\, c^{10} \cos 4\varphi + \dfrac{15}{64}\, c^{10} \cos 8\varphi$

6. Some other methods to find $S_{p,q}$

Suppose we know that representation (1.6) holds for the solution $W_m(\boldsymbol{y})$ to the Dirichlet problem (1.4). Then we can find polynomial $S_{p,q}(\boldsymbol{y})$ using a lot of methods different from that developed in Section 2.

First, we can use the degenerate elliptic partial differential equation (4.4). This equation needs no boundary condition to be solved, hence, the method of undetermined coefficients is applicable.

Second, for solving the Poisson differential equation (4.3) with zero boundary values the Ritz method [6, 7] ideally suits.

Third, for solving the above problem for the Poisson differential equation the Green integral formula is applicable, where the Green function is given in [5].

The above methods give required polynomials through a huge bulk of computational work, contrary to the method of Section 2 or its improved version of Section 4. In other words, the above methods are implicit ones.

7. Examples on other methods to find $S_{p,q}$

Polynomials $S_{p,q}$ obtained by any method of Section 6 are fully expanded. Some of them are placed in Table 4.

Table 4. Some fully expanded polynomials $S_{p,q}(\boldsymbol{x})$

No	$S_{p,q}(\boldsymbol{x})$
1	$2^6\,S_{7,0} = +63\,x^5 - 42\,x^3y^2 + 7\,xy^4 + 56\,c^2x^3 - 28\,c^2xy^2 + 35\,c^4x$
4	$2^7\,S_{0,8} = -x^6 + 29\,x^4y^2 - 99\,x^2y^4 + 127\,y^6$
	$+ 7\,c^2x^4 - 98\,c^2x^2y^2 + 119\,c^2y^4 - 21\,c^4x^2 + 91\,c^4y^2 + 35\,c^6$
9	$2^7\,S_{3,5} = -8\,x^5y + 64\,x^3y^3 + 8\,xy^5 + 4\,c^2x^3y + 20\,c^2xy^3 + 12\,c^4xy$
14	$2^{11}\,S_{6,6} = +x^{10} - 67\,x^8y^2 + 562\,x^6y^4 + 562\,x^4y^6 - 67\,x^2y^8 + y^{10}$
	$+ c^2x^8 - 68\,c^2x^6y^2 + 630\,c^2x^4y^4 - 68\,c^2x^2y^6 + c^2y^8$
	$- 5\,c^4x^6 + 105\,c^4x^4y^2 + 105\,c^4x^2y^4 - 5\,c^4y^6$
	$- 5\,c^6x^4 + 110\,c^6x^2y^2 - 5\,c^6y^4 + 10\,c^8x^2 + 10\,c^8y^2 + 10\,c^{10}$

8. Conclusions

1. An explicit solver for the Dirichlet problem for the Laplace equation in a disk in polynomials is presented.

2. The solver essentially uses representation (1.3) for the solution to the problem but admits very high level of packing independent variables.

3. The level of packing independent variables is influenced by the number of essential coefficients of the solution.

4. The number of essential coefficients of the solution equals the number of non-zero Fourier coefficients of function ΔR_m restricted to the boundary of the disk, where R_m is the boundary polynomial.

9. Acknowledgements

We express our sincere gratefulness to Prof. P. K. Kogut for his assistance and support during submitting the article.

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Received 05.06.2019

Journal of Optimization, Differential Equations and Their Applications

Volume 27 Issue 1 June 2019

For notes

Journal of Optimization, Differential Equations and Their Applications

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Volume 27 **Issue 1** **June 2019**

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Journal of Optimization, Differential Equations and Their Applications

Volume 27 **Issue 1** **June 2019**

ISSN 2617-0108

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