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UNIFORM ATTRACTORS FOR VANISHING VISCOSITY APPROXIMATIONS OF NON-AUTONOMOUS COMPLEX FLOWS

Nataliia V. Gorban,* Oleksiy V. Kapustyan,† Pavlo O. Kasyanov,‡
Olha V. Khomenko,§ Liliia S. Paliichuk,¶ José Valero,‖
Michael Z. Zgurovsky**

Abstract. In this paper we prove the existence of uniform global attractors in the strong topology of the phase space for semiflows generated by vanishing viscosity approximations of some class of non-autonomous complex fluids.

Key words: non-Newtonian fluids, parabolic equations, global attractors, infinite-dimensional dynamical systems..

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1. Introduction

In this paper we consider a non-autonomous evolution problem which appears in the investigation of the model of concentrated suspensions (proposed by Hebraud and Lequex [12]) with non-autonomous coefficients. More precisely, the unknown function $p(x,t)$, representing probability density, satisfies the following equation:

$$\frac{\partial p}{\partial t} = -b(t)\frac{\partial p}{\partial x} + D(p)\frac{\partial^2 p}{\partial x^2} - \chi_{\mathbb{R}\setminus[-1,1]}(x)p + \frac{D(p)}{\alpha}\delta_0(x), \tag{1.1}$$

where $\alpha > 0$ is a parameter, $\chi_{\mathbb{R}\setminus[-1,1]}$ is the characteristic function of the open set $\mathbb{R}\setminus[-1,1]$, δ_0 is the Dirac delta function with support at the origin,

$$D(f) = \alpha \int\limits_{|x|>1} f(x)dx,$$

*Institute for Applied System Analysis, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, nata_gorban@i.ua

†Taras Shevchenko National University of Kyiv, Kyiv, Ukraine, alexkap@univ.kiev.ua

‡Institute for Applied System Analysis, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, kasyanov@i.ua

§Institute for Applied System Analysis, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, olgkhomenko@ukr.net

¶Institute for Applied System Analysis, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, lili262808@gmail.com

‖Universidad Miguel Hernandez de Elche, Centro de Investigación Operativa, Avda. Universidad s/n, 03202-Elche (Alicante), Spain, jvalero@umh.es

**National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"Peremogy ave., 37, 03056, Kyiv, Ukraine, mzz@kpi.ua

and the function $b(t)$ is assumed to be non-autonomous. Moreover, mechanical background of the model requires boundedness with respect to the time of the average stress function

$$\tau(t) = \int_{\mathbb{R}} xp(t,x)dx.$$

Existence and uniqueness results for such model were proved in [4]. The theory of global attractors was applied first for (1.1) in Amigó et al. [1], where the existence of global unbounded attractors with respect to the weak topology was proved for the case $b(t) \equiv 0$. Numerical aspects were investigated in [2,13]. The key point in [4,13] was the analysis of the so-called vanishing viscosity approximation system, where the diffusion coefficient was everywhere positive. In [3,5–10,14–22] the existence of global attractor in the strong topology of the phase space for m-semiflow generated by vanishing viscosity approximation was proved. Only autonomous (i.e. $b(t) \equiv const$) case was considered. In the present paper we extend results from [14] to much more general non-autonomous case, using the uniform global attractor approach [11,23–26].

2. Setting of the problem and preliminaries

Let $\alpha > 0$ be a positive constant, $0 \le \varepsilon \ll 1$ be a small parameter, and $b : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Consider the following evolution problem with non-degenerate diffusion:

$$\frac{\partial p}{\partial t} = -b(t)\frac{\partial p}{\partial x} + (D(p)+\varepsilon)\frac{\partial^2 p}{\partial x^2} - \chi_{\mathbb{R}\setminus[-1,1]}(x)p + \frac{D(p)}{\alpha}\delta_0(x), \text{ a.e. in } \mathbb{R}\times\mathbb{R}_+; \tag{2.1}$$

$$p(x,t) \ge 0, \text{ a.e. in } \mathbb{R}\times\mathbb{R}_+; \tag{2.2}$$

$$\int_{\mathbb{R}} p(x,t)dx = 1, \text{ a.e. in } \mathbb{R}_+; \tag{2.3}$$

$$\int_{\mathbb{R}} |x|p(x,t)dx < \infty, \text{ a.e. in } \mathbb{R}. \tag{2.4}$$

Suppose that b is an essentially bounded function, that is, there exists a constant $B > 0$ such that

$$|b(t)| \le B \text{ for a.e. } t > 0. \tag{2.5}$$

Further we will use the following notation:

$$L^p = L^p(\mathbb{R}), \;\; H^1 = H^1(\mathbb{R}), \;\; H^{-1} = (H^1)^*,$$

for each $1 \le p \le \infty$. Let $\langle\cdot,\cdot\rangle$ be the pairing on $H^{-1}\times H^1$ (on $L^q \times L^p$ respectively with $p \ge 1$ and $1 < q \le \infty$ such that $\frac{1}{p}+\frac{1}{q} = 1$) that coincides with the inner product on L^2, that is,

$$\langle f,u\rangle = \int_{\mathbb{R}} f(x)u(x)dx,$$

for each $f \in L^2$ and $u \in H^1$ (for each $f \in L^q$ and $u \in L^p$, respectively).

Let $0 \le \tau < T < \infty$ be arbitrary fixed. A solution of equation (2.1) on a finite time interval $[\tau, T]$ is defined as follows.

Definition 2.1. Let $0 < \varepsilon \ll 1$. A function $p \in L^\infty(\tau, T; L^1 \cap L^2) \cap L^2(\tau, T; H^1)$ with $\frac{\partial p}{\partial t} \in L^2(\tau, T; H^{-1})$ is called a (weak) solution of equation (2.1) on $[\tau, T]$, if the equality

$$\int_\tau^T \left(\langle \frac{\partial p}{\partial t}, \eta \rangle + b(t) \langle \frac{\partial p}{\partial x}, \eta \rangle + (D(p(\,\cdot\,, t)) + \varepsilon) \langle \frac{\partial p}{\partial x}, \frac{\partial \eta}{\partial x} \rangle + \int_{|x|>1} p \cdot \eta \, dx \right) dt$$
$$= \int_\tau^T \frac{D(p(\,\cdot\,, t))}{\alpha} \langle \delta_0, \eta \rangle dt, \tag{2.6}$$

holds for each $\eta \in L^2(\tau, T; H^1)$.

Remark 2.1. We note that the right hand-side of equality (2.6) is equal to

$$\int_\tau^T \frac{D(p(t))}{\alpha} \eta(0, t) dt.$$

Remark 2.2. Let $0 < \varepsilon \ll 1$, and p be a solution of equation (2.1) on $[\tau, T]$. Since $p \in L^2(\tau, T; H^1)$ and $\frac{\partial p}{\partial t} \in L^2(\tau, T; H^{-1})$, then $p \in C([\tau, T]; L^2)$, and, therefore, the following initial condition

$$p|_{t=\tau} = p_\tau(x), \text{ a.e. in } \mathbb{R}, \tag{2.7}$$

makes sense for $p_\tau \in L^1 \cap L^2$.

Let

$$X := \{p \in L^2(\mathbb{R}) \,:\, \int_{\mathbb{R}} |x| \, |p(x)| \, dx < \infty\},$$

which is a Banach space with the norm

$$\|p\|_X := \|p\|_{L^2} + \int_{\mathbb{R}} |x| \, |p(x)| \, dx, \quad p \in X.$$

Remark 2.3. The embedding $X \subset L^1 \cap L^2$ is continuous. Moreover, $X = \overline{L}^1 \cap L^2$, where

$$\overline{L}^1 := \{p \in L^1 \,:\, \int_{\mathbb{R}} |x| \, |p| \, dx < \infty\}$$

is a Banach space with the following norm:

$$\|p\|_{\overline{L}^1} := \int_{\mathbb{R}} (1 + |x|) \, |p| \, dx, \quad p \in \overline{L}^1.$$

We understand condition (2.4) in the sense of the following definition.

Definition 2.2. The solution p of equation (2.1) on $[\tau, T]$ satisfies condition (2.4) on $[\tau, T]$ if $xp \in L^\infty(\tau, T; L^1)$.

Remark 2.4. Let p be a solution of equation (2.1) on $[\tau, T]$. Then $xp \in L^\infty(\tau, T; L^1)$ if and only if $p \in L^\infty(\tau, T; X)$. Moreover, since $p \in L^\infty(0, T; X)$, $p \in C([0, T]; L^2)$, and $X \subset L^2$, we have that $p \in C([0, T]; X_w)$.

Let $0 < \varepsilon \ll 1$ be arbitrary fixed. Cancès et al. [4, Proposition 2.1] proved that for each p_τ such that

$$p_\tau \in L^1 \cap L^\infty,\ \ p_\tau \geq 0,\ \int_{\mathbb{R}} p_\tau(x)dx = 1,\ \int_{\mathbb{R}} |x| p_\tau(x)dx < \infty, \tag{2.8}$$

problem (2.1)–(2.4), (2.7) on $[\tau, T]$ has a unique solution p. Moreover,

$$\begin{aligned} p \in L^\infty(\mathbb{R} \times (\tau, T)),\ \ \sigma p \in L^\infty\left(0, T; L^1\right), \\ p \in C([\tau, T]; L^2 \cap L^1),\ \ D(p) \in C([\tau, T]), \end{aligned}$$

and

$$\int_{\mathbb{R}} p\,(t, \sigma)\, d\sigma = 1,\ \ p\,(t) \geq 0 \text{ for all } t \geq 0. \tag{2.9}$$

Therefore, the phase space for this problem can be defined as follows:

$$H := \mathrm{cl}_X E,\ \ E := \{p \in X\ :\ p \in L^\infty,\ p \geq 0,\ \int_{\mathbb{R}} p(x)dx = 1\},$$

where cl_X is the closure in the space X (see Amigó et al. [1]). The convexity of E implies the equality $H = \mathrm{cl}_{X_w} E$.

Remark 2.5. For $0 < \varepsilon \ll 1$ it is easy to show that for every $p_\tau \in E$ $p \in C([\tau, T]; (L^1 \cap L^\infty)_w)$. In particular, we have that $p(t) \in E$ for each $t \in [\tau, T]$. Therefore, for each $p \in H$ the following two conditions hold: (a) $p(x) \geq 0$ for a.e. $x \in \mathbb{R}$, and (b) $\int_{\mathbb{R}} p(x)dx = 1$ [1, p. 212]. Moreover, for each $0 < \varepsilon \ll 1$, $0 \leq \tau < T < \infty$, and $p_\tau \in H$ there exists no more than one solution p of problem (2.1)-(2.3), (2.7) on $[\tau, T]$.

The main goal of the present paper is to show the existence of uniform global attractors in the strong topology of the phase space H for the m-semiflow generated by the non-autonomous problem (2.1)–(2.4).

3. Existence and properties of solutions

In this section we provide results from [14] about existence and topological properties of (2.1)-(2.4).

Let $\mathcal{K}^+_{\tau,\varepsilon}$ ($\mathcal{D}^+_{\tau,\varepsilon}$) denotes the family of all globally defined solutions of problem (2.1)-(2.3) ((2.1)-(2.4)) on $[\tau, \infty)$ with $p\,(\tau) \in H$. By definition, $\mathcal{D}^+_{\tau,\varepsilon} \subseteq \mathcal{K}^+_{\tau,\varepsilon}$

Lemma 3.1. [14, Lemma 3.1] *There exists a constant $C > 0$ such that, if*

$$0 \le \varepsilon \ll 1,\ \tau \ge 0 \text{ and } p \in \mathcal{K}^+_{\tau,\varepsilon} \text{ with } p(\tau) \in H,$$

then $p \in \mathcal{D}^+_{\tau,\varepsilon}$ and the following inequality holds:

$$\|p(t)\|_{\overline{L}^1} \le \|p(\tau)\|_{\overline{L}^1} e^{-\frac{1}{2}(t-\tau)} + C, \tag{3.1}$$

for each $t \ge \tau$. Moreover, for each $\delta > 0$ and a bounded set (in $\overline{L}^1$) $K \subset H$ there exist constants $T = T(\delta, K) > 0$ and $\bar{k} = \bar{k}(\delta, K) > 0$ such that for each $0 \le \varepsilon \ll 1$, $\tau \ge 0$, and $p \in \mathcal{K}^+_{\tau,\varepsilon}$ with $p(\tau) \in K$ the following inequality holds:

$$\int_{|x|>2k} p(x,t)|x| dx \le \delta, \tag{3.2}$$

for each $t \ge \tau + T$ and $k \ge \bar{k}$.

Remark 3.1. According to Lemma 3.1, each globally defined solution p of problem (2.1)–(2.3) on $[\tau, \infty)$ with $\tau \ge 0$, $0 \le \varepsilon \ll 1$, and $p(\tau) \in H$, belongs to $L^\infty(\tau, \infty; \overline{L}^1)$. In particular, the following equality holds:

$$\mathcal{D}^+_{\tau,\varepsilon} = \{p \in \mathcal{K}^+_{\tau,\varepsilon} : p(\tau) \in H\}.$$

The following result guaranties existence and dissipativity for the problem (2.1)-(2.4).

Theorem 3.1. *Let $0 < \varepsilon \ll 1$. Then for every $p_\tau \in H$ problem (2.1)–(2.4), (2.7) on $[\tau, T]$ has a unique solution p. Moreover, $p \in C([\tau, T]; H)$. Moreover, there exists $R_0 > 0$ such that for an arbitrary bounded (in L^2) set $K \subset H$ and for arbitrary $\varepsilon \in (0, 1)$ there exists a moment of time $T = T(K, \varepsilon)$ such that for every $\tau \ge 0$ and $p \in \mathcal{D}^+_{\tau,\varepsilon}$ satisfying $p(\tau) \in K$ the following inequality holds:*

$$\|p(t)\|_{L^2} \le R_0, \tag{3.3}$$

for each $t \ge \tau + T$.

The next result guaranties the continuous properties of solutions of (2.1)-(2.4).

Theorem 3.2. [14, Lemma 3.3] *Let $0 \le \tau < T < \infty$, $p^n_\tau \in H$, $b_n \in L^\infty(\tau, T)$, and $0 < \varepsilon_n \ll 1$ for each $n = 0, 1, \dots$. Suppose that $|b_n(t)| \le B$ for a.e. $t \in (\tau, T)$ and $p^n \in C([\tau, T]; H_w)$ be a solution of problem (2.1)–(2.4), (2.7) on $[\tau, T]$ with parameters $p^n_\tau, \varepsilon_n, b_n$, for each $n \ge 1$. If*

$$p^n_\tau \to p^0_\tau \text{ in } H_w,\ \varepsilon_n \to \varepsilon_0 > 0,\ b_n \to b_0 \text{ weakly-star in } L^\infty(\tau, T),$$

then there exists a solution $p \in C([\tau, T]; H_w)$ of problem (2.1)–(2.4), (2.7) on $[\tau, T]$ with parameters $p^0_\tau, \varepsilon_0, b_0$, such that up to a subsequence the following convergence holds:

$$p^n \to p \text{ in } C([\tau, T]; H_w). \tag{3.4}$$

Moreover, if $p^n_\tau \to p^0_\tau$ in H, then the following statements hold:

(a) $p, p^n \in C([\tau, T]; H)$ *for each* $n \geq 1$;

(b) the following convergence holds for the entire sequence:

$$p^n \to p \text{ in } L^2(\tau, T; H^1), \tag{3.5}$$

$$p^n \to p \text{ in } C([\tau, T]; H). \tag{3.6}$$

If, additionally, $b_n \to b_0$ *in the Lebesgue measure on* $[\tau, T]$, *then*

$$\frac{\partial p^n}{\partial t} \to \frac{\partial p}{\partial t} \text{ in } L^2(\tau, T; H^{-1}). \tag{3.7}$$

4. Existence and properties of uniform global attractors in the non-autonomous case

To characterize the uniform long-time behavior of solutions for non-autonomous dissipative dynamical system consider the *united trajectory space* $\mathcal{K}^+_{\varepsilon,\cup}$ for the family of solutions $\{\mathcal{K}^+_{\varepsilon,\tau}\}_{\tau\geq 0}$ shifted to zero:

$$\mathcal{K}^+_{\varepsilon,\cup} := \bigcup_{\tau\geq 0} \left\{ T(h)y(\,\cdot + \tau\,) \,:\, y(\,\cdot\,) \in \mathcal{K}^+_{\varepsilon,\tau},\, h \geq 0 \right\}, \tag{4.1}$$

and the *extended united trajectory space* for the family $\{\mathcal{K}^+_{\varepsilon,\tau}\}_{\tau\geq 0}$:

$$\mathcal{K}^+_{\varepsilon} := \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)} \left[\mathcal{K}^+_{\varepsilon,\cup} \right], \tag{4.2}$$

where $\mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}[\,\cdot\,]$ is the closure in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Since $T(h)\mathcal{K}^+_{\varepsilon,\cup} \subseteq \mathcal{K}^+_{\varepsilon,\cup}$ for each $h \geq 0$, then

$$T(h)\mathcal{K}^+_{\varepsilon} \subseteq \mathcal{K}^+_{\varepsilon} \text{ for each } h \geq 0, \tag{4.3}$$

due to

$$\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(T(h)u, T(h)v) \leq \rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(u, v) \text{ for each } u, v \in C^{\mathrm{loc}}(\mathbb{R}_+; H),$$

where $\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ is the standard metric on Fréchet space $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Therefore the set

$$\mathbb{X} := \{y(0) \,:\, y \in \mathcal{K}^+_{\varepsilon}\} \tag{4.4}$$

is closed in H. We endow this set $\mathbb{X}$ with metric

$$\rho_{\mathbb{X}}(x_1, x_2) = \|x_1 - x_2\|_X, \quad x_1, x_2 \in \mathbb{X}.$$

Then we obtain that ($\mathbb{X}$,ρ) is a Polish space (complete separable metric space).

Let us define the multivalued semiflow (*m-semiflow*) $V_\varepsilon : \mathbb{R}_+ \times \mathbb{X} \to 2^{\mathbb{X}}$:

$$V_\varepsilon(t, y_0) := \{y(t) \,:\, y(\cdot) \in \mathcal{K}^+_{\varepsilon} \text{ and } y(0) = y_0\}, \quad t \geq 0,\, y_0 \in \mathbb{X}. \tag{4.5}$$

According to (4.3) and (4.4) for each $t \geq 0$ and $y_0 \in \mathbb{X}$ the set $V_\varepsilon(t, y_0)$ is nonempty. Moreover, the following two conditions hold:

(i) $V_\varepsilon(0,\cdot) = I$ is the identity map;

(ii) $V_\varepsilon(t_1+t_2, y_0) \subseteq V_\varepsilon(t_1, V_\varepsilon(t_2, y_0))$, $\forall t_1, t_2 \in \mathbb{R}_+$, $\forall y_0 \in \mathbb{X}$,

where $V_\varepsilon(t, D) = \underset{y\in D}{\cup} V_\varepsilon(t, y)$, $D \subseteq \mathbb{X}$.

We denote by $\mathrm{dist}_{\mathbb{X}}(C, D) = \sup_{c\in C} \inf_{d\in D} \rho_{\mathbb{X}}(c, d)$ the *Hausdorff semidistance* between nonempty subsets C and D of the Polish space $\mathbb{X}$. Recall that the compact set $\Theta_\varepsilon \subset \mathbb{X}$ is a *global attractor* of the m-semiflow V_ε if it satisfies the following conditions:

(i) Θ_ε attracts each bounded subset $B \subset \mathbb{X}$, i.e.

$$\mathrm{dist}_{\mathbb{X}}(V_\varepsilon(t, B), \Theta_\varepsilon) \to 0, \quad t \to +\infty; \tag{4.6}$$

(ii) Θ_ε is negatively semi-invariant set, that is, $\Theta_\varepsilon \subseteq V_\varepsilon(t, \Theta_\varepsilon)$ for each $t \geq 0$.

In this paper we examine the uniform long-time behavior of solution sets $\{\mathcal{K}^+_{\tau,\varepsilon}\}_{\tau\geq 0}$ in the strong topology of the natural phase space H (as time $t \to +\infty$ for a fixed $\varepsilon > 0$) in the sense of the existence of a compact global attractor for m-semiflow V_ε generated by the family of solution sets $\{\mathcal{K}^+_{\tau,\varepsilon}\}_{\tau\geq 0}$ and their shifts.

Theorem 4.1. *For each $\varepsilon > 0$ the m-semiflow (4.5) has the connected stable global attractor Θ_ε in the phase space $\mathbb{X}$. Moreover, Θ_ε is bounded in H uniformly in ε.*

Proof. Due to Theorems 3.1, 3.2 and classical results about existence of global attractors (see [21]) it is sufficient to prove that V_ε is asymptotically compact, that is,

every sequence $\{\bar{\xi}_n \in V_\varepsilon(t_n, p_0^n)\}$ is precompact in H,

where $t_n \nearrow +\infty$, $\|p_0^n\|_X \leq r$.

Let $\bar{\xi}_n \in V_\varepsilon(t_n, p_0^n)$. Then $\exists \xi_n : \|\xi_n - \bar{\xi}_n\|_{\mathbb{X}} < \frac{1}{n}$ and $\xi_n = p_n(t_n)$, p_n is a solution of (2.1)–(2.4) with $p_n(0) = p_0^n$ and $b_n(\,\cdot\,) := b(\,\cdot\, + \tau_n)$, $\tau_n \geq 0$. Therefore, from Theorem 3.1

$$\|p_n(t)\|_X \leq R_0 + r, \ \forall\ n \geq 1,\ t \geq 0. \tag{4.7}$$

So we can claim that $\{\xi_n\}$ is precompact in H_w. Indeed, since $\|\xi_n\|_{L^2} \leq R_0 + r$ then up to subsequence $\xi_n \to \xi$ in L^2_w. Let us prove that up to a subsequence $\xi_n \to \xi$ in $\overline{L}^1_w$. Since $\xi_n = p_n(t_n)$, then (3.2) yields that for each $\delta > 0$ there exist $k(\delta) \geq 1$, $n(\delta) \geq 1$ such that

$$\int_{|x|>k} \xi_n(x)|x|dx < \frac{\delta}{3},\ \forall\ k \geq k(\delta),\ n \geq n(\delta).$$

According to Amigó et al. [1, Lemma 6.1]

$$(\overline{L^1})^* = \{\varphi = (1+|x|)u \,:\, u \in L^\infty\}.$$

Thus, we set $d_n(x) = (1 + |x|)\xi_n(x)$ and prove that $\{d_n\}$ is a Cauchy sequence in L^1_w, because

$$\left|\int_{\mathbb{R}}(d_n(x) - d_m(x))u(x)dx\right| \leq \left|\int_{|x|\leq k}(1 + |x|)(\xi_n(x) - \xi_m(x))u(x)dx\right|$$
$$+ 2\|u\|_{L^\infty}\left(\int_{|x|>k}\xi_n(x)|x|dx + \int_{|x|>k}\xi_m(x)|x|dx\right) < \delta,$$

for each $u \in L^\infty$ and $n, m \geq N = N(\delta, k)$. Since the space L^1 is weakly complete, then up to a subsequence $d_n \to d$ in L^1_w for some $d \in L^1$. Thus

$$\xi_n \to \bar{\xi} = \frac{d}{1 + |x|} \quad \text{in} \quad \overline{L}^1_w.$$

If we consider the restriction of ξ_n to each interval $[-k, k]$, then we deduce that $\bar{\xi} = \xi$ and up to a subsequence $\xi_n \to \xi$ in H_w.

Now let us prove this convergence in the strong topology of H. Consider a smooth real function θ that satisfies the following three conditions:

$$\begin{array}{lll} \text{(a)} & \theta(s) = 0, & |s| \leq 1; \\ \text{(b)} & 0 \leq \theta(s) \leq 1, & |s| \in [1, 2]; \\ \text{(c)} & \theta(s) = 1, & |s| \geq 2, \end{array} \tag{4.8}$$

and define for $k > 1$

$$\rho_k(x) = \theta(\frac{x}{k}).$$

According to Amigó et al. [1, pp. 215–216] after multiplying (2.1) by $\rho_k(x)p_n$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho_k(x)p_n^2dx + b_n(t)\int_{\mathbb{R}}\rho_k(x)p_n\frac{\partial p_n}{\partial x}dx \\ &+(D(p_n(\cdot,t)) + \varepsilon_n)\Big(\int_{\mathbb{R}}\rho_k(x)\left(\frac{\partial p_n}{\partial x}\right)^2dx \\ &+\frac{1}{k}\int_{\mathbb{R}}\theta'(\frac{x}{k})p_n\frac{\partial p_n}{\partial x}dx\Big) + \int_{\mathbb{R}}\rho_k(x)p_n^2dx = 0. \end{aligned} \tag{4.9}$$

Integrating by parts we deduce

$$b_n(t)\int_{\mathbb{R}}(\rho_k(x)p_n\frac{\partial p_n}{\partial x}dx = -\frac{b_n(t)}{2k}\int_{\mathbb{R}}\theta'(\frac{x}{k})p_n^2dx,$$

$$\frac{1}{k}\int_{\mathbb{R}}\theta'(\frac{x}{k})p_n\frac{\partial p_n}{\partial x}dx = -\frac{1}{2k^2}\int_{\mathbb{R}}\theta''(\frac{x}{k})p_n^2dx.$$

Then from (4.9) we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\rho_k(x)p_n^2dx + \int_{\mathbb{R}}\rho_k(x)p_n^2dx \leq \left(\frac{B\beta}{2k} + \frac{(\alpha+1)\beta}{2k^2}\right)\int_{\mathbb{R}}p_n^2dx, \tag{4.10}$$

where $\beta := \max_{|s|\in[1,2]} \{|\theta'(s)| + |\theta''(s)|\}$.

Combining (4.7) and (4.10) we deduce from Gronwall's Lemma that for some positive constant $C = C(r)$

$$\int_{|x|>2k} p_n^2(x,t)dx \le e^{-2t}r^2 + \frac{C(r)}{k}, \ \forall\ t \ge 0,\ \ n \ge 1,\ \ k > 1. \tag{4.11}$$

On the other hand, for every solution of (2.1)–(2.4) we have the following energy equality (for details see the proof of Lemma 3.2):

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (p(x,t))^2 dx + (D(p(\,\cdot\,,t)) + \varepsilon)\int_{\mathbb{R}} \left(\frac{\partial p(x,t)}{\partial x}\right)^2 dx + \int_{|x|>1} (p(x,t))^2 dx = \frac{D(p(\,\cdot\,,t))}{\alpha}\langle \delta_0, p(\,\cdot\,,t)\rangle. \tag{4.12}$$

Let us consider the functions

$$\bar{p}_n(t) = p_n(t + (t_n - 1)),\ \ t \ge 0.$$

Then $\bar{p}_n$ is a solution of (2.1)–(2.4) with $\bar{b}_n(\,\cdot\,) := b_n(\,\cdot + t_n - 1) = b(\,\cdot + t_n - 1 + \tau_n)$, $\bar{p}_n(0) = p_n(t_n - 1)$, $\bar{p}_n(1) = \xi_n$ and $\bar{p}_n$ satisfies (4.7), (4.9), (4.12). Moreover, similarly to the previous arguments we deduce that up to subsequence

$$\bar{p}_n(0) = p_n(t_n - 1) \to \bar{p}_0 \ \text{ in } \ H_w.$$

Hence, from Lemma 3.2 we obtain for every $T > 1$ that

$$\bar{p}_n \to \bar{p} \ \text{ in } \ C([0,T]; H_w), \tag{4.13}$$

where $\bar{p}$ is a solution of (2.1)–(2.4) with $\bar{p}(0) = \bar{p}_0$ and some $\bar{b} \in L^\infty(0,+\infty)$ such that $\bar{b}_n \to b$ weakly star in $L^\infty(0,T)$ for each $T > 0$. In particular, $|\bar{b}(t)| \le B$ for a.e. $t > 0$.

Since $\varepsilon > 0$ is fixed, we can derive from (4.7), (4.12) and the Aubin-Lions theorem [16] that for every $k > 1$ up to subsequence

$$\bar{p}_n \to \bar{p} \ \text{ in } \ L^2(0,T; L^2(-k,k)).$$

In particular,

$$\bar{p}_n(t) \to \bar{p}(t) \ \text{ in } \ L^2(-k,k) \ \text{ for a.a. } t \in (0,T).$$

By a diagonal procedure we obtain that up to a subsequence and for some $\tau \in (0,1)$,

$$\bar{p}_n(\tau) \to \bar{p}(\tau) \ \text{ in } \ L^2(-k,k),\ \forall\ k \ge 1. \tag{4.14}$$

From (4.11) we get

$$\int_{|x|>2k} \bar{p}_n^2(x,\tau)dx \le e^{-2(\tau+t_n-1)}r^2 + \frac{C(r)}{k}, \ \forall\ n \ge 1,\ \ k > 1. \tag{4.15}$$

Combining (3.2), (4.14), (4.15) we have

$$\bar{p}_n(\tau) \to \bar{p}(\tau) \quad \text{in} \quad X.$$

Then the second part of Theorem 3.2 guarantees the convergence

$$\bar{p}_n \to \bar{p} \quad \text{in} \quad C([\tau, T]; H).$$

In particular,

$$\xi_n = \bar{p}_n(1) \to \bar{p}(1) \quad \text{in } H.$$

Thus we obtain the required precompactness of $\{\xi_n\}$ and, therefore, the existence of the connected, stable global attractor Θ_ε. □

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ON INDIRECT APPROACH TO THE SOLVABILITY OF QUASI-LINEAR DIRICHLET ELLIPTIC BOUNDARY VALUE PROBLEM WITH BMO-ANISOTROPIC P-LAPLACIAN

Peter I. Kogut*, Olha P. Kupenko†‡

Abstract. We study here Dirichlet boundary value problem for a quasi-linear elliptic equation with anisotropic p-Laplace operator in its principle part and L^1-control in coefficient of the low-order term. As characteristic feature of such problem is a specification of the matrix of anisotropy $A = A^{sym} + A^{skew}$ in BMO-space. Since we cannot expect to have a solution of the state equation in the classical Sobolev space $W_0^{1,p}(\Omega)$, we specify a suitable functional class in which we look for solutions and prove existence of weak solutions in the sense of Minty using a non standard approximation procedure and compactness arguments in variable spaces.

Key words: Anisotropic p-Laplacian, approximation procedure, weak solutions, BMO-coefficients.

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1. Introduction

In this paper we deal with the following boundary value problem

$$\begin{cases} -\Delta_p(A,y) + |y|^{p-2}yu = -\operatorname{div} f \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ \qquad u \in L^1(\Omega), \quad u(x) \geq 0 \text{ a.e. in } \Omega\ , \end{cases} \tag{1.1}$$

where

$$-\Delta_p(A,y) = -\operatorname{div}\left(|(\nabla y, A\nabla y)|^{\frac{p-2}{2}} A\nabla y\right) \tag{1.2}$$

is the anisotropic p-Laplacian, $2 \leq p < +\infty$, A is the matrix of anisotropy, $y_d \in L^2(\Omega)$ and $f \in L^\infty(\Omega;\mathbb{R}^N)$ are given distributions.

The interest to elliptic equations whose principal part is an anisotropic p-Laplace operator arises from various applied contexts related to composite materials such as nonlinear dielectric composites, whose nonlinear behavior is modeled by the so-called power-low (see, for instance, [1, 21] and references therein). From mathematical point of view, the interest of anisotropic p-Laplacian lies on its

*Department of Differential Equations, Oles Honchar Dnipro National University, 72, Gagarin av., Dnipro, 49010, Ukraine, `p.kogut@i.ua`

†Department of System Analysis and Control, National Mining University, 19, Yavornitskii av., 49005 Dnipro

‡Institute of Applied System Analysis, National Academy of Sciences and Ministry of Education and Science of Ukraine, 37/35, Peremogy av., IASA, 03056 Kyiv, Ukraine, `kupenko.olga@gmail.com`

nonlinearity and an effect of degeneracy, which turns out to be the major difference from the standard Laplacian on $\mathbb{R}^N$. As characteristic feature of boundary value problem (1.1) is a specification of the matrix of anisotropy $A = B + D$, where $B := A^{sym} = (A+A^t)/2$ and $D := A^{skew} = (A-A^t)/2$, and the control $u \in L^1(\Omega)$. In particular, we assume that the matrix A is such that

$$\alpha^2(x)I \le B(x) \le \beta^2(x)I \quad \text{a. e. in } \ \Omega,$$

where $\alpha, \beta \in L^1(\Omega)$, $\beta(x) \ge \alpha(x) \ge 0$ almost everywhere in Ω, $\alpha \notin L^\infty(\Omega)$, $\alpha^{-1} \in L^1(\Omega)$, and α, α^{-1}, and β extended by zero outside of Ω are in $BMO(\mathbb{R}^N)$.

We note that these assumptions on the class of admissible matrices are essentially weaker than they usually are in the literature (see, for instance, [8, 9, 11, 19, 20]). In fact, we deal with the Dirichlet boundary value problem (BVP) for degenerate anisotropic elliptic equation with unbounded coefficients in its principal part and with L^1-bounded control in the coefficient of the low-order term. It is well-known that such BVP can exhibit the so-called Lavrentieff phenomenon, non-uniqueness of the weak solutions as well as other surprising consequences (see, for instance, [2,4]). As a result, the existence, uniqueness, and variational properties of the weak solution to the above BVP usually are drastically different from the corresponding properties of solutions to the elliptic equations with coercive L^∞-matrices of anisotropy (we refer to [6, 26–28, 31] for the details and other results in this field). Another distinguishing feature of the boundary value problem (3.1)–(3.2) is the fact that the skew-symmetric part D of the matrix A is merely measurable and its sub-multiplicative norm belongs to the BMO-space (rather than the space $L^\infty(\Omega)$). This circumstance can entail a number of pathologies with respect to the standard properties of BVPs for elliptic equations with anisotropic p-Laplacian even with 'a good' symmetric part A and a smooth right-hand side f. In particular, the unboundedness of the skew-symmetric part of matrix $A \in \mathfrak{M}_{ad}$ can have a reflection in non-uniqueness of weak solutions to the corresponding boundary value problem. For more details and other types of solutions to elliptic equations with unbounded coefficients we refer to [7, 14–16, 33]. So, in contrast to the paper [32], where the author consider the case of well-posed Dirichlet boundary value problem for a quasi-linear elliptic equation with unbounded coefficients in its principal part, we deal with an ill-posed boundary value problem.

We introduce a special functional space $\mathbb{X}_{u,B}$ related to a given control u and symmetric part B of matrix A, and prove (see Theorem 4.1) that the original boundary value problem admits weak solutions in the sense of Minty. Moreover, we show that for every control $u \in L^1(\Omega)$, a weak solutions (in the sense of Minty) to the corresponding BVP can be obtained as the limit of solutions to coercive problems with bounded coefficients, using any L^∞-approximation of BMO-matrix A. Such solutions are called approximation solutions in [33]. Their characteristic feature is the fact that they lay in variable space $\mathbb{X}_{u,B}$ and, in general, do not satisfy the energy equality but rather some energy inequality. We also derive a priori estimates for such solutions that do not depend on the skew-symmetric part D of matrix A. As a bi-product of our approach, we derive the conditions guaranteeing the equality $H^{1,p}_{0,B}(\Omega) = W^{1,p}_{0,B}(\Omega)$, i.e. we establish the density of smooth compactly supported functions in $W^{1,p}_{0,B}(\Omega)$.

2. Notation and Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$ with a Lipschitz boundary. Let p be a real number such that $2 \leq p < \infty$, and let $q = p/(p-1)$ be the conjugate of p. Let $\mathbb{M}^N$ be the set of all $N \times N$ real matrices. We denote by $\mathbb{S}^N_{skew}$ and $\mathbb{S}^N_{sym}$ the set of all skew-symmetric and symmetric matrices, respectively. We always identify each matrix $A \in \mathbb{M}^N$ with the decomposition $A = B + D$, where $B := \frac{1}{2}\left(A + A^t\right) \in \mathbb{S}^N_{sym}$ and $D := \frac{1}{2}\left(A - A^t\right) \in \mathbb{S}^N_{skew}$. Moreover, applying the Cholesky method to the symmetric part of matrix A (see Isaacson and Keller [30]), we deduce the existence of a lower triangular matrix L such that $B(x) := \frac{1}{2}\left(A(x) + A^t(x)\right) = L^t(x)L(x)$. In what follows, by matrix norm in $\mathbb{M}^N$ we mean a sub-multiplicative norm

$$\|A\| := \sup_{\substack{|\xi|\neq 0\\ \xi\in\mathbb{R}^N}} \left\{\frac{|A\xi|}{|\xi|}\right\} = \left(\text{maximal eigenvalue of } \ A^tA\right)^{1/2} \quad \text{a.e. in } \ \Omega.$$

BMO-Functions Defined on Bounded Domains. We recall that a function g on $\mathbb{R}^N$ belongs to the space $BMO(\mathbb{R}^N)$ if $g \in L^1_{loc}(\mathbb{R}^N)$ and

$$\|g\|_{BMO(\mathbb{R}^N)} := \sup \frac{1}{|Q|}\int_Q |g - g_Q|\, dx < +\infty,$$

where $g_Q = ⨍_Q g\, dx := \frac{1}{|Q|}\int_Q g\, dx$, $Q = Q(x,r)$ is a ball centered at x and of radius $\ell(Q) = r$, and the supremum is taken over all balls $Q \subset \mathbb{R}^N$. Obviously, $L^\infty(\mathbb{R}^N) \subset BMO(\mathbb{R}^N)$. As an example of unbounded function in $BMO(\mathbb{R}^N)$, one can take $\ln|x|$.

For our further analysis, we make use of the following result: if $g \in BMO(\mathbb{R}^N)$ then the John-Nirenberg estimate

$$⨍_Q |g - g_Q|^p\, dx \leq C_{p,\Omega}\|g\|_{BMO(\mathbb{R}^N)} \quad \text{for all } \ p \geq 1 \tag{2.1}$$

holds for any ball $Q \subset \mathbb{R}^N$ (see [13]).

Let $L^1(\Omega)^{\frac{N(N+1)}{2}} = L^1\left(\Omega; \mathbb{S}^N_{sym}\right)$ be the space of measurable absolutely integrable functions whose values are symmetric matrices. By $BMO(\Omega; \mathbb{S}^N_{skew})$ we denote the space of all skew-symmetric matrices $D = [d_{ij}]$ (the-so-called matrices of bounded mean oscillation) such that $D \in L^1(\Omega; \mathbb{S}^N_{skew})$ and their sub-multiplicative norm extended by zero to the entire $\mathbb{R}^N$ is in $BMO(\mathbb{R}^N)$. The similar specification holds for the space $BMO(\Omega; \mathbb{M}^N)$.

Matrices with Degenerate Eigenvalues. Let α, β be given elements of $L^1(\Omega)$ satisfying the conditions

$$\alpha^{-1} \in L^1(\Omega), \quad \alpha^{-1} \notin L^\infty(\Omega), \quad 0 \le \alpha(x) \le \beta(x) \text{ a.e. in } \Omega, \tag{2.2}$$
$$\alpha, \alpha^{-1}, \beta \ \text{ extended by zero outside of } \Omega \text{ are in } \ BMO(\mathbb{R}^N). \tag{2.3}$$

Remark 2.1. As immediately follows from the John-Nirenberg estimate (2.1) and assumption (2.3), we have

$$\begin{aligned}
\|\alpha^{-1}\|^r_{L^r(\Omega)} &\le 2^{r-1}\int_\Omega |\alpha^{-1} - \alpha^{-1}_Q|^r\,dx + 2^{r-1}\left(\frac{1}{|Q|}\int_Q \alpha^{-1}\,dx\right)^r |\Omega| \\
&\le 2^{r-1}|Q|\left[\fint_Q |\alpha^{-1} - \alpha^{-1}_Q|^r\,dx + \frac{|\Omega|}{|Q|^{r+1}}\left(\int_\Omega \alpha^{-1}\,dx\right)^r\right] \\
&\overset{\text{by (2.1)}}{\le} C_{Q,r}\left(\|\alpha^{-1}\|_{BMO(\mathbb{R}^N)} + \|\alpha^{-1}\|^r_{L^1(\Omega)}\right) \quad \forall\, r > 1.
\end{aligned} \tag{2.4}$$

Here, Q is a ball such that $\Omega \subset Q$, and $\alpha^{-1}_Q = \fint_Q \alpha^{-1}\,dx$. The similar estimates hold true for α and β. So, we can suppose that $\alpha, \alpha^{-1}, \beta \in L^r(\Omega)$ for all $r \ge 1$ provided the conditions (2.2)–(2.3) hold true.

We define the class of matrices $\mathfrak{M}_{ad}$ as follows

$$\mathfrak{M}_{ad}(\Omega) = \left\{ A \in \mathbb{M}^N \;\middle|\; \begin{array}{c} A = B + D = \frac{1}{2}\left(A + A^t\right) + \frac{1}{2}\left(A - A^t\right), \\ \alpha^2\|\eta\|^2 \le (\eta, B\eta) \le \beta^2\|\eta\|^2 \text{ a.e. in } \Omega\ \forall\, \eta \in \mathbb{R}^N, \\ B(x) = L^t(x)L(x) \quad \text{a.e. in } \Omega, \\ D \in BMO(\Omega; \mathbb{S}^N_{skew}), \\ \alpha \text{ and } \beta \text{ satisfy conditions (2.2)–(2.3).} \end{array} \right\}. \tag{2.5}$$

Remark 2.2. Here, in view of the estimate $(\eta, B\eta) \ge \alpha^2\|\eta\|^2$ a.e. in $\Omega\ \forall\, \eta \in \mathbb{R}^N$, L is a triangular matrix with positive (a.e. in Ω) diagonal elements. Moreover, for a fixed $A \in \mathfrak{M}_{ad}$, conditions (2.2)–(2.5) imply the following inequalities:

$$\|L\|_{BMO(\Omega;\mathbb{M}^N)} \le \|\beta\|_{BMO(\mathbb{R}^N)} < +\infty, \tag{2.6}$$
$$(B(x)\xi, \xi) = |L(x)\xi|^2 \le \beta^2(x)|\xi|^2 \quad \text{a. e. in } \ \Omega,\ \forall\, \xi \in \mathbb{R}^N \tag{2.7}$$
$$\left|L^{-1}(x)\xi\right|^2 \le \alpha^{-2}(x)|\xi|^2 \quad \text{a. e. in } \ \Omega,\ \forall\, \xi \in \mathbb{R}^N, \tag{2.8}$$

and, therefore,

$$\|L(x)\| \le \beta(x) \quad \text{and} \quad \|L^{-1}(x)\| \le \alpha^{-1}(x) \quad \text{a. e. in } \ \Omega, \tag{2.9}$$
$$L, L^{-1} \in BMO(\Omega; \mathbb{M}^N). \tag{2.10}$$

Weighted Sobolev Spaces. To each matrix $A \in \mathfrak{M}_{ad}(\Omega)$ we can formally associate two weighted Sobolev spaces: $W^{1,p}_{0,B}(\Omega)$ and $H^{1,p}_{0,B}(\Omega)$, where $W^{1,p}_{0,B}(\Omega)$ is the set of functions $y \in W^{1,1}_0(\Omega)$ for which the norm

$$\|y\|_{W^{1,p}_{0,B}(\Omega)} = \Big(\int_\Omega \big(|y|^p + |(\nabla y, B\nabla y)|^{\frac{p}{2}} \big)\, dx \Big)^{1/p} \tag{2.11}$$

is finite, and $H^{1,p}_{0,B}(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.11).

As follows from the definition of the class $\mathfrak{M}_{ad}$ and estimates

$$\int_\Omega |y|\, dx \le \Big(\int_\Omega |y|^p\, dx \Big)^{1/p} |\Omega|^{1/q} \le C\|y\|_{W^{1,p}_{0,B}(\Omega)}, \ \forall\, y \in W^{1,p}_{0,B}(\Omega), \tag{2.12}$$

$$\begin{aligned} \int_\Omega |\nabla y|\, dx &\le \Big(\int_\Omega |\nabla y|^p \alpha^p\, dx \Big)^{1/p} \Big(\int_\Omega \alpha^{-q}\, dx \Big)^{1/q} \\ &\le \Big(\int_\Omega |(\nabla y, B(x)\nabla y)|^{p/2}\ dx \Big)^{1/p} \|\alpha^{-1}\|_{L^q(\Omega)} \le C\|y\|_{W^{1,p}_{0,B}(\Omega)}, \end{aligned} \tag{2.13}$$

the space $W^{1,p}_{0,B}(\Omega)$ is complete with respect to the norm $\|\cdot\|_{W^{1,p}_{0,B}(\Omega)}$. It is clear that $H^{1,p}_{0,B}(\Omega)$ and $W^{1,p}_{0,B}(\Omega)$, for $p \ge 2$, are uniformly convex reflexive Banach spaces such that $H^{1,p}_{0,B}(\Omega) \subseteq W^{1,p}_{0,B}(\Omega)$ (see, for instance [10]). In general, the identity $W^{1,p}_{0,B}(\Omega) = H^{1,p}_{0,B}(\Omega)$ is not always valid (for the corresponding examples, we refer to [5]).

Further we make use of the following observation. If we introduce the parameter p_s by $p_s := ps/(s+1) < p$ with a certain $s > 0$ and use the Hölder inequality with the parameter $r = \frac{s+1}{s} = \frac{p}{p_s} > 1$, we obtain

$$\begin{aligned} \int_\Omega |\nabla y|^{p_s}\, dx &= \int_\Omega |\nabla y|^{p_s} \alpha^{p_s} \alpha^{-p_s}\, dx \\ &\le \Big(\int_\Omega |\nabla y|^p \alpha^p\, dx \Big)^{p_s/p} \Big(\int_\Omega \alpha^{-s-1}\, dx \Big)^{1/(s+1)} \\ &\le \Big(\int_\Omega |(\nabla y, B(x)\nabla y)|^{p/2}\ dx \Big)^{s/(s+1)} \|\alpha^{-1}\|_{L^{s+1}(\Omega)} \\ &\overset{\text{by (2.4)}}{\le} C\|y\|^{\frac{p_s}{p}}_{H^{1,p}_{0,B}(\Omega)}, \end{aligned} \tag{2.14}$$

$$\int_\Omega |y|^{p_s}\, dx \le \Big(\int_\Omega |y|^p\, dx \Big)^{s/(s+1)} |\Omega|^{1/(s+1)} \le C\|y\|^{\frac{p_s}{p}}_{H^{1,p}_{0,B}(\Omega)}. \tag{2.15}$$

Hence, each function $y \in H^{1,p}_{0,B}(\Omega)$ belongs to the non-weighted space $W^{1,p_s}_0(\Omega)$. Combining this fact with the Sobolev embedding theorem, we deduce:

$$\text{if } s > \frac{N}{p} \text{ then } p^*_s = \frac{Np_s}{N - p_s} > p,$$

and, therefore, we have the compact embedding

$$\begin{gathered} W^{1,p_s}_0(\Omega) \hookrightarrow L^r(\Omega) \ \text{ and } \ H^{1,p}_{0,B}(\Omega) \hookrightarrow L^r(\Omega), \\ 1 \le r < p^*_s = \tfrac{Nps}{(N-p)s+N}. \end{gathered} \tag{2.16}$$

Moreover, as follows from (2.16) and (2.14), the following the weighted Friedrichs inequality

$$\begin{aligned}\|y\|_{L^p(\Omega)} &\overset{\text{by (2.16)}}{\leq} C\|y\|_{W_0^{1,p_s}(\Omega)} = C\|\nabla y\|_{L^{p_s}(\Omega)^N}\\ &\overset{\text{by (2.14)}}{\leq} C\|\alpha^{-1}\|^{1/p_s}_{L^{s+1}(\Omega)}\left(\int_\Omega |(\nabla y, B(x)\nabla y)|^{p/2}\,dx\right)^{1/p}\end{aligned}$$

holds true for each $y \in H^{1,p}_{0,B}(\Omega)$. Hence, the norm

$$\|y\|_{H^{1,p}_{0,B}(\Omega)} = \Big(\int_\Omega |(\nabla y, B\nabla y)|^{\frac{p}{2}}\,dx\Big)^{1/p} \tag{2.17}$$

on the space $H^{1,p}_{0,B}(\Omega)$ is equivalent to the norm $\|\cdot\|_{W^{1,p}_{0,B}(\Omega)}$ defined by (2.11).

Weak Convergence in Variable L^p-Spaces Associated with $\mathbb{S}^N_{sym}$-Matrices. Let $\{B_k\}_{k\in\mathbb{N}}$ and B be a given collection of $\mathbb{S}^N_{sym}$-matrices such that

$$B_k, B \in L^1(\Omega;\mathbb{S}^N_{sym}) \text{ and } B_k \rightharpoonup B \text{ in } L^1(\Omega;\mathbb{S}^N_{sym}). \tag{2.18}$$

Let $L^p(\Omega, B\,dx)^N$, with $p \geq 2$, be the Lebesgue space of measurable vector-valued functions $f(x) \in \mathbb{R}^N$ on Ω such that

$$\|f\|_{L^p(\Omega,B\,dx)^N} = \Big(\int_\Omega |(f, Bf)|^{\frac{p}{2}}\,dx\Big)^{1/p} < +\infty.$$

We say that a sequence $\{v_k \in L^p(\Omega, B_k\,dx)^N\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty}\int_\Omega |(v_k, B_k v_k)|^{\frac{p}{2}}\,dx < +\infty.$$

Definition 2.1. A bounded sequence $\{v_k \in L^p(\Omega, B_k\,dx)^N\}_{k\in\mathbb{N}}$ is weakly convergent to a function $v \in L^p(\Omega, B\,dx)^N$ in variable space $L^p(\Omega, B_k\,dx)^N$ if

$$\lim_{k\to\infty}\int_\Omega (\varphi, B_k v_k)\,dx = \int_\Omega (\varphi, Bv)\,dx \quad \forall\,\varphi \in C_0^\infty(\Omega)^N. \tag{2.19}$$

Definition 2.2. A sequence $\{v_k \in L^p(\Omega, B_k\,dx)^N\}_{k\in\mathbb{N}}$ is said to be strongly convergent to a function $v \in L^p(\Omega, A\,dx)^N$ if

$$\lim_{k\to\infty}\int_\Omega (b_k, B_k v_k)\,dx = \int_\Omega (b, Bv)\,dx \tag{2.20}$$

whenever $b_k \rightharpoonup b$ in $L^q(\Omega, B_k\,dx)^N$ as $k \to \infty$, where $q = p/(p-1)$ is the Holder conjugate of p.

Remark 2.3. Note that in the case $B_k \equiv B$, Definitions 2.1–2.2 leads to the well-known notion of convergence in weighted Lebesgue space $L^p(\Omega, B\,dx)^N$.

The main properties of the weak and strong convergence in $L^p(\Omega, B_k\,dx)^N$ can be expressed as follows (see [17, 18] for the details):

Proposition 2.1. If a sequence $\left\{v_k \in L^p(\Omega, B_k\,dx)^N\right\}_{k\in\mathbb{N}}$ is bounded and condition (2.18) holds true, then it is compact with respect to the weak convergence in $L^p(\Omega, B_k\,dx)^N$.

Proposition 2.2. If the sequence $\left\{v_k \in L^p(\Omega, B_k\,dx)^N\right\}_{k\in\mathbb{N}}$ converges weakly to $v \in L^p(\Omega, B\,dx)^N$ and condition (2.18) holds true, then

$$\liminf_{k\to\infty} \int_\Omega |(v_k, B_k v_k)|^{\frac{p}{2}}\, dx \geq \int_\Omega |(v, Bv)|^{\frac{p}{2}}\, dx. \tag{2.21}$$

Proposition 2.3. Assume the condition (2.18) holds true. Then the weak convergence of a sequence $\left\{v_k \in L^p(\Omega, B_k\,dx)^N\right\}_{k\in\mathbb{N}}$ to $v \in L^p(\Omega, B\,dx)^N$ and

$$\lim_{k\to\infty} \int_\Omega |(v_k, B_k v_k)|^{\frac{p}{2}}\, dx = \int_\Omega |(v, Bv)|^{\frac{p}{2}}\, dx \tag{2.22}$$

are equivalent to the strong convergence of $\{v_k\}_{k\in\mathbb{N}}$ to v in $L^p(\Omega, B_k\,dx)^N$.

We make also use of the following inequality that was established by Maz'ya in 1972 [23]. If μ is a positive Radon measure, then

$$\left(\int_\Omega |\varphi|^r\, d\mu\right)^{1/r} \leq C_M \int_\Omega |\nabla\varphi|\, dx \quad \forall\, \varphi \in C_0^\infty(\Omega),\ \forall\, r \in [1,\infty), \tag{2.23}$$

with the best constant

$$C_M = \sup_{\Omega'\subset\Omega} \frac{\mu(\Omega')^{1/r}}{\mathcal{H}^{N-1}(\partial\Omega')} \tag{2.24}$$

where the supremum in (2.23) is taken over all open subsets of Ω, with C^∞-boundary, such that $\overline{\Omega'} \subset \Omega$.

3. Setting of the Boundary Value Problem

Let $y_d \in L^2(\Omega)$ and $f \in L^\infty(\Omega)^N$ be given distributions. For a fixed $A \in \mathfrak{M}_{ad}$, we consider the following boundary value problem:

$$-\operatorname{div}\left(|(\nabla y, A\nabla y)|^{\frac{p-2}{2}} A\nabla y\right) + |y|^{p-2} y u = -\operatorname{div} f \quad \text{in } \Omega, \tag{3.1}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

$$u \in L^1(\Omega), \quad u(x) \geq 0 \ \text{ a.e. in } \ \Omega, \tag{3.3}$$

where we adopt u as a given control function.

It is worth to notice that, in view of the definition of the set $\mathfrak{M}_{ad}$, we deal with a boundary value problem for degenerate quasi-linear elliptic equation with singular coefficients. It means that even for symmetric matrices of coefficients $A \in \mathfrak{M}_{ad}$

this problem can exhibit the Lavrentieff phenomenon (i.e. $W^{1,p}_{0,B}(\Omega) \neq H^{1,p}_{0,B}(\Omega)$) and, as a consequence, non-uniqueness of the weak solutions. Thus, the original boundary value problem (3.1)–(3.2) is ill-posed, in general.

The another distinguishing feature of the boundary value problem (3.1)–(3.2) is the fact that the skew-symmetric part D of the matrix $A \in \mathfrak{M}_{ad}$ is merely measurable and belongs to the space $BMO\big(\Omega;\mathbb{M}^N\big)$ (rather than the space of bounded matrices $L^\infty\big(\Omega;\mathbb{M}^N\big)$). This circumstance can entail a number of pathologies with respect to the standard properties of BVPs for elliptic equations with anisotropic p-Laplacian even with 'a good' symmetric part B of A and a smooth right-hand side f. In particular, the unboundedness of the skew-symmetric part of matrix $A \in \mathfrak{M}_{ad}$ can have a reflection in non-uniqueness of weak solutions to the corresponding boundary value problem. For more details and other types of solutions to elliptic equations with unbounded coefficients we refer to [7,14–16,33].

We associate to the boundary value problem (3.1)–(3.2) the following space $\mathbb{X}_{u,B} = H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx)$. Here, $L^p(\Omega, u\,dx)$ is a usual Banach space with respect to the measure $d\mu = u\,dx$. Since $u \in L^1(\Omega)$ and $u(x) \geq 0$ a.e. in Ω, it follows that μ is a positive Radon measure and, hence, the space $H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx)$ is well defined and it is a Banach space with respect to the norm (see [3])

$$\begin{aligned}\|y\|_{\mathbb{X}_{u,B}} &= \left(\int_\Omega |(\nabla y, B\nabla y)|^{\frac{p}{2}}\,dx + \int_\Omega |y|^p u\,dx\right)^{1/p}\\ &= \left(\|y\|^p_{H^{1,p}_{0,B}(\Omega)} + \|y\|^p_{L^p(\Omega,u\,dx)}\right)^{1/p}.\end{aligned}$$

Definition 3.1. We say that, for a fixed control u and given distributions $A \in \mathfrak{M}_{ad}$, and $f \in L^\infty(\Omega)^N$, a function $y = y(A,u,f)$ is a weak solution (in the sense of Minty) to boundary value problem (3.1)–(3.2) if $y \in \mathbb{X}_{u,B}$ and the inequality

$$\begin{aligned}\int_\Omega |(\nabla\varphi, A\nabla\varphi)|^{\frac{p-2}{2}} (A\nabla\varphi, \nabla\varphi - \nabla y)\,dx + \int_\Omega |\varphi|^{p-2}\varphi(\varphi - y)u\,dx&\\ \geq \int_\Omega (f, \nabla\varphi - \nabla y)\,dx& \quad (3.4)\end{aligned}$$

holds for any $\varphi \in C^\infty_0(\Omega)$.

To begin with, let us show that this definition makes a sense. Indeed, by the initial assumptions and Hölder's inequality, we have

$$\begin{aligned}\int_\Omega (f,\nabla\varphi - \nabla y)\,dx &= \int_\Omega ((L^{-1})^t f, L\nabla\varphi - L\nabla y)\,dx\\ &\leq \|f\|_{L^\infty(\Omega)^N} \int_\Omega \|L^{-1}\| |L\nabla\varphi - L\nabla y|\,dx\\ &\overset{\text{by (2.9), (2.17)}}{\leq} \|f\|_{L^\infty(\Omega)^N} \|\alpha^{-1}\|_{L^q(\Omega)} \|\varphi - y\|_{H^{1,p}_{0,B}(\Omega)} \leq C\|\varphi - y\|_{\mathbb{X}_{u,B}} \quad (3.5)\end{aligned}$$

and

$$\int_\Omega |\varphi|^{p-2}\varphi(\varphi-y)u\,dx \le \|\varphi\|^{p-1}_{L^p(\Omega,u\,dx)}\|\varphi-y\|_{L^p(\Omega,u\,dx)} \le C\|\varphi-y\|_{\mathbb{X}_{u,B}}. \quad (3.6)$$

As for the first term in (3.4), we observe that

$$|(\nabla\varphi, A\nabla\varphi)|^{\frac{p-2}{2}} = |(L\nabla\varphi, \big[\underbrace{I+(L^t)^{-1}DL^{-1}}_{T}\big]L\nabla\varphi)|^{\frac{p-2}{2}} \le \|T\|^{\frac{p-2}{2}}|L\nabla\varphi|^{p-2}$$

and, therefore,

$$\begin{aligned}\int_\Omega &|(\nabla\varphi, A\nabla\varphi)|^{\frac{p-2}{2}}(A\nabla\varphi, \nabla\varphi-\nabla y)\,dx\\ &\le \int_\Omega \|T\|^{\frac{p-2}{2}}|L\nabla\varphi|^{p-2}\big(TL\nabla\varphi, L\nabla\varphi-L\nabla y\big)\,dx\\ &\le \int_\Omega \|T\|^{\frac{p}{2}}|L\nabla\varphi|^{p-1}|L\nabla\varphi-L\nabla y|\,dx\\ &\le \|\varphi\|^{p-1}_{C^1(\Omega)}\int_\Omega \|T\|^{\frac{p}{2}}\beta^{p-1}|L\nabla\varphi-L\nabla y|\,dx\\ &\le \|\varphi\|^{p-1}_{C^1(\Omega)}\left(\int_\Omega \|T\|^{\frac{pq}{2}}\beta^p\,dx\right)^{1/q}\|\varphi-y\|_{H^{1,p}_{0,B}(\Omega)}. \quad (3.7)\end{aligned}$$

Since,

$$\begin{aligned}\int_\Omega \|T\|^{\frac{pq}{2}}\beta^p\,dx &\le \int_\Omega \big(1+\alpha^{-2}\|D\|\big)^{\frac{pq}{2}}\beta^p\,dx\\ &\le 2^{pq-1}\int_\Omega\Big(\beta^p+\big(\alpha^{-q}\beta\big)^p\|D\|^{\frac{pq}{2}}\Big)\,dx\\ &\le 2^{pq-1}\Big[\|\beta\|^p_{L^p(\Omega)}+\|\alpha^{-1}\|^{pq}_{L^{4pq}(\Omega)}\|\beta\|^p_{L^{4p}(\Omega)}\|D\|^{\frac{pq}{2}}_{L^{pq}(\Omega;\mathbb{S}^N_{skew})}\Big]\\ &\overset{\text{by (2.4)}}{<} +\infty,\end{aligned}$$

it follows from (3.7) that

$$\int_\Omega |(\nabla\varphi, A\nabla\varphi)|^{\frac{p-2}{2}}(A\nabla\varphi, \nabla\varphi-\nabla y)\,dx \le C\|\varphi-y\|_{\mathbb{X}_{u,B}}. \quad (3.8)$$

Thus, the well posedness of each term in the variational inequality (3.4) and, hence, the consistency of the definition of the weak solution in the sense of Minty to the considered boundary value problem, obviously follows from the estimates (3.5)-(3.6), (3.8).

Remark 3.1. The estimate (3.8) and the fact that $(\nabla\varphi(x), D(x)\nabla\varphi(x)) = 0$ a.e. in Ω by the skew-symmetry property of D, imply that the variational inequality

(3.4) can be rewritten as follows

$$\int_\Omega |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}}(A\nabla\varphi, \nabla\varphi - \nabla y)\,dx + \int_\Omega |\varphi|^{p-2}\varphi(\varphi - y)u\,dx \geq \int_\Omega (f, \nabla\varphi - \nabla y)\,dx. \quad (3.9)$$

Getting inspired by this, we call a function $y \in \mathbb{X}_{u,B}$ a weak solution (in the sense of Minty) to boundary value problem (3.1)–(3.2) if it satisfies the inequality (3.9) for every test function $\varphi \in C_0^\infty(\Omega)$.

Taking this remark into account, it is reasonable to consider another definition of the weak solution to the given boundary value problem, in the terms of distributions, which appears more natural:

$y \in \mathbb{X}_{u,B}$ is the distributional solution to (3.1)–(3.2) if the integral identity

$$\int_\Omega |(\nabla y, B\nabla y)|^{\frac{p-2}{2}}(A\nabla y, \nabla\varphi)\,dx + \int_\Omega |y|^{p-2}y\varphi u\,dx = \int_\Omega (f, \nabla\varphi)\,dx \quad (3.10)$$

holds true for every $\varphi \in C_0^\infty(\Omega)$.

In spite of the fact that the relations between these definitions are very intricate for general matrix $A \in \mathfrak{M}_{ad}$ (for an example when these definitions lead to the different solutions even for linear equations, we refer to [25]), we can leverage the integral identity (3.10) for the following estimate

$$\begin{aligned}
&\left|\int_\Omega |(\nabla y, B\nabla y)|^{\frac{p-2}{2}}(A\nabla y, \nabla\varphi)\,dx\right| \\
&\quad\leq \int_\Omega |y|^{p-1}u^{\frac{p-1}{p}}|\varphi|u^{\frac{1}{p}}\,dx + \int_\Omega |(L^{-1})^t f||L\nabla\varphi|\,dx \\
&\quad\leq \|y\|^{p-1}_{L^p(\Omega, u\,dx)}\|\varphi\|_{L^p(\Omega, u\,dx)} + \|f\|_{L^\infty(\Omega)^N}\|\alpha^{-1}\|_{L^q(\Omega)}\|\varphi\|_{H^{1,p}_{0,B}(\Omega)} \\
&\quad\leq \left[\|y\|^{p-1}_{L^p(\Omega, u\,dx)} + \|f\|_{L^\infty(\Omega)^N}\|\alpha^{-1}\|_{L^q(\Omega)}\right]\|\varphi\|_{\mathbb{X}_{u,B}} \\
&\quad= C\,(y, u, B, f)\,\|\varphi\|_{\mathbb{X}_{u,B}}. \qquad (3.11)
\end{aligned}$$

Remark 3.2. As follows from (3.11), a weak solution to the considered problem in the sense of distribution belongs to the special subset $D(\mathbb{X}_{u,B})$ of the space $\mathbb{X}_{u,B} := H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx)$, elements of which possess the property (3.11). As a result, if $y \in D(\mathbb{X}_{u,B})$ then the mapping

$$\varphi \mapsto [y, \varphi]_A := \int_\Omega |(\nabla y, B\nabla y)|^{\frac{p-2}{2}}(A\nabla y, \nabla\varphi)\,dx$$

can be defined for all test functions $\varphi \in \mathbb{X}_{u,B}$ using the standard rule

$$[y, z]_A = \lim_{k\to\infty}[y, \varphi_k]_A$$

where $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ and $\varphi_k \to z$ strongly in $\mathbb{X}_{u,B}$ (it is the case when we essentially use the fact that $C_0^\infty(\Omega)$ is dense in $H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx)$). In particular, if $y \in D(\mathbb{X}_{u,B})$, then we can define the value $[y,y]_A$ and this one is finite for every $y \in D(\mathbb{X}_{u,B})$, although the "integrand"

$$|(\nabla y, B\nabla y)|^{\frac{p}{2}} + |(\nabla y, B\nabla y)|^{\frac{p-2}{2}}(D\nabla y, \nabla y)$$

needs not be integrable on Ω, in general. As a result, we can derive form (3.10) the energy equality for distributional solutions

$$[y,y]_A + \int_\Omega |y|^p u\,dx = \int_\Omega (f, \nabla y)\,dx. \tag{3.12}$$

However, as it follows from definition of the form $[y,\varphi]_A$, the value $[y,y]_A$ is not equal to $\|y\|^p_{H^{1,p}_{0,B}(\Omega)}$, in general, and it does not preserve the inequality

$$[y,y]_A \geq \|y\|^p_{H^{1,p}_{0,B}(\Omega)} \quad \text{for all } y \in D(\mathbb{X}_{u,B}).$$

Hence, even if the relation $H^{1,p}_{0,B}(\Omega) = W^{1,p}_{0,B}(\Omega)$ holds true, the energy equality (3.12) does not allow us to derive a reasonable a priory estimate in $\|\cdot\|_{\mathbb{X}_{u,B}}$-norm for the weak solutions in the sense of distributions.

4. On Solvability of Boundary Value Problem (3.1)–(3.3)

Our main intension in this section is to show that boundary value problem admits a weak solution due to the approximation approach. It is clear that the condition $A \in \mathfrak{M}_{ad}(\Omega)$ ensures the existence of the sequence of matrices $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}_{ad}(\Omega) \cap L^\infty(\Omega;\mathbb{M}^N)$ such that $A_k \to A$ strongly in $L^1(\Omega;\mathbb{M}^N)$. With that in mind we give a few auxiliary results.

Lemma 4.1. *Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}_{ad}(\Omega)$ and $A \in \mathfrak{M}_{ad}(\Omega)$ be matrices such that*

$$A_k \in L^\infty(\Omega;\mathbb{M}^N) \quad \forall\, k \in \mathbb{N}, \tag{4.1}$$

$$A_k \to A \quad \textit{strongly in} \quad L^1(\Omega;\mathbb{M}^N), \tag{4.2}$$

$$\begin{gathered}(\eta, A_k\eta) \geq \alpha_k^2|\eta|^2 \quad \textit{a.e. in } \Omega\ \forall\, \eta \in \mathbb{R}^N \\ \textit{and for some positive } \alpha_k \in \mathbb{R},\ \alpha_k \geq \alpha(x).\end{gathered} \tag{4.3}$$

Then

$$L_k^{-1} \to L^{-1} \quad \textit{and} \quad T_k \to T \quad \textit{strongly in} \quad L^1(\Omega;\mathbb{M}^N), \tag{4.4}$$

where

$$\begin{gathered}B_k := \frac{1}{2}(A_k + A_k^t) = L_k^t L_k,\ B := \frac{1}{2}(A + A^t) = L^t L, \\ T_k := I + (L_k^t)^{-1} D_k L_k^{-1},\ T := I + (L^t)^{-1} D L^{-1}, \\ D_k := \frac{1}{2}(A_k - A_k^t),\ D := \frac{1}{2}(A - A^t).\end{gathered} \tag{4.5}$$

Remark 4.1. The simplest way to construct a sequence $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}_{ad}(\Omega)$, possessing the properties (4.1)–(4.3), is to set

$$A_k = k^{-1}I + [\max\{\min\{a_{ij}, k\}, -k\}]_{i,j=1}^N$$

or apply the procedure of the direct Steklov smoothing to a given matrix $A \in \mathfrak{M}_{ad}(\Omega)$ with some positive compactly supported smooth kernel (see, for instance, [15]).

Proof. The conditions (4.1)–(4.3) ensure that $B_k^{-1} \in L^\infty(\Omega; \mathbb{S}^N_{sym})$ for all $k \in \mathbb{N}$ and (up to a subsequence)

$$D_k(x) \to D(x) \ \text{ and } \ L_k^{-1}(x) \to L^{-1}(x) \quad \text{a.e. in } \ \Omega.$$

Moreover, since $\alpha_k \geq \alpha$ a.e. in Ω, it follows that

$$\|L_k^{-1}(x)\| \leq \alpha_k^{-1} \leq \alpha^{-1}(x) \ \text{ a.e. in } \ \Omega,$$

where $\alpha^{-1} \in L^1(\Omega)$ (see (2.2)). Hence, the sequence $\left\{L_k^{-1}\right\}_{k\in\mathbb{N}}$ is equi-integrable. In view of the definition of the class $\mathfrak{M}_{ad}(\Omega)$, the same conclusion can be made for the sequence of skew-symmetric matrices $\left\{(L_k^t)^{-1} D_k L_k^{-1}\right\}_{k\in\mathbb{N}}$. As a result, the property (4.4) is a direct consequence of Lebesgue's Theorem. □

Lemma 4.2. *Let $f \in L^\infty(\Omega)^N$ be a given distribution, and let $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}_{ad}(\Omega)$ and $A \in \mathfrak{M}_{ad}(\Omega)$ be matrices satisfying the properties* (4.1)–(4.3)*. Then, for an arbitrary smooth function $\varphi \in C_0^\infty(\Omega)$, the sequences*

$$\left\{v_k := |(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} L_k^{-1} T_k L_k \nabla\varphi\right\}_{k\in\mathbb{N}} \quad \text{and} \quad \left\{w_k := B_k^{-1} f\right\}_{k\in\mathbb{N}}$$

are bounded in $L^q(\Omega, B_k\,dx)^N$ and

$$v_k \to v = |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} L^{-1} T L \nabla\varphi \text{ strongly in variable } L^q(\Omega, B_k\,dx)^N, \quad (4.6)$$

$$w_k \to w = B^{-1} f \text{ strongly in variable } L^q(\Omega, B_k\,dx)^N, \quad (4.7)$$

where the matrices T_k and T are defined by (4.5)*.*

Proof. Indeed, by definition of the space $L^q(\Omega, B_k\,dx)^N$, we have

$$\begin{aligned}
\|v_k\|^q_{L^q(\Omega, B_k\,dx)^N} &= \int_\Omega |(v_k, B_k v_k)|^{\frac{q}{2}}\,dx = \int_\Omega |L_k v_k|^q\,dx \\
&= \int_\Omega \left| |(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} T_k L_k \nabla\varphi \right|^q dx \leq \|\varphi\|^p_{C^1(\Omega)} \int_\Omega \left[\|L_k\|^{p-1}\|T_k\|\right]^q dx \\
&\leq \|\varphi\|^p_{C^1(\Omega)} \int_\Omega \left[\beta^{p-1}\left(1 + \alpha^{-2}\|D_k\|\right)\|\right]^q dx \\
&\leq 2^{q-1}\|\varphi\|^p_{C^1(\Omega)} \int_\Omega \beta^p \left(1 + \alpha^{-2q}\|D_k\|^q\right) dx \\
&\leq 2^{q-1}\|\varphi\|^p_{C^1(\Omega)} \left[\|\beta\|^p_{L^p(\Omega)} + \|\beta\|^p_{L^{3p}(\Omega)} \|\alpha^{-1}\|^{2q}_{L^{6q}(\Omega)} \|D\|^q_{L^{3q}(\Omega)}\right] \\
&\overset{\text{by (2.4)}}{\leq} \text{const} < +\infty. \qquad (4.8)
\end{aligned}$$

Hence, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $L^q(\Omega, B_k\,dx)^N$.

Further we notice that, by the initial assumption (4.2), Lemma 4.1, and BMO-properties of the matrices L, L^{-1}, and D, we see that the sequence

$$\left\{|(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} T_k L_k \nabla\varphi\right\}_{k\in\mathbb{N}}$$

is equi-integrable and

$$|(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} T_k L_k \nabla\varphi \to |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} TL\nabla\varphi \quad \text{a.e. in} \;\; \Omega$$

for any $\varphi \in C_0^\infty(\Omega)$. Hence, by Lebesgue's Theorem, we have the strong convergence

$$|(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} T_k L_k \nabla\varphi \to |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} TL\nabla\varphi \quad \text{in} \;\; L^1(\Omega;\mathbb{R}^N). \tag{4.9}$$

As a result, this implies

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega (\nabla\psi, B_k v_k)\,dx &= \lim_{k\to\infty}\int_\Omega |(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} (\nabla\psi, T_k L_k\nabla\varphi)\,dx\\ &\overset{\text{by (4.9)}}{=} \int_\Omega |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} (\nabla\psi, TL\nabla\varphi)\,dx\\ &= \int_\Omega (\nabla\psi, Bv)\,dx, \quad \forall\,\psi\in C_0^\infty(\Omega).\end{aligned} \tag{4.10}$$

Thus, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is weakly convergent in $L^q(\Omega, B_k\,dx)^N$ to the vector-valued function $v = |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} L^{-1}TL\nabla\varphi$.

It remains to show that the sequence $\{v_k\}_{k\in\mathbb{N}}$ is strongly convergent to v. To do so, we make use of Proposition 2.3. Following this assertion, it is enough to prove the equality

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |\,(v_k, B_k v_k)\,|^{\frac{q}{2}}\,dx &= \lim_{k\to\infty}\int_\Omega |L_k v_k|^q\,dx\\ = \lim_{k\to\infty}\int_\Omega &\left||(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} T_k L_k\nabla\varphi\right|^q dx\\ = \int_\Omega &\left||(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} TL\nabla\varphi\right|^q dx = \int_\Omega (v, Bv)^{\frac{q}{2}}\,dx.\end{aligned} \tag{4.11}$$

In view of the estimate

$$\left||(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} T_k L_k\nabla\varphi\right| \le \|L_k\nabla\varphi\|^{p-1}\|T_k\| \le \beta^{p-1}\|T\||\nabla\varphi|^{p-1}$$

and the fact that the term $\left(\beta^{p-1}\|T\||\nabla\varphi|^{p-1}\right)^q = \beta^p\|T\|^q|\nabla\varphi|^p$ is in $L^1(\Omega)$ by Remark 2.1, we see that the sequence $\left\{|\,(v_k, B_k v_k)\,|^{\frac{q}{2}}\right\}_{k\in\mathbb{N}}$ is equi-integrable. On the other hand, property (4.2) and Lemma 4.1 imply that, within a subsequence,

$$|(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} T_k L_k \to |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} TL \;\; \text{almost everywhere in } \Omega.$$

Therefore, the equality (4.11) is a direct consequence of Lebesgue Dominated Theorem. Thus, the strong convergence in variable space $L^q(\Omega, B_k\,dx)^N$ of the sequence $\{v_k\}_{k\in\mathbb{N}}$ is established.

The property (4.7) can be proved following the same arguments. $\square$

For our further analysis, we make use of the following concept.

Definition 4.1. We say that a bounded sequence

$$\left\{(A_k, y_k) \in \mathfrak{M}_{ad}(\Omega) \times \left[H^{1,p}_{0,B_k}(\Omega) \cap L^p(\Omega, u\,dx)\right]\right\}_{k\in\mathbb{N}} \tag{4.12}$$

w-converges to the pair $(A, y) \in \mathfrak{M}_{ad}(\Omega) \times \left[H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx)\right]$ as $k \to \infty$ (in symbols, $(A_k, y_k) \stackrel{w}{\longrightarrow} (A, y)$) if

$$A_k \to A \quad \text{in} \;\; L^1(\Omega; \mathbb{M}^N),$$
$$y_k \to y \quad \text{in} \;\; L^p(\Omega) \;\; \text{and weakly in weighted space} \;\; L^p(\Omega, u\,dx),$$
$$\nabla y_k \rightharpoonup \nabla y \quad \text{in the variable space} \;\; L^p(\Omega, B_k\,dx)^N.$$

In particular, as follows from this definition, if $(A_k, y_k) \stackrel{w}{\to} (A, y)$, then

$$\lim_{k\to\infty} \int_\Omega \|A_k\|\,dx = \int_\Omega \|A\|\,dx,$$
$$\lim_{k\to\infty} \int_\Omega y_k \varphi u\,dx = \int_\Omega y\varphi u\,dx \quad \forall\,\varphi \in C_0^\infty(\Omega),$$
$$\lim_{k\to\infty} \int_\Omega (\xi, B_k \nabla y_k)\;dx = \int_\Omega (\xi, B\nabla y)\;dx \quad \forall\,\xi \in C_0^\infty(\Omega)^N.$$

In order to motivate this definition, we give the following result.

Lemma 4.3. *Let* $\left\{(A_k, y_k) \in \mathfrak{M}_{ad}(\Omega) \times \left[H^{1,p}_{0,B_k}(\Omega) \cap L^p(\Omega, u\,dx)\right]\right\}_{k\in\mathbb{N}}$ *be a sequence with the following properties:*

(i) $A_k \in L^\infty(\Omega; \mathbb{M}^N)\ \forall\, k \in \mathbb{N}$, *and there exists a matrix* $A \in \mathfrak{M}_{ad}(\Omega)$ *such that* $A_k \to A$ *in* $L^1(\Omega; \mathbb{M}^N)$*;*

(ii) $\left\{y_k \in H^{1,p}_{0,B_k}(\Omega) \cap L^p(\Omega, u\,dx)\right\}_{k\in\mathbb{N}}$ *are bounded sequences, i.e.*

$$\sup_{k\in\mathbb{N}} \int_\Omega \left(u|y_k|^p + (\nabla y_k, B_k \nabla y_k)^{\frac{p}{2}}\right) dx < +\infty; \tag{4.13}$$

Then, within a subsequence, the original sequence is w*-convergent. Moreover, each* w*-limit pair* (A, y) *belongs to the set* $\mathfrak{M}_{ad}(\Omega) \times \left[H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx)\right]$.

Proof. To begin with, we note that the conditions (i)–(ii) and estimates (2.12)–(2.13) immediately imply the boundedness of the sequence

$$\left\{ y_k \in H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx) \right\}_{k\in\mathbb{N}}$$

in $W^{1,1}(\Omega;\mathbb{M}^N)$ and in variable spaces $H^{1,p}_{0,B_k}(\Omega)$ and $L^p(\Omega, u\,dx)$. Moreover, due to the inequalities (2.14)–(2.15), we have the compact embedding

$$H^{1,p}_{0,B_k}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all} \quad 1 \le r < p_s^* = \frac{Nps}{(N-p)s + N}.$$

Since $p_s^* = \frac{Np_s}{N-p_s} > p$ provided $s > \frac{N}{p}$, it follows that the sequence $\{y_k\}_{k\in\mathbb{N}}$ is compact with respect to the norm topology of $L^p(\Omega)$.

Thus, combining this fact with the compactness criterium for the weak convergence in variable spaces (see Proposition 2.1), we can deduce the existence of a pair $(y, z) \in L^p(\Omega) \times L^p(\Omega, u\,dx) \times L^p(\Omega, B\,dx)^N$ such that, within a subsequence of $\{y_k\}_{k\in\mathbb{N}}$, we have

$$y_k \to y \quad \text{in} \quad L^p(\Omega), \tag{4.14}$$

$$y_k \rightharpoonup z \quad \text{in} \quad L^p(\Omega, u\,dx), \tag{4.15}$$

$$\nabla y_k \rightharpoonup v \quad \text{in the variable space} \quad L^p(\Omega, B_k\,dx)^N. \tag{4.16}$$

Our aim is to show that $y = z$, $v = \nabla y$, and as a consequence $y \in H^{1,p}_{0,B}(\Omega) \cap L^p(\Omega, u\,dx)$. With that in mind, we note that for every measurable subset $K \subset \Omega$, the estimate

$$\begin{aligned} \int_K |\nabla y_k|\,dx &\le \Big(\int_K |L_k \nabla y_k|^p\,dx\Big)^{\frac{1}{p}} \Big(\int_K \alpha^{-q}\,dx\Big)^{\frac{1}{q}} \\ &\le \Big(\int_\Omega |(\nabla y_k, B_k \nabla y_k)|^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}} \Big(\int_K \alpha^{-q}\,dx\Big)^{\frac{1}{q}} \\ &\overset{\text{by (4.13)}}{\le} C|K|^{\frac{1}{2q}} \|\alpha^{-1}\|_{L^{2q}(\Omega)} \\ &\overset{\text{by (2.4)}}{\le} C_1 |K|^{\frac{1}{2q}} \Big(\|\alpha\|^{2q}_{L^1(\Omega)} + \|\alpha^{-1}\|_{BMO(\mathbb{R}^N)} \Big)^{\frac{1}{2q}} \end{aligned}$$

implies equi-integrability of the family $\{|\nabla y_k|_{\mathbb{R}^N}\}$. Combining this fact with estimate (2.13) and property (ii), we deduce that the sequence $\{|\nabla y_k|\}_{k\in\mathbb{N}}$ is weakly compact in $L^1(\Omega)$. Since, for an arbitrary $\xi \in C_0^\infty(\Omega)^N$, we have

$$B_k^{-1}\xi \to B^{-1}\xi \quad \text{strongly in variable } L^q(\Omega, B_k\,dx)^N \tag{4.17}$$

by Lemma 4.2, it follows that

$$\begin{aligned} \int_\Omega (\xi, \nabla y_k)\,dx &= \int_\Omega \big(B_k^{-1}\xi, B_k \nabla y_k\big)\,dx \\ &\overset{\text{by (4.16), (4.17), and (2.20)}}{\longrightarrow} \int_\Omega \big(B^{-1}\xi, Bv\big)\,dx \\ &= \int_\Omega (\xi, v)\,dx \quad \forall\, \xi \in C_0^\infty(\Omega)^N. \end{aligned}$$

Thus, in view of the weak compactness property of $\{\nabla y_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega)^N$, we conclude

$$\nabla y_k \rightharpoonup v \text{ in } L^1(\Omega;\mathbb{R}^N) \quad \text{as } n\to\infty. \tag{4.18}$$

Since $y_k \in W^{1,1}(\Omega)$ for all $k\in\mathbb{N}$ and the Sobolev space $W^{1,1}(\Omega)$ is complete, (4.14) and (4.18) imply $\nabla y = v$, and consequently $y \in H^{1,p}_{0,B}(\Omega)$.

To end the proof, it remains to establish the equality $y = z$ a.e. in Ω. Since the sequence $\{y_k \in L^p(\Omega, u\,dx)\}_{k\in\mathbb{N}}$ is bounded and for any measurable set $K\subseteq\Omega$, we have

$$\int_K y_k u\,dx \le \left(\int_\Omega |y|^p u\,dx\right)^{1/p}\left(\int_K u\,dx\right)^{1/q},$$

it follows that the sequence $\{y_k u\}_{k\in\mathbb{N}}$ is equi-integrable and weakly compact in $L^1(\Omega)$ and, hence, the weak convergence (4.15) is equivalent to the weak convergence

$$y_k u \rightharpoonup zu \text{ in } L^1(\Omega). \tag{4.19}$$

Further, we note that

$$\begin{aligned}\int_\Omega |\varphi|\,u\,dx &\le \sup_{\Omega'\subset\Omega}\frac{\int_{\Omega'}|u|\,dx}{\mathcal{H}^{N-1}(\partial\Omega')}\int_\Omega|\nabla\varphi|\,dx\\ &\le \sup_{\Omega'\subset\Omega}\frac{\|u\|_{L^1(\Omega')}}{\mathcal{H}^{N-1}(\partial\Omega')}\left(\int_\Omega|L\nabla\varphi|^p\,dx\right)^{1/p}\left(\int_\Omega\alpha^{-q}\,dx\right)^{1/q}\\ &\le \text{const}\,\|\varphi\|_{H^{1,p}_{0,B}(\Omega)} \quad \forall\,\varphi\in C_0^\infty(\Omega)\end{aligned}$$

by Maz'ya inequality (2.23). Since the set $C_0^\infty(\Omega)$ is dense in $H^{1,p}_{0,B}(\Omega)$, it follows that the family $\{u(y_k-y)\}_{k\in\mathbb{N}}$ is weakly compact in $L^1(\Omega)$. Taking into account the compactness of the embedding $H^{1,p}_{0,B}(\Omega)\hookrightarrow L^p(\Omega)$ and the weak convergence $y_k \rightharpoonup y$ in $L^p(\Omega)$, we can suppose that $y_k\to y$ almost everywhere in Ω. Hence, $u(y_k-y)\to 0$ a.e. in Ω. Then the strong convergence $u(y_k-y)\to 0$ in $L^1(\Omega)$ immediately follows from the Lebesgue Theorem. Thus, in order to conclude the desired equality $y=z$, it is enough to combine this inference with the property (4.19). The proof is complete. □

We are now in a position to prove the main result of this section. Namely, we show that the boundary value problem (3.1)–(3.3) admits a weak solution.

Theorem 4.1. *For given $f\in L^\infty(\Omega)^N$, $u\in L^1(\Omega)$, $u\ge 0$ a.e. in Ω, $\gamma>0$, and for an arbitrary matrix $A\in\mathfrak{M}_{ad}$, there exists a weak solution $y\in\mathbb{X}_{u,B}$ (in the sense of Minty) to boundary value problem* (3.1)–(3.2) *with an a priori estimate*

$$\|y\|_{\mathbb{X}_{u,B}} \le \left(C_{Q,q}\|f\|_{L^\infty(\Omega)^N}\right)^{\frac{1}{p-1}}\left(\|\alpha^{-1}\|_{BMO(\mathbb{R}^N)}+\|\alpha^{-1}\|^q_{L^1(\Omega)}\right)^{\frac{1}{p}} \tag{4.20}$$

and the energy relation

$$\int_\Omega |(\nabla y, B\nabla y)|^{\frac{p}{2}}\,dx + \int_\Omega |y|^p u\,dx \le \int_\Omega (f,\nabla y)\,dx. \tag{4.21}$$

Proof. Let $u \in \mathfrak{U}_{ad}$ be an arbitrary admissible control. For a given matrix $A \in \mathfrak{M}_{ad}$ let us consider an approximation $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}_{ad}(\Omega)$ with properties (4.1)–(4.3), and the corresponding variational problem

$$\begin{gathered}\text{Find } y_k \in W_0^{1,p}(\Omega) \text{ such that}\\ \int_\Omega |(\nabla y_k, A_k \nabla y_k)|^{\frac{p-2}{2}} (A_k \nabla y_k, \nabla \varphi)\, dx + \int_\Omega |y_k|^{p-2} y_k \varphi u\, dx \\ = \int_\Omega (f, \nabla \varphi)\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega).\end{gathered} \tag{4.22}$$

Since $A_k \in L^\infty(\Omega; \mathbb{M}^N)$, it follows that $(\nabla y_k, A_k \nabla y_k) = (\nabla y_k, B_k \nabla y_k)$. Hence, by the well-known result of quasi-linear elliptic equations (see [29, Theorem 2.14]), for every $k \in \mathbb{N}$, the problem (4.22) admits a unique weak solution $y_k \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |(\nabla y_k, B_k \nabla y_k)|^{\frac{p}{2}}\, dx + \int_\Omega |y_k|^p u\, dx = \int_\Omega (f, \nabla y_k)\, dx \tag{4.23}$$

and

$$\begin{gathered}\int_\Omega |(\nabla \varphi, B_k \nabla \varphi)|^{\frac{p-2}{2}} (A_k \nabla \varphi, \nabla \varphi - \nabla y_k)\, dx + \int_\Omega |\varphi|^{p-2} \varphi (\varphi - y_k) u\, dx \\ \geq \int_\Omega (f, \nabla \varphi - \nabla y_k)\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega).\end{gathered} \tag{4.24}$$

It is clear that the energy equality (4.23) leads to the following estimate

$$\begin{aligned}\|y_k\|^p_{\mathbb{X}_{u,B_k}} &:= \int_\Omega |(\nabla y_k, B_k \nabla y_k)|^{\frac{p}{2}}\, dx + \int_\Omega |y_k|^p u\, dx \leq \int_\Omega |(L_k^{-1})^t f| |L_k \nabla y_k|\, dx \\ &\leq \|f\|_{L^\infty(\Omega)^N} \|\alpha^{-1}\|_{L^q(\Omega)} \|y_k\|_{H^{1,p}_{0,B_k}(\Omega)} \\ &\leq C_{Q,q} \|f\|_{L^\infty(\Omega)^N} \left(\|\alpha^{-1}\|_{BMO(\mathbb{R}^N)} + \|\alpha^{-1}\|^q_{L^1(\Omega)} \right)^{\frac{1}{q}} \|y_k\|_{\mathbb{X}_{u,B_k}}.\end{aligned}$$

Hence, the sequence $\{y_k\}_{k\in\mathbb{N}}$ is bounded in variable space $\mathbb{X}_{u,B_k}$,

$$\begin{aligned}\|y_k\|_{\mathbb{X}_{u,B_k}} \leq &\left(C_{Q,q} \|f\|_{L^\infty(\Omega)^N} \right)^{\frac{1}{p-1}} \\ &\times \left(\|\alpha^{-1}\|_{BMO(\mathbb{R}^N)} + \|\alpha^{-1}\|^q_{L^1(\Omega)} \right)^{\frac{1}{p}}, \quad \forall\, k \in \mathbb{N},\end{aligned} \tag{4.25}$$

and, by Lemma 4.3, we can suppose the existence of an element $y \in \mathbb{X}_{u,B}$ such that (within a subsequence) y is subjected to the estimate (4.20) and

$$y_k \rightharpoonup y \quad \text{in} \quad L^p(\Omega, u\, dx), \tag{4.26}$$

$$\nabla y_k \rightharpoonup \nabla y \quad \text{in the variable space} \quad L^p(\Omega, B_k\, dx)^N. \tag{4.27}$$

We are now in a position to pass to the limit in (4.24) as $k \to \infty$. With that in mind we make use of Lemma 4.2. In particular, we utilize the properties (4.6)–(4.7). Then, it follows from Definition 2.2 and (4.26)–(4.27) that

$$\int_\Omega (f, \nabla\varphi - \nabla y_k)\, dx = \int_\Omega \left(B_k^{-1} f, B_k \left(\nabla\varphi - \nabla y_k\right)\right)\, dx$$
$$\stackrel{k\to\infty}{\to} \int_\Omega \left(B^{-1} f, B \left(\nabla\varphi - \nabla y\right)\right)\, dx = \int_\Omega (f, \nabla\varphi - \nabla y)\, dx,$$

$$\begin{aligned}\int_\Omega &|(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} (A_k\nabla\varphi, \nabla\varphi - \nabla y_k)\, dx \\ &= \int_\Omega \Big(|(\nabla\varphi, B_k\nabla\varphi)|^{\frac{p-2}{2}} L_k^{-1} T_k L_k \nabla\varphi, B_k\left(\nabla\varphi - \nabla y_k\right)\Big)\, dx \\ &\stackrel{k\to\infty}{\to} \int_\Omega \Big(|(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} L^{-1} T L \nabla\varphi, B\left(\nabla\varphi - \nabla y_k\right)\Big)\, dx \\ &= \int_\Omega |(\nabla\varphi, B\nabla\varphi)|^{\frac{p-2}{2}} (A\nabla\varphi, \nabla\varphi - \nabla y)\, dx.\end{aligned}$$

Taking into account that

$$\int_\Omega |\varphi|^{p-2}\varphi(\varphi - y_k) u\, dx \stackrel{k\to\infty}{\longrightarrow} \int_\Omega |\varphi|^{p-2}\varphi(\varphi - y) u\, dx$$

by (4.26) and definition of the weak convergence in $L^p(\Omega, u\, dx)$, we can pass to the limit in (4.24) as $k \to \infty$ and readily obtain the desired relation (3.9). Thus, y is a weak solution to the boundary value problem (3.1)–(3.3). As for the energy inequality (4.21), it follows from (4.23) and the weak convergence properties (4.26)–(4.27). □

Remark 4.2. As follows from approximation procedure that was used in the proof of Theorem 4.1, it always leads to some weak solution of the original boundary value problem. Such solutions are called approximation solutions in [33]. The characteristic feature of such solutions is the fact that they satisfy energy inequality (4.21) and their a priori estimate (4.20) does not depend on the skew-symmetric part $D \in BMO(\Omega; \mathbb{S}^N_{skew})$ of matrix $A \in \mathfrak{M}_{ad}(\Omega)$. Moreover, it is unknown in general whether approximation solutions are the weak solutions to the boundary value problem (3.1)–(3.2) in the sense of distributions and belong to the set $D(\mathbb{X}_{u,B})$.

5. On Density of Smooth Compactly Supported Functions in $W^{1,p}_{0,B}(\Omega)$

The aim of this section is to find out the sufficient conditions guaranteeing the equality $H^{1,p}_{0,B}(\Omega) = W^{1,p}_{0,B}(\Omega)$. With that in mind, it is enough to check whether, for each $A \in \mathfrak{M}_{ad}(\Omega)$ and $p \geq 2$, the set of smooth compactly supported functions $C_0^\infty(\Omega)$ is dense in $W^{1,p}_{0,B}(\Omega)$.

Let $f \in W^{1,p}_{0,B}(\Omega)$ be an arbitrary function. For any $\delta > 0$, we set

$$\Omega_\delta := \{x \in \Omega \ : \ \mathrm{dist}\,(x, \partial\Omega) > \delta\}$$
$$\text{and } \zeta_\delta(x) = \int_{\Omega_{3\delta/4}} \omega_{\delta/4}(|x-y|)\,dy, \quad \forall\, x \in \mathbb{R}^N,$$

where

$$\omega(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right), & 0 \leq |x| < 1, \\ 0, & || \geq 1, \end{cases}$$

with

$$C = \left(\int_{B_1(0)} \exp\left(\frac{1}{|x|^2-1}\right)\,dx\right)^{-1}$$

and

$$\omega_\delta(|x|) = \frac{1}{\delta}\omega(|x|/\delta), \quad \forall\, x \in \mathbb{R}^N,$$

so that $\omega_\delta \in C_0^\infty(B_\delta(0))$, $\int_{\mathbb{R}^N} \omega_\delta(x)\,dx = 1$, $\omega_\delta(|x|) \geq 0\ \forall\, x \in \mathbb{R}^N$.

Then, the following properties of ζ_δ are well-known [24, Theorem 1.4.2]:

(i) $0 \leq \zeta_\delta(x) \leq 1$ for all $x \in \mathbb{R}^N$;

(ii) $\zeta_\delta(x) = 1$ for all $x \in \Omega_\delta$;

(iii) $\zeta_\delta(x) = 0$ outside of $\Omega_{\delta/2}$;

(iv) $\left|\frac{\partial \zeta_\delta(x)}{\partial x_i}\right| \leq \frac{C}{\delta}\ \forall\, x \in \mathbb{R}^N$, $i = 1, \dots, N$, where C is a positive constant independent of δ.

Setting $f^\delta(x) := f(x)\zeta_\delta(x)$, we see that $f^\delta = 0$ outside of $\Omega_{\delta/2}$. Before proceeding further, we make use of the following auxiliary result.

Lemma 5.1. *Assume that, in addition to* (2.2)–(2.3)*, the functions* α *and* β *satisfy the condition*

$$\alpha^{-1}, \beta \in L^\infty(\Omega \setminus \Omega_\delta), \qquad \textit{where } \ \Omega_\delta := \{x \in \Omega \ : \ \mathrm{dist}\,(x, \partial\Omega) > \delta\} \tag{5.1}$$

for some $\delta > 0$ *small enough. Then for given* $A \in \mathfrak{M}_{ad}(\Omega)$ *and* $f \in W^{1,p}_{0,B}(\Omega)$*, we have*

$$f^\delta \in W^{1,p}_{0,B}(\Omega) \quad \textit{and} \quad \|f - f^\delta\|^p_{W^{1,p}_{0,B}(\Omega)} = o(1) \quad \textit{as } \delta \to 0. \tag{5.2}$$

Proof. Indeed, the inclusion $f^\delta \in W^{1,p}_{0,B}(\Omega)$ is a direct consequence of the property $f^\delta = 0$ outside of $\Omega_{\delta/2}$ and the following estimate

$$\begin{aligned}
\|f^\delta\|^p_{W^{1,p}_{0,B}(\Omega)} &= \int_\Omega \left(|f^\delta|^p + \left|(\nabla f^\delta, B\nabla f^\delta)\right|^{\frac{p}{2}}\right) dx \\
&= \int_\Omega \left(|f\zeta_\delta|^p + |L\left(\zeta_\delta \nabla f + f\nabla\zeta_\delta\right)|^p\right) dx \\
&\le \int_\Omega \left(|f|^p + p\,|L\nabla f|^p + p|f|^p\beta^p|\nabla\zeta_\delta|^p\right) dx \\
&\le (1+p)\|f\|^p_{W^{1,p}_{0,B}(\Omega)} + p\|\beta\|^p_{L^\infty(\Omega\setminus\Omega_\delta)} \left(\sqrt{\frac{C^2 N}{\delta^2}}\right)^p \int_{\Omega\setminus\Omega_\delta} |f|^p\,dx \\
&\le C(\delta)\|f\|^p_{W^{1,p}_{0,B}(\Omega)}
\end{aligned}$$

which is valid for δ small enough (see (5.1)).

As for the asymptotic behaviour of the difference $f - f\zeta_\delta = f(1-\zeta_\delta)$, we provide this analysis utilizing the following chain of estimates

$$\begin{aligned}
\|f - f\zeta_\delta\|^p_{W^{1,p}_{0,B}(\Omega)} &= \int_\Omega |f(1-\zeta_\delta)|^p\,dx + \int_\Omega |(1-\zeta_\delta)L\left(\nabla f\right) - fL\left(\nabla\zeta_\delta\right)|^p\ dx \\
&\le \int_{\Omega\setminus\Omega_\delta} |f|^p\,dx + p\int_{\Omega\setminus\Omega_\delta} |(\nabla f, B\nabla f)|^{\frac{p}{2}}\ dx \\
&\quad + p\int_{\Omega\setminus\Omega_\delta} |f|^p\beta^p|\nabla\zeta_\delta|^p\,dx \\
&\le (1+p)\|f\|^p_{W^{1,p}_{0,B}(\Omega\setminus\Omega_\delta)} \\
&\quad + p\|\beta\|^p_{L^\infty(\Omega\setminus\Omega_\delta)} \left(\frac{C\sqrt{N}}{\delta}\right)^p \int_{\Omega\setminus\Omega_\delta} |f|^p\,dx. \qquad (5.3)
\end{aligned}$$

In order to estimate the last term in (5.3), we make use of the Maz'ya inequality (2.23). This gets

$$\begin{aligned}
\Big(\int_{\Omega\setminus\Omega_\delta} |f|^p\,dx\Big)^{\frac{1}{p}} &\le \sup_{\Omega'\subset\Omega\setminus\Omega_\delta} \frac{\mathcal{L}^N(\Omega')^{\frac{1}{p}}}{\mathcal{H}^{N-1}(\partial\Omega')} \int_{\Omega\setminus\Omega_\delta} |\nabla f|\,dx \\
&\le \sup_{\Omega'\subset\Omega\setminus\Omega_\delta} \frac{\mathcal{L}^N(\Omega')^{\frac{1}{p}}}{\mathcal{H}^{N-1}(\partial\Omega')} \int_{\Omega\setminus\Omega_\delta} |L\nabla f|\alpha^{-1}\,dx \\
&\le \sup_{\Omega'\subset\Omega\setminus\Omega_\delta} \frac{\mathcal{L}^N(\Omega')^{\frac{1}{p}}}{\mathcal{H}^{N-1}(\partial\Omega')} \|\alpha^{-1}\|_{L^\infty(\Omega\setminus\Omega_\delta)} \mathcal{L}^N(\Omega')^{\frac{1}{q}} \left(\int_{\Omega\setminus\Omega_\delta} |L\nabla f|^p\,dx\right)^{\frac{1}{p}} \\
&\le \|\alpha^{-1}\|_{L^\infty(\Omega\setminus\Omega_\delta)} \sup_{\Omega'\subset\Omega\setminus\Omega_\delta} \frac{\mathcal{L}^N(\Omega')}{\mathcal{H}^{N-1}(\partial\Omega')} \|f\|_{W^{1,p}_{0,B}(\Omega\setminus\Omega_\delta)}. \qquad (5.4)
\end{aligned}$$

Since $\mathcal{L}^N(\Omega') \le C^*\delta\mathcal{H}^{N-1}(\partial\Omega')$ for δ small enough and with C^* independent of δ, it follows from (5.4) that

$$\int_{\Omega\setminus\Omega_\delta} |f|^p\,dx \le \text{const}\,\delta^p\,\|f\|^p_{W^{1,p}_{0,B}(\Omega\setminus\Omega_\delta)}.$$

Thus, from (5.3) we finally deduce

$$\|f - f\zeta_\delta\|^p_{W^{1,p}_{0,B}(\Omega)} \le \widehat{C}\|f\|^p_{W^{1,p}_{0,B}(\Omega\setminus\Omega_\delta)} = o(1) \quad \text{as } \delta \to 0. \tag{5.5}$$

□

Taking this result into account and following the standard rule, we define the smoothing of f^δ:

$$\left(f\zeta_\delta\right)_\varepsilon(x) := \int_{\mathbb{R}^N} \omega_\varepsilon(x-y)f(y)\zeta_\delta(y)\,dy = (\omega_\varepsilon * f^\delta)(x), \quad \forall\, x \in \mathbb{R}^N. \tag{5.6}$$

Then $\left(f\zeta_\delta\right)_\varepsilon(x) = 0$ has a compact support in Ω provided $\varepsilon < \delta/2$. Since $\left(f\zeta_\delta\right)_\varepsilon \in C_0^\infty(\Omega)$ and $W^{1,p}_{0,B}(\Omega) \subset W^{1,p_s}(\Omega)$ with continuous embedding for all $p_s < p$ (see estimates (2.14)–(2.15)), it follows from the classical theory of Sobolev spaces that $\left(f\zeta_\delta\right)_\varepsilon \to f\zeta_\delta$ in $W^{1,p_s}(\Omega)$ as $\varepsilon \to 0$ and, therefore, up to a subsequence, we can suppose that $\left(f\zeta_\delta\right)_\varepsilon \to f\zeta_\delta$ almost everywhere in Ω. Let us show that $\left(f\zeta_\delta\right)_\varepsilon \rightharpoonup f$ in $W^{1,p}_{0,B}(\Omega)$. Indeed, we can deduce from (5.6) that

$$|\nabla \left(f^\delta\right)_\varepsilon(x)| \le C_1 M(\nabla f^\delta)(x), \quad \forall\, \varepsilon > 0, \tag{5.7}$$

where $M(f)(x) = \sup\limits_Q \dfrac{1}{|Q|}\displaystyle\int_Q |f(y)|\,dy$ is the Hardy-Littlewood maximal function. It is also known that [12, p.174]

$$\alpha, 1/\alpha \in \bigcap_{r>1} A_r \quad \Leftrightarrow \quad \ln\alpha \in \text{closure}_{BMO} L^\infty(\mathbb{R}^N). \tag{5.8}$$

Since $\ln\alpha \in \text{closure}_{BMO}\, L^\infty(\mathbb{R}^N)$ is equivalent to $\ln\alpha^p \in \text{closure}_{BMO}\, L^\infty(\mathbb{R}^N)$, it follows from (5.8) and (2.2)–(2.3) that $\alpha^p, \beta^p \in A_p$. Then, by the celebrated Mackengoupt theorem [22], we have

$$\alpha^p \in A_p \quad \Leftrightarrow \quad \int_{\mathbb{R}^N} |M(\nabla f^\delta)|^p \alpha^p\,dx \le C(\alpha,p)\int_{\mathbb{R}^N} |\nabla f^\delta|^p \alpha^p\,dx,$$
$$\beta^p \in A_p \quad \Leftrightarrow \quad \int_{\mathbb{R}^N} |M(\nabla f^\delta)|^p \beta^p\,dx \le C(\beta,p)\int_{\mathbb{R}^N} |\nabla f^\delta|^p \beta^p\,dx.$$

Since the norms $|\xi|$ and $\sqrt{(\xi, B\xi)}$ are equivalent in $\mathbb{R}^N$, it follows that

$$\beta^p, \alpha^p \in A_p \;\Leftrightarrow\; \int_{\mathbb{R}^N} |\left(M(\nabla f^\delta), BM(\nabla f^\delta)\right)|^{\frac{p}{2}}\,dx \le C_2\int_{\mathbb{R}^N} |\left(\nabla f^\delta, B\nabla f^\delta\right)|^{\frac{p}{2}}\,dx \tag{5.9}$$

for some positive constant C_2 depending on α, β, and p. Using the fact that each of the matrices $A \in \mathfrak{M}_{ad}(\Omega)$ is assumed to be zero-extended outside of Ω, we deduce from (5.7) and (5.9)

$$\begin{aligned}\int_\Omega |\left(\nabla \left(f^\delta\right)_\varepsilon, B\nabla \left(f^\delta\right)_\varepsilon\right)|^{\frac{p}{2}}\, dx &= \int_{\mathbb{R}^N} |\left(\nabla \left(f^\delta\right)_\varepsilon, B\nabla \left(f^\delta\right)_\varepsilon\right)|^{\frac{p}{2}}\, dx\\ &\le C\int_{\mathbb{R}^N} |\left(\nabla f^\delta, B(\nabla f^\delta)\right)|^{\frac{p}{2}}\, dx\\ &= C\int_\Omega |\left(\nabla f^\delta, B(\nabla f^\delta)\right)|^{\frac{p}{2}}\, dx \le C\|f^\delta\|^p_{W^{1,p}_{0,B}(\Omega)} < +\infty. \quad (5.10)\end{aligned}$$

Following the similar reasoning, it can be shown that

$$\int_\Omega |\left(f^\delta\right)_\varepsilon|^p\, dx \le C\int_\Omega |f^\delta|^p\, dx \le C\|f^\delta\|^p_{W^{1,p}_{0,B}(\Omega)} < +\infty. \quad (5.11)$$

Hence, the sequence $\left\{\left(f^\delta\right)_\varepsilon\right\}_{\varepsilon>0}$ is bounded in $\|\cdot\|_{W^{1,p}_{0,B}(\Omega)}$-norm. Therefore, in view of the pointwise convergence: $(f\zeta_\delta)_\varepsilon \to f\zeta_\delta$ almost everywhere in Ω, we can deduce the weak convergence $(f\zeta_\delta)_\varepsilon \rightharpoonup f\zeta_\delta$ in $W^{1,p}_{0,B}(\Omega)$. Then by Mazur's theorem, the element $f^\delta := f\zeta_\delta$ can be attained in the strong topology of $W^{1,p}_{0,B}(\Omega)$ by the convex combinations of $\left\{\left(f^\delta\right)_\varepsilon\right\}_{\varepsilon>0}$. It means that for any given $\eta > 0$ it can be found a convex combination $f^\delta_* \in C^\infty_0(\Omega)$ of a finite number of elements of the sequence $\left\{\left(f^\delta\right)_\varepsilon\right\}_{\varepsilon>0}$ such that

$$\|f^\delta_* - f^\delta\|_{W^{1,p}_{0,B}(\Omega)} < \frac{\eta}{2}.$$

Besides, the property (5.2) implies that

$$\|f - f^\delta\|_{W^{1,p}_{0,B}(\Omega)} < \frac{\eta}{2} \quad \text{for } \delta \text{ small enough.}$$

Hence, for a given function $f \in W^{1,p}_{0,B}(\Omega)$ and arbitrary positive η, we have

$$\|f - f^\delta_*\|_{W^{1,p}_{0,B}(\Omega)} < \eta.$$

Thus, we can formulate the obtained result as follows:

Theorem 5.1. *Assume the set of admissible matrices $\mathfrak{M}_{ad}(\Omega)$ is such that in addition to its definition in the form (2.5), the condition (5.1) holds true for some positive small enough parameter δ. Then the set of smooth compactly supported functions $C^\infty_0(\Omega)$ is dense in $W^{1,p}_{0,B}(\Omega)$ or, what is equivalent, we have the equality $H^{1,p}_{0,B}(\Omega) = W^{1,p}_{0,B}(\Omega)$.*

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OPTIMAL CONTROL PROBLEM FOR SOME DEGENERATE VARIATION INEQUALITY: ATTAINABILITY PROBLEM

Nina V. Kasimova*

Abstract. We study an optimal control problem for degenerate elliptic variation inequality with degenerate weight function of potential type in the so-called class of H-admissible solutions. Using an appropriate regular algorithm of perturbation, we prove attainability of H-optimal pairs via optimal solutions of some non-degenerate perturbed optimal control problems.

Key words: optimal control problem, elliptic variation inequality, degenerate weight function of potential type, H-admissible solution, H-optimal solution, perturbation.

2010 Mathematics Subject Classification: 49J20, 49K20, 58J37, 35J50.

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1. Introduction

The aim of this paper is to study optimal control problems associated to degenerate elliptic variational inequalities in the so-called class of H-admissible solutions. Dealing with degenerate problems leads us to the concept of weighted Sobolev spaces such as $W(\Omega, \rho dx)$ (see for example [5]), where ρ is degenerate (in some sense) weight function, such that the differential operator associated to our problem is not coercive in the classical sense. Hence, the classical approach to investigate mentioned problems can't be used. In [17] was proposed an alternative method for solving optimal control problems for degenerate variational elliptic inequality, using Hardy-Poincare inequality.

It is known that smooth functions are, in general, not dense in the space $W(\Omega, \rho dx)$ that leads to the issues related to non-uniqueness of the setting of correspondent boundary value problem and as a consequence, to several possible settings of an optimal control problem associated to the mentioned control object. If we consider the space $H(\Omega, \rho dx)$ which is the closure of $C_0^\infty(\Omega)$ in $W(\Omega, \rho dx)$, then $H(\Omega, \rho dx) \neq W(\Omega, \rho dx)$, in general (see, for example [15]). In literature this fact is called the Lavrentiev phenomenon.

In applications a degenerate weight function ρ appears as the limit of the sequence of non-degenerate weights ρ^ε, for which the corresponding "approximate" problem is solvable. In this paper we interested in attainability of H-optimal solutions to degenerated problems via optimal solutions of non-degenerated problems, namely, we show that each optimal solution to the degenerate problem can

*Department of Integral and Differential Equations, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, Kyiv, Ukraine, 01601, zadoianchuk.nv@gmail.com

be attained by admissible solutions to perturbed problems, however there exists at least one optimal solution of degenerated problem which can be attained by optimal solutions to appropriate perturbed problems.

2. Notations and preliminaries

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be an open bounded set with regular boundary $\partial\Omega$ such, that $0 \in \mathbb{R}^N$ is an inner point of Ω. Hereafter we will denote a locally convex space of all infinitely differentiable functions with supports in Ω by $C_0^\infty(\Omega)$.

Let $\rho : \Omega \to \mathbb{R}$ be a given function such that: $\rho(x) > 0$ a.e. on Ω,

$$\rho \in L^1(\Omega),\ \rho^{-1} \in L^1(\Omega),\ \nabla \ln \rho \in L^2(\Omega; \mathbb{R}^N) \ \text{ i } \ \rho + \rho^{-1} \notin L^\infty(\Omega). \tag{2.1}$$

Hereafter, we assume that there exists a closed subset $\mathcal{O}$ of the set Ω such that

$$\operatorname{dist}(\mathcal{O}, \partial\Omega) = \varepsilon, \quad \rho > \varepsilon \ \text{ м.с. в } \ \Omega \setminus \mathcal{O}, \quad \text{i} \quad \rho \in L^\infty(\Omega \setminus \mathcal{O}) \tag{2.2}$$

for some $\varepsilon > 0$. In other words we assume that conditions (2.1) are not typical for boundary layer of the set Ω.

Weighted spaces. We call a nonnegative function ρ with properties (2.1)–(2.2) degenerate and consider weighted Hilbert spaces $L^2(\Omega, \rho\, dx)$ and $L^2(\Omega, \rho^{-1}\, dx)$, saying that

$$f \in L^2(\Omega, \rho\, dx) \text{ if } \|f\|^2_{L^2(\Omega, \rho\, dx)} = \int_\Omega f^2 \rho\, dx < +\infty,$$

and $g \in L^2(\Omega, \rho^{-1} dx)$ if $\|g\|^2_{L^2(\Omega, \rho^{-1}\, dx)} = \int_\Omega g^2 \rho^{-1}\, dx < +\infty.$

We define the space $W = W(\Omega, \rho dx)$ as a set of functions $y \in W_0^{1,1}$ for which the norm

$$\|y\|_\rho := \left(\int_\Omega y^2 \rho\, dx + \int_\Omega |\nabla y|^2_{\mathbb{R}^N} \rho\, dx \right)^{1/2} \tag{2.3}$$

is finite, and the space $H = H(\Omega, \rho dx)$ as the closure of the space $C_0^\infty(\Omega)$ with respect to the norm (2.3).

Note, that spaces W and H are reflexive Banach spaces with respect to the norm (2.3) due to the estimate

$$\int\limits_\Omega |\nabla y| dx \leq \left(\int\limits_\Omega \rho |\nabla y|_2^2 dx \right)^{1/2} \left(\int\limits_\Omega \rho^{-1} dx \right)^{1/2} \leq C\|y\|_\rho,$$

where $|\eta|_2 = \left(\sum\limits_{k=1}^N |\eta_k|^2 \right)^{1/2}$.

Since the smooth functions are in general not dense in the weighted Sobolev space W, it follows that $H \neq W$; that is for a "typical" degenerate weight ρ the identity $W = H$ is not always valid (for corresponding examples we refer to

[1,12,13]). However, if ρ is a non-degenerate weight function, that is, ρ is bounded between two positive constants, then it is easy to verify that $W = H = H_0^1(\Omega)$. We recall that the dual space of H is $H^* = W^{-1,2}(\Omega, \rho^{-1}dx)$ (for more details see [5]).

Remark 2.1. [16, Remark 1] In the case when the weight $\rho^{-1} \in L^1(\Omega)$, the space $H(\Omega, \rho dx)$ is continuously embedded into the space $W_0^{1,1}(\Omega)$.

Let us consider the next concept [17]

Definition 2.1. We say $\rho : \Omega \to \mathbb{R}$ is the weight function of potential type if ρ satisfies conditions (2.1)–(2.2) and there exists such constant $\widehat{C}(\Omega) > 0$, that the following inequality is fulfilled:

$$-\widehat{C}(\Omega) \leq -\triangle \ln \rho(x) - \frac{1}{2}|\nabla \ln \rho|^2_{\mathbb{R}^N} < \frac{2\lambda_*}{|x|^2_{\mathbb{R}^N}} = \frac{(N-2)^2}{2|x|^2_{\mathbb{R}^N}} \quad \text{in} \quad \Omega. \tag{2.4}$$

In this case the function $V(x) = -\triangle \ln \rho(x) - \frac{1}{2}|\nabla \ln \rho|^2_{\mathbb{R}^N}$ is called Hardy potential for the weighted function ρ.

Elliptic Variational Inequalities.

Let V be a Banach space and $K \subset V$ be a closed convex subset. Suppose also that $A : K \to V^*$ is a nonlinear operator and $f \in V^*$ is a given element of the dual space.

Let us consider the following variational problem: to find an element $y \in K$ such that

$$\langle Ay, v - y\rangle_V \geq \langle f, v - y\rangle_V, \quad \forall v \in K. \tag{2.5}$$

Referring to [9], we make use of the following assumptions.

Hypothesis 1. There exists a reflexive Banach space X such that $X \subset V^*$, the imbedding $X \hookrightarrow V^*$ is continuous, and X is dense in V^*.

Hypothesis 2. There can be found a duality mapping $J : X \to X^*$ such that $\forall y \in K$, $\forall \varepsilon > 0$ there exists an $y_\varepsilon \in K$ such that $A(y_\varepsilon) \in X$ and

$$y_\varepsilon + \varepsilon J\left(A(y_\varepsilon)\right) = y.$$

Theorem 2.1. *[9, Theorem 8.7] Assume that Hypothesis 1 and Hypothesis 2 hold true. Let operator $A : V \to V^*$ be monotone, semicontinuous, bounded and satisfy the following assumption: there exist an element $v_0 \in K$ such that*

$$\frac{\langle Ay, y - v_0\rangle_V}{\|y\|_V} \to +\infty \quad \text{as} \quad \|y\|_V \to \infty,\ y \in K.$$

Then for any solution y of variational inequality (2.5) the inclusion $Ay \in X$ takes plase provided $f \in X$.

Smoothing. Throughout the paper ε denotes a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence, while writing $\varepsilon \geq 0$, we also consider its limit $\varepsilon = 0$.

Definition 2.2. We say that a weight function ρ with properties (2.1)-(2.2) is approximated by non-degenerated weight functions $\{\rho^\varepsilon\}_{\varepsilon>0}$ on Ω if:

$$\rho^\varepsilon(x) > 0 \text{ a.e. in}\,\Omega,\;\; \rho^\varepsilon, (\rho^\varepsilon)^{-1} \in L^\infty(\Omega),\; \forall\varepsilon > 0, \tag{2.6}$$

$$\rho^\varepsilon \to \rho,\;\; (\rho^\varepsilon)^{-1} \to \rho^{-1} \;\text{ in }\; L^1(\Omega) \;\text{ as } \varepsilon \to 0. \tag{2.7}$$

Remark 2.2. The family $\{\rho^\varepsilon\}_{\varepsilon>0}$ satisfying properties (2.6)-(2.7) is called the non-degenerate perturbation of the weight function ρ.

Examples of such perturbations can be constructed using the classical smoothing. For instance, let Q be some positive compactly supported function such that $L^\infty\mathbb{R}^N$, $\int\limits_{\mathbb{R}^N} Q(x)dx = 1$, and $Q(x) = q(-x)$. Then, for a given weight function $\rho \in L^1_{loc}(\mathbb{R}^N)$, we can take $\rho^\varepsilon = (\rho)_\varepsilon$, where

$$(\rho)_\varepsilon(x) = \frac{1}{\varepsilon^N}\int\limits_{\mathbb{R}^N} Q\left(\frac{x-z}{\varepsilon}\right)\rho(z)dz = \int\limits_{\mathbb{R}^N} Q(z)\rho(x+\varepsilon z)dz. \tag{2.8}$$

In this case we say that the perturbation $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ of the original degenerate weight function ρ is conctructed by the "direct" smoothing scheme.

Lemma 2.1. *[10] If $\rho, \rho^{-1} \in L^1_{loc}(\mathbb{R}^N)$ then the "direct" smoothing $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ possesses properties (2.6)-(2.7).*

Weak compactness criterion in $L^1(\Omega)$. Throughout the paper we will often use the concepts of weak and strong convergence in $L^1(\Omega)$. Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $L^1(\Omega)$. We recall that $\{a_\varepsilon\}_{\varepsilon>0}$ is called equi-integrable if for any $\delta > 0$ there exists $\tau = \tau(\delta)$ such that $\int\limits_S |a_\varepsilon|dx < \delta$ for every $\varepsilon > 0$ and every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$. Then the following assertions are equivalent:

(i) A sequence $\{a_\varepsilon\}_{\varepsilon>0}$ is weakly compact in $L^1(\Omega)$.

(ii) The sequence $\{a_\varepsilon\}_{\varepsilon>0}$ is equi-integrable.

(iii) Given $\delta > 0$ there exists $\lambda = \lambda(\delta)$ such that $\sup\limits_{\varepsilon>0} \int\limits_{\{|a_\varepsilon|>\delta\}} |a_\varepsilon|dx < \delta$.

Theorem 2.2. *(Lebesgue's Theorem). If a bounded sequence $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ is equi-integrable and $a_\varepsilon \to a$ almost everywhere on Ω, then $a_\varepsilon \to a$ in $L^1(\Omega)$.*

Radon measures and convergence in variable spaces. By a nonnegative Radon measure on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω. The space of all nonnegative Radon measures on Ω will be denoted by $\mathcal{M}_+(\Omega)$. If μ is a nonnegative Radon measure on Ω, we will use

$L^r(\Omega, d\mu)$, $1 \le r \le \infty$, to denote the usual Lebesgue space with respect to the measure μ with the corresponding norm

$$\|f\|_{L^r(\Omega,d\mu)} = \left(\int_\Omega |f(x)|^r d\mu \right)^{1/r}.$$

Let $\{\mu_\varepsilon\}_{\varepsilon>0}$, μ be Radon measures such that μ_ε is $*$-weakly convergent to μ in $\mathcal{M}_+(\Omega)$; that is,

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi d\mu_\varepsilon = \int_\Omega \varphi d\mu \quad \forall \varphi \in C_0(\mathbb{R}^N), \tag{2.9}$$

where $C_0(\mathbb{R}^N)$ is the space of all compactly supported continuous functions. A typical example of such measures is $d\mu_\varepsilon = \rho^\varepsilon(x)dx$, $d\mu = \rho(x)dx$, where $0 \le \rho^\varepsilon \rightharpoonup \rho$ in $L^1(\Omega)$. Let us recall the definition and main properties of convergence in the variable L^2-space [13].

1. A sequence $\{v_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)\}$ is called bounded if

$$\limsup_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^2 d\mu_\varepsilon < +\infty.$$

2. A bounded sequence $\{v_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)\}$ converges weakly to $v \in L^2(\Omega, d\mu)$ if

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon \varphi d\mu_\varepsilon = \int_\Omega v\varphi d\mu$$

for any $\varphi \in C_0^\infty(\Omega)$ and we write $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon)$.

3. The strong convergence $v_\varepsilon \to v$ in $L^2(\Omega, d\mu_\varepsilon)$ means that $v \in L^2(\Omega, d\mu)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon z_\varepsilon d\mu_\varepsilon = \int_\Omega vz d\mu \quad \text{as} \quad z_\varepsilon \rightharpoonup z \quad \text{in} \quad L^2(\Omega, d\mu_\varepsilon). \tag{2.10}$$

The following convergence properties in variable spaces hold:

(a) *Compactness criterium*: if a sequence is bounded in $L^2(\Omega, d\mu_\varepsilon)$, then this sequence is compact with respect to the weak convergence.

(b) *Property of lower semicontinuity*: if $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon)$, then

$$\liminf_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^2 d\mu_\varepsilon \ge \int_\Omega v^2 d\mu. \tag{2.11}$$

(c) *Criterium of strong convergence*: $v_\varepsilon \to v$ if and only if $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^2 d\mu_\varepsilon = \int_\Omega v^2 d\mu. \tag{2.12}$$

Let us recall some well-known results concerning the convergence in the variable space $L^2(\Omega, d\mu_\varepsilon)$.

Lemma 2.2. *[10, 13, 15] If $\{\rho^\varepsilon\}_{\varepsilon>0}$ is non-degenerate perturbation of the weight function $\rho(x) \geq 0$, then:*

(A1) $((\rho^\varepsilon)^{-1}) \to \rho^{-1}$ *in* $L^2(\Omega, \rho^\varepsilon dx)$.

(A2) [$v_\varepsilon \rightharpoonup v$ *in* $L^2(\Omega, \rho^\varepsilon dx)$]$\Rightarrow$ [$v_\varepsilon \rightharpoonup v$ *in* $L^1(\Omega)$].

(A3) If a sequence $\{v_\varepsilon \in L^2(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ *is bounded, then the weak convergence* $v_\varepsilon \to v$ *in* $L^2(\Omega, \rho^\varepsilon dx)$ *is equivalent to the weak convergence* $\rho^\varepsilon v_\varepsilon \rightharpoonup \rho v$ *in* $L^1(\Omega)$.

(A4) If $a \in L^\infty$ *and* $v_\varepsilon \rightharpoonup v$ *in* $L^2(\Omega, \rho^\varepsilon dx)$, *then* $av_\varepsilon \rightharpoonup av$ *in* $L^2(\Omega, \rho^\varepsilon dx)$.

Variable Sobolev spaces. Let $\rho(x)$ be a degenerate weight function and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of the function ρ in the sense of Definition 2.2. We denote by $H(\Omega, \rho^\varepsilon dx)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\rho^\varepsilon}$. Since for every ε the function ρ^ε is non-degenerate, that is, ρ^ε is bounded between two positive constants, the space $H(\Omega, \rho^\varepsilon dx)$ (and the spaces $L^2(\Omega, \rho^\varepsilon dx)$ and $L^2(\Omega, (\rho^\varepsilon)^{-1} dx)$) coincides with the classical Sobolev space $H_0^1(\Omega)$ (with $L^2(\Omega)$).

Definition 2.3. We say that a sequence $\{y_\varepsilon \in H(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ converges weakly to an element $y \in W$ as $\varepsilon \to 0$, if the following hold: (i) This sequence is bounded. (ii) $y_\varepsilon \rightharpoonup y$ in $L^2(\Omega, \rho^\varepsilon dx)$. (iii) $\nabla y_\varepsilon \rightharpoonup \nabla y$ in $L^2(\Omega, \rho^\varepsilon dx)^N$.

Compensated Compactness Lemma in variable Lebesgue and Sobolev spaces. Let p, q such that $2 \leq p < \infty$, $1/p + 1/q{=}1$ and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function ρ. We associate to every ρ^ε the space

$$X(\Omega, \rho^\varepsilon dx) = \left\{ \vec{f} \in L^q(\Omega, \rho^\varepsilon dx)^N |\ \mathrm{div}\left(\rho^\varepsilon \vec{f}\right) \in L^q(\Omega) \right\}\ \forall \varepsilon > 0 \tag{2.13}$$

with the norm

$$\|\vec{f}\|_{X(\Omega,\rho^\varepsilon dx)} = \left(\|\vec{f}\|^q_{L^q(\Omega,\rho^\varepsilon dx)^N} + \|\mathrm{div}\left(\rho^\varepsilon \vec{f}\right)\|^q_{L^q(\Omega)} \right)^{1/q}.$$

We say that a sequence $\left\{ \vec{f}_\varepsilon \in X(\Omega, \rho^\varepsilon dx) \right\}_{\varepsilon>0}$ is bounded if

$$\limsup_{\varepsilon\to 0} \|\vec{f}_\varepsilon\|_{X(\Omega,\rho^\varepsilon dx)} < +\infty.$$

In order to discuss the problem of H-attainability we need the following result.

Lemma 2.3. *[3] Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function $\rho(x) > 0$. Let* $\left\{ \vec{f} \in L^q(\Omega, \rho^\varepsilon dx)^N \right\}_{\varepsilon>0}$ *and* $\{g_{\varepsilon \in H(\Omega,\rho^\varepsilon dx)}\}_{\varepsilon>0}$ *be sequences such that* $\{\vec{f_\varepsilon}\}_{\varepsilon>0}$ *is bounded in the variable space* $X(\Omega, \rho^\varepsilon dx)$, $\vec{f_\varepsilon} \rightharpoonup \vec{f}$ *weakly in* $L^q(\Omega, \rho^\varepsilon dx)^N$, $\{g_\varepsilon\}_{\varepsilon>0}$ *is bounded in the variable space* $H(\Omega, \rho^\varepsilon dx)$, $g_\varepsilon \rightharpoonup g$ *in* $L^p(\Omega)$, *and* $\nabla g_\varepsilon \rightharpoonup \nabla g$ *in* $L^p(\Omega, \rho^\varepsilon dx)^N$. *Then*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi \left(\vec{f}_\varepsilon, \nabla g_\varepsilon \right)_{\mathbb{R}^N} \rho^\varepsilon dx = \int_\Omega \varphi \left(\vec{f}, \nabla g \right)_{\mathbb{R}^N} \rho dx,\ \forall \varphi \in C_0^\infty(\Omega). \tag{2.14}$$

Further, we consider a special "lifting" operator

$$T_\varepsilon : L^p(\Omega, \rho dx) \to L^p(\Omega, \rho^\varepsilon dx)$$

defined as follows

$$\int_\Omega T_\varepsilon y \varphi \rho^\varepsilon dx = \int_\Omega y(\varphi)_\varepsilon \rho dx \ \forall \varphi \in C_0^\infty(\Omega), \ \forall \varepsilon > 0. \tag{2.15}$$

Firstly this operator was constructed in [14] for the case of an arbitrary measure. Let us consider the following well-known result.

Lemma 2.4. *[10, Lemma 7.2] Let $\rho \in L^1_{loc}(\mathbb{R}^N)$ be a degenerate weight function and let $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of ρ. Then for every element $y \in L^p(\Omega, \rho dx)$ there exists a sequence $\{T_\varepsilon y \in L^p(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ such that $T_\varepsilon y \to y$ in $L^p(\Omega, \rho^\varepsilon dx)$.*

Let us recall that a function $a \in L^2(\Omega, \rho dx)$ and a vector $b \in L^2(\Omega, \rho dx)^N$ are related by the equality

$$\operatorname{div}(\rho b) = a \quad \text{if} \quad \int_\Omega (b, \nabla\varphi)_{\mathbb{R}^N} \rho dx = -\int_\Omega a\varphi\rho dx \ \forall \varphi \in C_0^\infty(\Omega). \tag{2.16}$$

In a similar way, for $a^\varepsilon \in L^2(\Omega, \rho^\varepsilon dx)$ and $b \in L^2(\Omega, \rho^\varepsilon dx)^N$, we have

$$\operatorname{div}(\rho^\varepsilon b^\varepsilon) = a^\varepsilon \quad \text{if} \quad \int_\Omega (b^\varepsilon, \nabla\varphi)_{\mathbb{R}^N} \rho^\varepsilon dx = -\int_\Omega a^\varepsilon\varphi\rho^\varepsilon dx \ \forall \varphi \in C_0^\infty(\Omega). \tag{2.17}$$

Note that by arguments of completion, the above identities can be extended to test functions from H and $H(\Omega, \rho^\varepsilon dx)$, respectively.

Lemma 2.5. *[10, Lemma 7.3] If $a \in L^2(\Omega, \rho dx)$ and $b \in L^2(\Omega, \rho dx)^N$ are related by (2.16), then $a^\varepsilon = T_\varepsilon a$ and $b^\varepsilon = T_\varepsilon b$ are related by (2.17).*

Following [10,11] we can give a dual description of the weighted Sobolev space H. Let us consider two spaces: the first is X^2_ρ as the closure of the set $\{(y, \nabla y),\ y \in C_0^\infty(\Omega)\}$ in $L^2(\Omega, \rho dx) \times L^2(\Omega, \rho dx)^N$, hence, the elements of this space are pairs (y, v), where y is a function in H and $v = \nabla y$ is its gradient. The second space $\tilde{X}^2_\rho$ consists of pairs (y, v), where $y \in L^2(\Omega, \rho dx)$ abd $v \in L^2(\Omega, \rho dx)^N$ are such that

$$\int_\Omega ya\rho dx = -\int_\Omega (v, b)_{\mathbb{R}^N} \rho dx \tag{2.18}$$

for any (a, b) satisfying the conditions

$$a \in L^2(\Omega, \rho dx), \ b \in L^2(\Omega, \rho dx)^N, \ a = \operatorname{div}(\rho b) \tag{2.19}$$

It is easy to see that X^2_ρ and $\tilde{X}^2_\rho$ are closed in $L^2(\Omega, \rho dx)^{N+1}$ and $X^2_\rho \subseteq \tilde{X}^2_\rho$. Moreover, from [10, Lemma 7.4] (or [11, Theorem 1]) we have that $X^2_\rho = \tilde{X}^2_\rho$.

The next Theorem establishes the possibility of passing to the limit as $\varepsilon \to 0$ in variable space $H(\Omega, \rho^\varepsilon dx)$.

Theorem 2.3. *[10, Theorem 7.1] Let $\rho^\varepsilon = (\rho)_\varepsilon$ be a direct smoothing of a degenerate weight $\rho \in L^1_{loc}(\mathbb{R}^N)$ and let $y^\varepsilon \in H(\Omega, \rho^\varepsilon dx)$, $y^\varepsilon \rightharpoonup y$ in $L^2(\Omega, \rho^\varepsilon dx)$, $\nabla y^\varepsilon \rightharpoonup v$ in $L^2(\Omega, \rho^\varepsilon dx)^N$. Then $y \in H$ and $v = \nabla y$.*

3. Setting of the Optimal Control Problem

Let K be a non-empty convex closed subset of the space W, and let K be sequentialy closed with respect to the norm

$$\|y\|^2 := \int_\Omega y^2 \rho\, dx + \int_\Omega \left|\nabla y + \frac{y}{2}\nabla \ln \rho\right|^2_{\mathbb{R}^N} \rho\, dx. \tag{3.1}$$

Let $y_{ad} \in L^2(\Omega)$, $f \in L^2(\Omega, \rho^{-1}\, dx)$ and $u_0 \in L^2(\Omega, \rho^{-1}\, dx)$ be given distribution, and U_∂ be a non-empty convex closed subset in $L^2(\Omega, \rho^{-1}\, dx)$ such that

$$U_\partial = \{u \in L^2(\Omega, \rho^{-1}\, dx) : \ \|u - u_0\|_{L^2(\Omega, \rho^{-1}\, dx)} \le R\}. \tag{3.2}$$

Hereinafter functions $u \in U_\partial$ are considered to be admissible controls.

The main object we deal with in the paper is the following optimal control problem for the variational inequality with control in the right hand side:

$$I(u, y) = \frac{1}{2}\|y - y_{ad}\|^2_{L^2(\Omega, \rho dx)} \to \inf, \tag{3.3}$$

$$u \in U_\partial, \ y \in K, \tag{3.4}$$

$$\int_\Omega (\nabla y, \nabla v - \nabla y)_{\mathbb{R}^N}\, \rho\, dx \ge \int_\Omega (f + u)\,(v - y)\; dx, \quad \forall\, v \in K. \tag{3.5}$$

Let us consider the following linear operator related to the variational inequality (3.5):

$$A : W^{1,2}_0(\Omega; \rho\, dx) \to \left(W^{1,2}_0(\Omega; \rho\, dx)\right)^*,$$

that is defined by the rule:

$$\left\langle Ay,\ v - y\right\rangle_{H(\Omega; \rho dx)} = \int_\Omega (\nabla y, \nabla v - \nabla y)_{\mathbb{R}^N}\, \rho\, dx \quad \forall v \in K.$$

Here

$$\langle \cdot, \cdot \rangle_{H(\Omega; \rho dx)} : (H(\Omega; \rho dx))^* \times H(\Omega; \rho dx) \to \mathbb{R}$$

is the duality pairing. It is clear that

$$Ay = -\operatorname{div}(\rho(x)\nabla y).$$

Similarly to [4] let us consider the next definitions.

Definition 3.1. We say that a function $y = y(u, f) \in K$ is a W-solution to degenerate variational inequality (3.4)-(3.5) if

$$\langle -\operatorname{div}(\rho(x)\nabla y), v - y\rangle_W \ge \langle f + u, v - y\rangle_W \tag{3.6}$$

holds for any $v \in K$.

Definition 3.2. Let $\tilde{K}$ be a closure in the space $C_0^\infty(\Omega)$ of the set $K \cap C_0^\infty(\Omega)$. We say that a function $y = y(u, f) \in \tilde{K}$ is an H-solution to variational inequality (3.4)-(3.5) if

$$\langle -\operatorname{div}(\rho(x)\nabla y), v - y\rangle_{H(\Omega;\rho dx)} \geq \langle f + u, v - y\rangle_{H(\Omega;\rho dx)} \tag{3.7}$$

holds for any $v \in \tilde{K}$.

Remark 3.1. It is easy to say that the set $\tilde{K} \subset H$ is closed and convex.

Let us remark that in the case when the function ρ is a weight function of potential type in the sense of Definition 2.1 we can prove the existence and uniqueness of W-solution for the inequality (3.4)-(3.5), namely the following result takes place:

Theorem 3.1. *[17, Теорема 2] Let $\rho : \Omega \to \mathbb{R}_+$ be a weight function of potential type. Then for given $f \in L^2(\Omega, \rho^{-1}dx)$ and $u \in U_\partial$ the variational inequality* (3.4)–(3.5) *has unique solution $y = y(u, f) \in K$ such that $y = z/\sqrt{\rho}$ and $z \in H_0^1(\Omega)$.*

Remark 3.2. Similar result with Theorem 3.1 concerning existence and uniqueness of H-solution to problem (3.4)-(3.5) can be easily obtained using similar argumentation.

Taking this fact into account we can introduce two sets of admissible pairs to the optimal control problem (3.3)-(3.5):

$$\Xi_W = \{(u, y) \in U_\partial \times W \,|\, y \in K,\ (u, y) \text{ are related by } (3.6)\}, \tag{3.8}$$

$$\Xi_H = \{(u, y) \in U_\partial \times H \,|\, y \in \tilde{K},\ (u, y) \text{ are related by } (3.7)\}. \tag{3.9}$$

Hence for the given control object described by relations (3.4)-(3.5) with both fixed control constrains ($u \in U_\partial$) and fixed cost functional (3.3), we have two different statement of the original optimal control problem, namely

$$\left\langle \inf_{(u,y)\in\Xi_W} I(u, y) \right\rangle \quad \text{and} \quad \left\langle \inf_{(u,y)\in\Xi_H} I(u, y) \right\rangle.$$

Having assumed that $W \neq H$ for a given degenerate weight function $\rho \geq 0$, we can come to the effect which is usually called the Lavrentieff phenomenon. It means that for some $u \in U_\partial$ and $f \in L^2(\Omega, \rho^{-1}dx)$ an H-solution to problem (3.4)-(3.5) does not coincide with its W-solution [13].

Remark 3.3. In view of Theorem 3.1 and Remark 3.2, the set Ξ_H is always nonempty.

Let us consider the following concept.

Definition 3.3. We say that a pair $(u^0, y^0) \in L^2(\Omega, \rho^{-1}dx) \times H$ is an H-optimal solution to problem (3.3)-(3.5) if $(u^0, y^0) \in \Xi_H$ and

$$I(u^0, y^0) = \inf_{(u,y)\in\Xi_H} I(u, y)$$

.

Note that optimal control problem (3.3)-(3.5) is solvable, namely the following result takes place.

Theorem 3.2. *Let $\rho(x) > 0$ be a degenerate weight function of potential type. Then the set of H-optimal solutions to problem* (3.3)-(3.5) *is non-empty* $\forall f \in L^2(\Omega, \rho^{-1}dx)$.

4. Attainability of H-optimal Solutions

In this section we propose a regular algorithm of approximation (perturbation) for the original degenerate optimal control problem (3.3)-(3.5) and it will be shown that H-optimal solutions of mentioned problem can be attained by optimal solutions of perturbed problems. Note that in view of Theorem 3.2 that the set of H-optimal solutions to the problem (3.3)-(3.5) is non-empty.

Let ρ be a degenerate weight function with properties (2.2)-(2.1), and let $\{\rho\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of ρ in the sense of Definition 2.2

Definition 4.1. We say that a bounded sequence

$$\{(u_\varepsilon, y_\varepsilon) \in \mathbb{Y}(\Omega, \rho^\varepsilon dx) = L^2(\Omega, (\rho^\varepsilon)^{-1}dx) \times H(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$$

w-converges to $(u, y) \in L^2(\Omega, \rho^{-1}dx) \times W$ in the variable space $\mathbb{Y}(\Omega, \rho^\varepsilon dx)$ as $\varepsilon \to 0$, if $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega, (\rho^\varepsilon)^{-1}dx)$, $y_\varepsilon \rightharpoonup y$ in $L^2(\Omega, \rho^\varepsilon dx)$, $\nabla y_\varepsilon \rightharpoonup \nabla y$ in $L^2(\Omega, \rho^\varepsilon dx)^N$.

Definition 4.2. We say that a minimization problem

$$\left\langle \inf_{(u,y)\in\Xi_H} I(u, y) \right\rangle \tag{4.1}$$

is a weak variational limit (or variational w-limit) of the sequence

$$\left\{ \left\langle \inf_{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon) \right\rangle;\ \Xi_\varepsilon \subset \mathbb{Y}(\Omega, \rho^\varepsilon dx),\ \varepsilon > 0 \right\}, \tag{4.2}$$

with respect to w-convergence in variable space $\mathbb{Y}(\Omega, \rho^\varepsilon dx)$, if the following conditions are satisfied:

(1) if $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, and a sequence $\{(u_k, y_k) \in \Xi_{\varepsilon_k}\}_{\varepsilon>0}$ w-converges to a pair (u, y), then

$$(u, y) \in \Xi_H;\ I(u, y) \leq \liminf_{k\to\infty} I_{\varepsilon_k}(u_k, y_k); \tag{4.3}$$

(2) for every pair $(u, y) \in \Xi_H$ and any value $\delta > 0$ there exists a realizing sequence $\{(\hat{u}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ such that

$$(\hat{u}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon\ \forall\varepsilon > 0,\ (\hat{u}_\varepsilon, \hat{y}_\varepsilon)\ w-\ \text{converges to}\ (\hat{u}, \hat{y}), \tag{4.4}$$

$$\|u - \hat{u}\|_{L^2(\Omega,\rho^{-1}dx)} + \|y - \hat{y}\|_\rho \leq \delta,\ I(u, y) \geq \limsup_{\varepsilon\to0} I_\varepsilon(\hat{u}_\varepsilon, \hat{y}_\varepsilon) - \delta. \tag{4.5}$$

The last definition is motivated by the following property of variational w-limits (for the details we refer to [2]).

Theorem 4.1. *Assume that* (4.1) *is a weak variational limit of the sequence* (4.2)*, and the constrained minimization problem* (4.1) *has a solution. Suppose* $\{(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\}$ *is a sequence of optimal pairs to* (4.2)*. Then there exists a pair* $(u^0, y^0) \in \Xi_H$ *such that* $(u^0_\varepsilon, y^0_\varepsilon)$ *w-converges to* (u^0, y^0)*, and*

$$\inf_{(u,y)\in\Xi_H} I(u,y) = I(u^0,y^0) = \lim_{\varepsilon\to 0} \inf_{(u_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(u_\varepsilon, y_\varepsilon).$$

Let us consider the sequences $\{K_\varepsilon\}_{\varepsilon>0}$ and $\{U^\varepsilon_\partial\}_{\varepsilon>0}$ of non-empty convex closed subsets, which sequentially converges to sets $\widetilde{K}$ and U_∂, respectively, in the sense of Kuratovski as $\varepsilon \to 0$ with respect to weak topology of spaces $H(\Omega, \rho^\varepsilon dx)$ and $L^2(\Omega, (\rho^\varepsilon)^{-1}dx)$, respectively, and let Hypothesis 2 hold true for $X = L^2(\Omega, (\rho^\varepsilon)^{-1}dx)$ and $V = H(\Omega, \rho^\varepsilon dx)$ $\forall \varepsilon > 0$. Taking into account Theorem 4.1, we consider the following collection of perturbed optimal control problems for non-degenerate elliptic variational inequalities:

$$\text{Minimize} \quad \left\{ I_\varepsilon(u,y) = \frac{1}{2}\int_\Omega |y(x) - y_{ad}|^2 dx \right\}, \tag{4.6}$$

$$u \in U^\varepsilon_\partial, \; y \in K_\varepsilon, \tag{4.7}$$

$$\langle -\operatorname{div}(\rho^\varepsilon(x)\nabla y), v - y\rangle_{H(\Omega;\rho^\varepsilon dx)} \geq \langle f + u, v - y\rangle_{H(\Omega;\rho^\varepsilon dx)} \; \forall v \in K_\varepsilon, \tag{4.8}$$

where the elements $y_{ad} \in L^2(\Omega)$, $f \in L^2(\Omega, \rho^{-1}dx) \subset L^2(\Omega, (\rho^\varepsilon)^{-1}dx)$ are the same as for original problem (3.3)-(3.5). For every $\varepsilon > 0$ we define Ξ_ε as a set of all admissible pairs to the problem (4.6)-(4.8), namely $(u, y) \in \Xi_H$ if and only if the pair (u, y) satisfies (4.7)-(4.8).

Let us discuss the optimality conditions for problem (4.6)-(4.8). Let $V = H(\Omega, \rho^\varepsilon dx)$, $H = L^2(\Omega)$. Taking into account suggestions of the section 2, we have that V and H are Hilbert spaces, and $V \hookrightarrow H$ continuously and V is dense in H. Let us denote by $(\cdot,\cdot)$ the scalar product in H. Let us identify H with its conjugated H^*, and let V^* be the space conjugated to V. Then $V \subset H \subset V^*$ and every space is dense in the next one and corresponding embeddings are continuous. Let $U = L^2(\Omega, (\rho^\varepsilon)^{-1}dx)$ be the control space (which coincides with $L^2(\Omega)$), U^ε_∂ is convex and closed in U by the construction. Let us consider an operator $A : V \to V^*$, $Ay = -\operatorname{div}(\rho^\varepsilon(x)\nabla y)$, and functions f and y_{ad} as in previous suggestions. For every control $u \in U$ the state $y(u)$ is defined as the solution to the following problem

$$Ay = f + u, \; y \in H(\Omega, \rho^\varepsilon dx). \tag{4.9}$$

Let us consider for every $u \in U$ the cost functional

$$J(u,y) = \frac{1}{2}\|y(u) - y_{ad}\|^2_H. \tag{4.10}$$

The optimal control problem is to find such pair $(u,y)\in U_\partial^\varepsilon\times H(\Omega,\rho^\varepsilon dx)$ that

$$J(u,y)=\inf_{(v,y(v))\in U_\partial^\varepsilon\times H(\Omega,\rho^\varepsilon dx)} J(v,y(v)) \text{ with conditions (4.9).} \tag{4.11}$$

It is known that the solution of the optimal control problem is characterized by the inequality

$$J'_u(u,y(u))(v-u)\geq 0,\ \forall v\in U_\partial^\varepsilon. \tag{4.12}$$

Since, A is an isomorphism of the space V to V^* (see for details [8]), then $y(u)=A^{-1}(f+u)$, and then

$$y'(u)(v-u)=A^{-1}(v-u)=y(v)-y(u).$$

Hence, (4.12) is equivalent to the following inequality:

$$(y(u)-y_{ad},y(v)-y(u))\geq 0,\ \forall v\in U_\partial^\varepsilon. \tag{4.13}$$

Let $A^*\in\mathcal{L}(V,V^*)$ be the conjugate operator to A and it is an isomorphism of V on V^* as well as A. For the control $v\in U_\partial^\varepsilon$ let us define the conjugate state $p(v)\in V$ by the next relation:

$$A^*p(v)=y(v)-y_{ad}. \tag{4.14}$$

Then

$$\begin{aligned}(A^*p(u),y(v)-y(u))&=(y(u)-y_{ad},y(v)-y(u))=(p(u),Ay(v)-Ay(u))\\ &=(p(u),v-u)=(p(u),v-u)_U=\int_\Omega p(u)(v-u)dx\geq 0,\end{aligned}$$

since $p(u)\in V\subset L^2(\Omega,\rho^\varepsilon dx)$, $v-u\in L^2(\Omega,(\rho^\varepsilon)^{-1}dx)$. Similarly to [1, Theorem 1.4], obtained results can be formulated as the following theorem.

Theorem 4.2. *Let $a(u,v)=(Au,v)$ be a bilinear continuous and coercive form on V, and cost functional be as in (4.10). The element $u\in U_\partial^\varepsilon$ is the optimal control if and only if the following relations are fulfilled:*

$$-div(\rho^\varepsilon(x)y)=f+u \quad in\ \Omega,\ y\in V,$$

$$-div(\rho^\varepsilon(x)p)=y-y_{ad} \quad in\ \Omega,\ p\in V,$$

$$\int_\Omega p(u)(v-u)dx\geq 0,\ \forall v\in U_\partial^\varepsilon.$$

Remark 4.1. Let us recall that sequential K-upper and K-lower limits of a sequence of sets $\{E_k\}_{k\in\mathbb{N}}$ are defined as follows, respectively:

$$K_s-\overline{lim}E_k=\{y\in X:\exists\sigma(k)\to\infty\ ,\exists y_k\to y,\ \forall k\in\mathbb{N}:\ y_k\in E_{\sigma(k)}\},$$

$$K_s-\underline{lim}E_k=\{y\in X:\exists y_k\to y\ \exists k\geq k_0\in\mathbb{N}:y_k\in E_k\}.$$

The sequence $\{E_k\}_{k\in\mathbb{N}}$ sequantially converges in the sense of Kuratovski to the set E (shortly, K_s-converges), if $E=K_s-\underline{lim}E_k=K_s-\overline{lim}E_k$.

Lemma 4.1. *Let $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho \geq 0$. Let $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible pairs to the problem (4.6)-(4.8). Then there exists a pair $\{(u^*, y^*)\}$ and a subsequence $\{(u_{\varepsilon_k}, y_{\varepsilon_k})\}_{k\in\mathbb{N}}$ of $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ such that $(u_{\varepsilon_k}, y_{\varepsilon_k})$ w-converges to $\{(u^*, y^*)\}$ as $k \to \infty$ and $(u^*, y^*) \in \Xi_H$.*

Proof. Let us consider the following variational inequality:

$$\langle -\mathrm{div}(\rho^\varepsilon \nabla y_\varepsilon), v_\varepsilon - y_\varepsilon \rangle_{H(\Omega,\rho^\varepsilon dx)} \geq \langle f + u_\varepsilon, v_\varepsilon - y_\varepsilon \rangle_{H(\Omega,\rho^\varepsilon dx)}, \ \forall v_\varepsilon \in K_\varepsilon. \quad (4.15)$$

Let us show the bondedness of the sequence $\{y_\varepsilon\}_{\varepsilon>0}$ in the space $H(\Omega, \rho^\varepsilon dx)$. Let us suppose that $\|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} \to \infty$ as $\varepsilon \to 0$. Then on the one hand

$$\begin{aligned} &\langle -\mathrm{div}(\rho^\varepsilon \nabla y_\varepsilon), y_\varepsilon - v_\varepsilon \rangle_{H(\Omega,\rho^\varepsilon dx)} \\ &\|f + u_\varepsilon\|_{L^2(\Omega,(\rho^\varepsilon)^{-1}dx)} \|y_\varepsilon - v_\varepsilon\|_{L^2(\Omega,\rho^\varepsilon dx)} \\ &\leq \|f + u_\varepsilon\|_{L^2(\Omega,(\rho^\varepsilon)^{-1}dx)} \|y_\varepsilon - v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}, \ \forall v_\varepsilon \in K_\varepsilon, \ \forall \varepsilon > 0. \end{aligned} \quad (4.16)$$

On the other hand , for arbitrary fixed element $v \in \tilde{K}$ let us consider the sequence $\{v_\varepsilon \in K_\varepsilon\}_{\varepsilon>0}$ such that $v_\varepsilon \rightharpoonup v$ in $H(\Omega, \rho^\varepsilon dx)$ (note, that such sequence always exists provided $\tilde{K} = K_s - \lim K_\varepsilon$), and taking into account the definition and properties of the space $H(\Omega, \rho^\varepsilon dx)$ and operator $A : H(\Omega, \rho^\varepsilon dx) \to (H(\Omega, \rho^\varepsilon dx))^*$, $Ay_\varepsilon = -\mathrm{div}(\rho^\varepsilon \nabla y_\varepsilon)$, we obtain such estimations:

$$\langle Ay_\varepsilon, y_\varepsilon \rangle_{H(\Omega,\rho^\varepsilon dx)} = \int\limits_\Omega (\nabla y_\varepsilon, \nabla y_\varepsilon)_{\mathbb{R}^N} \rho^\varepsilon dx \geq C_1 \|y_\varepsilon\|^2_{H(\Omega,\rho^\varepsilon dx)}, \ C_1 > 0,$$

$$\langle Ay_\varepsilon, y_\varepsilon - v_\varepsilon \rangle_{H(\Omega,\rho^\varepsilon dx)} \geq C_1 \|y_\varepsilon\|^2_{H(\Omega,\rho^\varepsilon dx)} - \|\nabla y_\varepsilon\|_{L^2(\Omega,\rho^\varepsilon dx)^N} \|\nabla v_\varepsilon\|_{L^2(\Omega,\rho^\varepsilon dx)^N}.$$

Hence, we have the following relations

$$\begin{aligned} &\frac{\langle -\mathrm{div}(\rho^{\varepsilon \nabla y_\varepsilon}), y_\varepsilon - v_\varepsilon \rangle_{H(\Omega,\rho^\varepsilon dx)}}{\|y_\varepsilon - v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}} \\ &\geq \frac{C_1 \|y_\varepsilon\|^2_{H(\Omega,\rho^\varepsilon dx)} - \|\nabla y_\varepsilon\|_{L^2(\Omega,\rho^\varepsilon dx)^N} \|\nabla v_\varepsilon\|_{L^2(\Omega,\rho^\varepsilon dx)^N}}{\|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} + \|v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}} \\ &\geq \frac{C_1 \|y_\varepsilon\|^2_{H(\Omega,\rho^\varepsilon dx)} - C_2 \|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} \|v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}}{\|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} + \|v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}} \\ &\geq \|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} \left(\frac{C_1 \|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} - C_2 \|v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}}{\|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} + \|v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}} \right) \\ &= \|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)} \left(\frac{C_1 - C_2 \frac{\|v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}}{\|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}}}{1 + \frac{\|v_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}}{\|y_\varepsilon\|_{H(\Omega,\rho^\varepsilon dx)}}} \right) \to \infty, \ \varepsilon \to 0, C_2 > 0 \end{aligned}$$

since the sequence $\{v_\varepsilon\}_{\varepsilon>0}$ is bounded in $H(\Omega, \rho^\varepsilon dx)$. The obtained contradiction with (4.16) implies that $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $H(\Omega, \rho^\varepsilon dx)$. Note that from definition of sets U_∂^ε we have that the sequence $\{u_\varepsilon \in U_\partial^\varepsilon\}_{\varepsilon>0}$ is bounded in the space $L^2(\Omega, (\rho^\varepsilon)^{-1}dx)$.

Hence, there exists a subsequence $\{\varepsilon_k\}$ of the sequence $\{\varepsilon\}$, converging to 0 and elements $u^* \in L^2(\Omega, \rho^{-1}dx)$, $y^* \in L^2(\Omega, \rho dx)$, $\vec{v} \in L^2(\Omega, \rho dx)^N$ such that $u_{\varepsilon_k} \rightharpoonup u^*$ in $L^2(\Omega, (\rho^\varepsilon)^{-1}dx)$, $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^2(\Omega, \rho^\varepsilon dx)$, $\nabla y_{\varepsilon_k} \rightharpoonup \vec{v}$ in $L^2(\Omega, (\rho^\varepsilon)^{-1}dx)^N$. By Theorem 2.3, we have that $y^* \in H$ and $v = \nabla y^*$ and, moreover, we have $y^* \in \tilde{K}$ and $u^* \in U_\partial$.

In order to prove the lemma, it is left to pass to the limit in the inequality (4.15) as $\varepsilon \to 0$. Let us take in Hypothesis 1 $V = H(\Omega, \rho^{\varepsilon_k}dx)$, $X = L^2(\Omega)$. In this case it is easy to see that the imbedding $X \hookrightarrow V^*$ is dense and continuous, and the imbedding $H(\Omega, \rho^{\varepsilon_k}dx) \hookrightarrow L^2(\Omega)$ is compact and dense (for details we refer to [7]). Since $f \in L^2(\Omega, \rho^{-1}dx) \subset L^2(\Omega, (\rho^{\varepsilon_k})^{-1}dx) \subset L^2(\Omega)$, then in view of Theorem 2.1 we have $\operatorname{div}(\rho^{\varepsilon_k}\nabla y_{\varepsilon_k}) \in L^2(\Omega)$ $\forall k \in \mathbb{N}$. Let us consider the next relation

$$\int_\Omega \operatorname{div}(\rho^{\varepsilon_k}\nabla y_{\varepsilon_k})\varphi dx = -\int_\Omega (\nabla y_{\varepsilon_k}, \nabla\varphi)_{\mathbb{R}^N}\rho^{\varepsilon_k}dx$$

$$\to -\int_\Omega (\nabla y^*, \nabla\varphi)_{\mathbb{R}^N}\rho dx = \int_\Omega \operatorname{div}(\rho\nabla y)\varphi dx, \ \forall\varphi \in C_0^\infty(\Omega), \quad \text{as} \quad k \to \infty.$$

Hence, $\operatorname{div}(\rho^{\varepsilon_k}\nabla y_{\varepsilon_k}) \rightharpoonup \operatorname{div}(\rho\nabla y)$ in $L^2(\Omega)$ so the sequence $\{\operatorname{div}(\rho^{\varepsilon_k}\nabla y_{\varepsilon_k})\}_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$.

Let us consider the sequence $g_{\varepsilon_k} := v_{\varepsilon_k} - y_{\varepsilon_k}$. We know that the sequence $\{g_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is bounded in $H(\Omega, \rho^{\varepsilon_k}dx)$ and $g_{\varepsilon_k} \rightharpoonup g := v - y^*$ in $H(\Omega, \rho^{\varepsilon_k}dx)$ as $k \to \infty$, where $\{v_{\varepsilon_k} \in K_{\varepsilon_k}\}_{k\in\mathbb{N}}$ weakly converges to $v \in \tilde{K}$ in $H(\Omega, \rho^{\varepsilon_k}dx)$. In view of properties of spaces $L^2(\Omega, \rho^{\varepsilon_k}dx)$ we have that the sequence $\{g_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ and $g_{\varepsilon_k} \rightharpoonup g := v - y^*$ in $L^2(\Omega)$. Taking into account Lemma 2.3 we obtain

$$\begin{aligned} &\langle -\operatorname{div}(\rho^{\varepsilon_k}(x)\nabla y_{\varepsilon_k}), v_{\varepsilon_k} - y_{\varepsilon_k}\rangle_{H(\Omega,\rho^{\varepsilon_k}dx)} \\ &\to \langle -\operatorname{div}(\rho(x)\nabla y), v - y^*\rangle_{H(\Omega,\rho dx)}, \quad \text{as} \quad k \to \infty. \end{aligned} \tag{4.17}$$

Let us consider the right hand side of the inequality (4.15).

$$\int_\Omega (f + u_{\varepsilon_k})(v_{\varepsilon_k} - y_{\varepsilon_k})dx = \int_\Omega f v_{\varepsilon_k}dx - \int_\Omega f y_{\varepsilon_k}dx + \int_\Omega u_{\varepsilon_k}v_{\varepsilon_k}dx - \int_\Omega u_{\varepsilon_k}y_{\varepsilon_k}dx.$$

Let us represent the last term by the following way:

$$-\int_\Omega u_{\varepsilon_k}y_{\varepsilon_k}dx \pm \int_\Omega u_{\varepsilon_k}y^*dx = -\int_\Omega u_{\varepsilon_k}(y_{\varepsilon_k} - y^*)dx - \int_\Omega u_{\varepsilon_k}y^*dx.$$

Since $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^2(\Omega, \rho^{\varepsilon_k} dx)$, $\nabla y_{\varepsilon_k} \rightharpoonup \nabla y^*$ in $L^2(\Omega, \rho^{\varepsilon_k} dx)^N$, then

$$\int_\Omega |y_{\varepsilon_k}| dx \le \left(\int_\Omega |y_{\varepsilon_k}|^2 \rho^{\varepsilon_k} dx \right)^{1/2} \left(\int_\Omega (\rho^{\varepsilon_k})^{-1} dx \right)^{1/2} \le \tilde{C}(|\Omega|)^{1/2},$$

$$\int_\Omega |\nabla y_{\varepsilon_k}|_2 dx \le \left(\int_\Omega |\nabla y_{\varepsilon_k}|^2 \rho^{\varepsilon_k} dx \right)^{1/2} \left(\int_\Omega (\rho^{\varepsilon_k})^{-1} dx \right)^{1/2} \le \hat{C}(|\Omega|)^{1/2}.$$

Therefore the sequence $\{y_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is equi-integrable on Ω and bounded in $W_0^{1,1}(\Omega)$. In view of compact embedding $W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$, there exists an element $\tilde{y}$ such that $y_{\varepsilon_k} \to \tilde{y}$ strongly in $L^1(\Omega)$. However, it is easy to see that $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^1(\Omega)$. Hence, $y^* = \tilde{y}$ a. e. on Ω. And we have that $\int_\Omega u_{\varepsilon_k}(y_{\varepsilon_k} - y^*) dx \to 0$, $k \to \infty$. Since $u_{\varepsilon_k} \rightharpoonup u^*$ in $L^2(\Omega, (\rho^{\varepsilon_k})^{-1} dx)$ and $y_{\varepsilon_k} \rightharpoonup y^*$ in $H(\Omega, \rho^{\varepsilon_k} dx)$, and $L^2(\Omega, (\rho^{\varepsilon_k})^{-1} dx)$ is the conjugate space to $L^2(\Omega, \rho^{\varepsilon_k} dx)$, it follows that

$$\int_\Omega f v_{\varepsilon_k} dx \to \int_\Omega f v dx, \ \int_\Omega f y_{\varepsilon_k} dx \to \int_\Omega f y^* dx,$$

$$\int_\Omega u_{\varepsilon_k} v_{\varepsilon_k} dx \to \int_\Omega u^* v dx, \ \int_\Omega u_{\varepsilon_k} y_{\varepsilon_k} \to \int_\Omega u^* y^* dx.$$

Hence, the limit inequality for the inequality (4.15) has the form:

$$\langle -\mathrm{div}(\rho(x)\nabla y^*), v - y^* \rangle_{H(\Omega,\rho dx)} \ge \langle f + u^*, v - y^* \rangle_{H(\Omega,\rho dx)}. \tag{4.18}$$

Moreover, in view of previous suggestions, we have

$$\lim_{k\to\infty} \langle -\mathrm{div}(\rho^{\varepsilon_k} \nabla y_{\varepsilon_k}), v_{\varepsilon_k} - y_{\varepsilon_k} \rangle_{H(\Omega,\rho^{\varepsilon_k} dx)}$$

$$= \langle -\mathrm{div}(\rho(x)\nabla y^*), v \rangle_{H(\Omega,\rho dx)} - \limsup_{k\to\infty} \int_\Omega (\nabla y_{\varepsilon_k}, \nabla y_{\varepsilon_k})_{\mathbb{R}^N} \rho_{\varepsilon_k} dx$$

$$\ge \langle f + u^*, v - y^* \rangle_{H(\Omega,\rho dx)},$$

or

$$\limsup_{k\to\infty} \int_\Omega (\nabla y_{\varepsilon_k}, \nabla y_{\varepsilon_k})_{\mathbb{R}^N} \rho_{\varepsilon_k} dx$$

$$\le \langle -\mathrm{div}(\rho(x)\nabla y^*), v \rangle_{H(\Omega,\rho dx)} - \langle f + u^*, v - y^* \rangle_{H(\Omega,\rho dx)}, \ \forall v \in \tilde{K}.$$

Having put in the last inequality $v = y^*$, we get

$$\limsup_{k\to\infty} \int_\Omega |\nabla y_{\varepsilon_k}|^2 \rho^{\varepsilon_k} dx \le \int_\Omega |\nabla y^*|^2 \rho dx,$$

that together with the property of the lower semicontinuity with respect to the weak convergence in $L^2(\Omega, \rho^{\varepsilon_k} dx)$, gives us that $\nabla y_{\varepsilon_k} \to \nabla y^*$ in $L^2(\Omega, \rho^{\varepsilon_k} dx)^N$, $k \to \infty$.The proof is complete. □

As an evident consequence of this lemma and the lower semicontinuity property of the cost functional (4.6) with respect to w-convergence in the variable space $\mathbb{Y}(\Omega, \rho^\varepsilon dx)$, we have the following conclusion.

Corollary 4.1. *Let $\{\varepsilon_k\}$ be a subsequence of indices $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, and let $\{(u_k, y_k) \in \Xi_{\varepsilon_k}\}_{k\in\mathbb{N}}$ be a sequence of admissible solutions to corresponding perturbed problems* (4.6)-(4.8) *such that (u_k, y_k) w-converges to (u, y). Then properties* (4.3) *are valid.*

To discuss properties (4.4)-(4.5), we give a result which is reciprocal in some sense to Lemma 4.1.

Lemma 4.2. *Let $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho(x) \geq 0$ and let $(u, y) \in \Xi_H$ be any admissible pair. Then there exists a relizing sequence $\{(\hat{u}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ such that*

$$(\hat{u}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon \ \forall\varepsilon > 0,\ \hat{u}_\varepsilon \rightharpoonup u \ \ in \ \ L^2(\Omega, (\rho^\varepsilon)^{-1} dx); \tag{4.19}$$

$$\hat{y}_\varepsilon \rightharpoonup y \ \ in \ \ L^2(\Omega, \rho^\varepsilon dx),\ \nabla \hat{y}_\varepsilon \to \nabla y \ \ in \ \ L^2(\Omega, \rho^\varepsilon dx)^N. \tag{4.20}$$

Proof. Let us construct the sequence $\{(\hat{u}_\varepsilon, \hat{y}_\varepsilon)\}_{\varepsilon>0}$ as follows:

$$\hat{u}_\varepsilon(x) = \int\limits_{\mathbb{R}^N} Q(z)u(x + \varepsilon z)dz, \tag{4.21}$$

$$\hat{y}_\varepsilon \in H(\Omega, \rho^\varepsilon dx) \ \text{ is an } H\text{-solution of (4.8) corresponding to } \ u = \hat{u}_\varepsilon. \tag{4.22}$$

Let us show that for every $\varepsilon > 0$ the pair $(\hat{u}_\varepsilon, \hat{y}_\varepsilon)$ is admissible to the corresponding problem (4.6)-(4.8). Indeed, as follows from [10] there exists $C > 0$ such that

$$\hat{u}_\varepsilon(x) \leq C \int\limits_\Omega u(x + \varepsilon z)dz.$$

Taking into account the last inequality, properties of functions ρ and u, using the replacement of variables in double integral, we have:

$$\|\hat{u}_\varepsilon\|^2_{L^2(\Omega,\rho^{-1}dx)} = \int\limits_\Omega \left(\int\limits_{\mathbb{R}^N} Q(z)u(x + \varepsilon z)dz \right)^2 \rho^{-1} dx$$

$$\leq \int\limits_\Omega \left(\int\limits_\Omega u(x + \varepsilon z)dz \right)^2 \rho^{-1} dx \leq C_1 \int\limits_\Omega \int\limits_\Omega u^2(x + \varepsilon z)\rho^{-1} dz dx$$

$$= C_2 \|u\|^2_{L^2(\Omega)} \|\rho^{-1}\|_{L^1(\Omega)} \leq C_3 \|u\|^2_{L^2(\Omega,\rho^{-1}dx)} \|\rho^{-1}\|_{L^1(\Omega)} < \infty,$$

where C_1, C_2, C_3 are some positive constants. Hence,

$$\hat{u}_\varepsilon \in L^2(\Omega, \rho^{-1} dx) \subset L^2(\Omega, (\rho^\varepsilon)^{-1} dx),$$

$\forall \varepsilon > 0$. Let $T_\varepsilon : L^2(\Omega, \rho dx) \to L^2(\Omega, \rho^\varepsilon dx)$ is a "lifting" operator, constructed in (2.15). Since $\rho^{-1}u \in L^2(\Omega, \rho dx)$ (for details we refer to [10]), then

$$\lim_{\varepsilon\to 0} \int_\Omega \hat{u}_\varepsilon \varphi (\rho^\varepsilon)^{-1} dx = \lim_{\varepsilon\to 0} \int_\Omega u(\varphi)_\varepsilon (\rho^\varepsilon)^{-1} dx$$

$$= \lim_{\varepsilon\to 0} \int_\Omega \rho^{-1} u(\varphi)_\varepsilon (\rho^\varepsilon)^{-1} \rho dx = \lim_{\varepsilon\to 0} \int_\Omega T_\varepsilon(\rho^{-1}u) \varphi (\rho^\varepsilon)^{-1} \rho^\varepsilon dx$$

$$= \lim_{\varepsilon\to 0} \int_\Omega T_\varepsilon(\rho^{-1}u) \varphi dx = \int_\Omega u \varphi \rho^{-1} dx.$$

Taking into account properties of "lifting" operator (see Theorem 2.4), we have that $\hat{u}_\varepsilon \rightharpoonup u$ in $L^2(\Omega, (\rho^\varepsilon)^{-1} dx)$. In view of the definition of U_∂^ε, we have that $\hat{u}_\varepsilon \in U_\partial^\varepsilon$. Thus, we conclude that the sequence $\{(\hat{u}_\varepsilon, \hat{y}_\varepsilon)\}_{\varepsilon>0} \in \Xi_\varepsilon$. As a result, following arguments of the proof of Lemma 4.1, we have that $\hat{y} \rightharpoonup y$ in $L^2(\Omega, \rho^\varepsilon dx)$ and $\nabla \hat{y}_\varepsilon \to \nabla y$ in $L^2(\Omega, \rho^\varepsilon dx)^N$ as $\varepsilon \to 0$, where $y = y(u)$, for any subsequence of $\{\hat{y}_\varepsilon \in H(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ and, hence, for the entire sequence. Here $(u, y) \in \Xi_H$ is a given H-admissible solution to problem (3.3)-(3.5). This concludes the proof. □

Corollary 4.2. *Lemma 4.2 implies the equality* $I(u, y) = \lim_{\varepsilon\to 0} I_\varepsilon(\hat{u}_\varepsilon, \hat{y}_\varepsilon)$.

As an obvious consequence of Definition 4.2, and Lemmas 4.1-4.2 with their Corollaries, we can give the following conclusion.

Theorem 4.3. *Let* $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ *be a "direct" smoothing of a degenerate weight function* $\rho(x) > 0$. *Then the minimization problem* (3.3)-(3.5) *is a weak variational limit of the sequence* (4.6)-(4.8) *as* $\varepsilon \to 0$ *with respect to the* w*-convergence in the variable space* $\mathbb{Y}(\Omega, \rho^\varepsilon dx)$.

5. General cinclusions

In this paper we substantiate the validity of an H-attainability concept. Note that it can be considered in the case of solvability of initial degenerate optimal control problem and corresponding approximate problems. In order to verify that the set of optimal solutions to initial degenerate OCP is not empty, we invoke the concept of degenerate weight function of potential type (see for details [17]). Also for non-degenerate perturbed OCPs we construct the optimality conditions. As far as we show that at least one optimal solution to the problem (3.3)-(3.5) can be attained by optimal solutions to perturbed problems (4.6)-(4.8), and therefore, we can apply the derived optimality system for $\varepsilon > 0$ small enough to characterise the attainable optimal pairs to the initial optimization problem.

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A FORMULATION OF AN EVOLUTION EQUATION GOVERNING MAGNETIC LINES

Vladimir L. Borsch*

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Abstract. It is shown that 'a free function' in the evolution equation of Hornig & Schindler for the magnetic induction (*Physics of Plasmas*, **3** (3), 781 – 791) has a unique representation, obtained in an explicit form. Some conclusions of the explicit formulation of the evolution equation are discussed.

Key words: magnetic induction, magnetic lines.

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1. Introduction to the formulation

The magnetic induction equation

$$\boldsymbol{B}_t + \nabla \times (\boldsymbol{B} \times \boldsymbol{u}) = \boldsymbol{0} \tag{1.1}$$

is a constitutive part of the governing equations of ideal MHD [5]. In (1.1) $\boldsymbol{u}(t,\boldsymbol{x})$ is the velocity field of a moving continuum, $\boldsymbol{B}(t,\boldsymbol{x})$ is the magnetic induction field, $(t,\boldsymbol{x})$ refers to an inertial frame of reference, and the lower index t indicates the partial derivative with respect to t.

The magnetic induction equation has been studying by many authors, but in the current study our concern is the following equation

$$\boldsymbol{B}_t + \boldsymbol{w} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{w} = \lambda \boldsymbol{B}\,, \tag{1.2}$$

derived by Hornig & Schindler [7] for the evolution of the $\boldsymbol{B}$-field and discussed in [2,3,11]. In (1.2) $\boldsymbol{w}(t,\boldsymbol{x})$ is the velocity of the magnetic lines (the vector lines of the $\boldsymbol{B}$-field), λ is 'a scalar free function'.

If the solenoidal nature of the $\boldsymbol{B}$-field ($\nabla \cdot \boldsymbol{B} = 0$) is accounted for in (1.1), then the former converts into the equation

$$\boldsymbol{B}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} + (\nabla \cdot \boldsymbol{u})\,\boldsymbol{B} = \boldsymbol{0}\,, \tag{1.3}$$

exactly the same as the Zorawski criterion [13] for the $\boldsymbol{B}$-field to be frozen in the moving continuum. It follows from (1.3) that $\boldsymbol{w}_\perp = \boldsymbol{u}_\perp$ and generally $\boldsymbol{w}_{||} \neq \boldsymbol{u}_{||}$, where $\boldsymbol{w}(t,\boldsymbol{x}) = \boldsymbol{w}_\perp(t,\boldsymbol{x}) + \boldsymbol{w}_{||}(t,\boldsymbol{x})$, $\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{u}_\perp(t,\boldsymbol{x}) + \boldsymbol{u}_{||}(t,\boldsymbol{x})$, and the lower

*Dept. of Differential Equations, Faculty of Mech & Math, Oles Honchar Dnipro National University, 72, Gagarin av., Dnipro, 49010, Ukraine, bvl@dsu.dp.ua

indices mean respectively local orthogonal and tangent directions to a magnetic line. But (1.2) is surely to differ from the criterion [13], and from this there stems our interest to the formulation (1.2) of the evolution equation for $\boldsymbol{B}(t,\boldsymbol{x})$ and especially to function λ.

2. Preliminaries of the formulation

Following [7] we introduce some diffeomorphic mappings to study the formulation (1.2) of the evolution equation for the magnetic induction.

The first one

$$\boldsymbol{x} = \boldsymbol{\varphi}_{\boldsymbol{u}}(t,\boldsymbol{X})\,, \qquad \boldsymbol{X} \in \mathcal{D}(t') \subseteq \mathbf{R}^3, \quad t \geqslant t', \tag{2.1}$$

maps domain $\mathcal{D}(t')$ onto domain $\mathcal{D}(t)$, $t \geqslant t'$, where $\boldsymbol{X}$ are coordinates parametrizing domain $\mathcal{D}(t')$ (sometimes called the Lagrangian independent variables), and

$$\boldsymbol{X} = \boldsymbol{\varphi}_{\boldsymbol{u}}(t',\boldsymbol{X}) \tag{2.2}$$

is the identical mapping. The inverse mapping

$$\boldsymbol{X} = \boldsymbol{\psi}_{\boldsymbol{u}}(t,\boldsymbol{x})\,, \qquad \boldsymbol{x} \in \mathcal{D}(t)\,, \quad t \geqslant t', \tag{2.3}$$

acts in the opposite direction (in time).

The partial derivative of (2.1) with respect to time is called the velocity of the mapping

$$\boldsymbol{v}(t,\boldsymbol{X}) = \frac{\partial \boldsymbol{\varphi}_{\boldsymbol{u}}(t,\boldsymbol{X})}{\partial t} \tag{2.4}$$

and is easily presented in the Eulerian independent variables $\boldsymbol{x}$

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{v}(t,\boldsymbol{\psi}_{\boldsymbol{u}}(t,\boldsymbol{x}))\,. \tag{2.5}$$

Diffeomorphism (2.1) is responsible for the motion of the continuum and is the only solution to the following Cauchy problem ‘in the whole’

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(t,\boldsymbol{x})\,, \\ \boldsymbol{x}(t') = \boldsymbol{X}'. \end{cases} \tag{2.6}$$

The second one

$$\boldsymbol{x} = \boldsymbol{\varphi}_{\boldsymbol{w}}(t,\boldsymbol{X})\,, \qquad \boldsymbol{X} \in \mathcal{D}(t') \subseteq \mathbf{R}^3, \quad t \geqslant t', \tag{2.7}$$

maps domain $\mathcal{D}(t')$ onto domain $\mathcal{D}(t)$, $t \geqslant t'$, and

$$\boldsymbol{X} = \boldsymbol{\varphi}_{\boldsymbol{w}}(t',\boldsymbol{X}) \tag{2.8}$$

is the identical mapping. The inverse mapping

$$\boldsymbol{X} = \boldsymbol{\psi}_{\boldsymbol{w}}(t, \boldsymbol{x}), \qquad \boldsymbol{x} \in \mathcal{D}(t), \quad t \geqslant t', \tag{2.9}$$

acts in the opposite direction (in time).

The partial derivative of (2.7) with respect to time is called the velocity of the mapping

$$\boldsymbol{w}(t, \boldsymbol{x}) = \left.\frac{\partial \boldsymbol{\varphi}_{\boldsymbol{w}}(t, \boldsymbol{X})}{\partial t}\right|_{\boldsymbol{X}=\boldsymbol{\psi}_{\boldsymbol{w}}(t,\boldsymbol{x})}. \tag{2.10}$$

Diffeomorphism (2.7) is responsible for the motion (evolution) of magnetic field $\boldsymbol{B}(t, \boldsymbol{x})$ and can be specified through velocity field $\boldsymbol{w}(t, \boldsymbol{x})$. Newcomb [9] and Stern [12] discussed some ways of specifying velocity filed $\boldsymbol{w}(t, \boldsymbol{x})$ having properties mentioned above.

The solutions to the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\sigma} = \boldsymbol{B}(t, \boldsymbol{x}), \\ \boldsymbol{x}(\sigma') = \boldsymbol{x}', \end{cases} \qquad \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{D}(t), \quad t \geqslant t', \tag{2.11}$$

are called the magnetic lines (of the $\boldsymbol{B}$ field) and are given as the following diffeomorphism (the third one)

$$\boldsymbol{x} = \boldsymbol{\varphi}_{\boldsymbol{B}}(t, \boldsymbol{x}', \sigma), \tag{2.12}$$

where σ is a scalar parameter along the magnetic lines.

Diffeomorphism (2.12) is the only solution to the Cauchy problem (2.11), since

$$\boldsymbol{B}(t, \boldsymbol{x}) = \left.\frac{\partial \boldsymbol{\varphi}_{\boldsymbol{B}}(t, \boldsymbol{x}', \sigma)}{\partial \sigma}\right|_{(\boldsymbol{x}',\sigma) \overset{(2.12)}{\to} \boldsymbol{x}}. \tag{2.13}$$

We note that $\boldsymbol{\varphi}_{\boldsymbol{u}}$, $\boldsymbol{\varphi}_{\boldsymbol{w}}$ and $\boldsymbol{\varphi}_{\boldsymbol{B}}$ are called sometimes the (phase) flows for fields $\boldsymbol{u}(t, \boldsymbol{x})$, $\boldsymbol{w}(t, \boldsymbol{x})$ and $\boldsymbol{B}(t, \boldsymbol{x})$, whereas in [3] $\boldsymbol{\varphi}_{\boldsymbol{w}}$ and $\boldsymbol{\varphi}_{\boldsymbol{B}}$ are called the generating functions for $\boldsymbol{w}(t, \boldsymbol{x})$ and $\boldsymbol{B}(t, \boldsymbol{x})$.

3. The main proposition of the formulation

Now we present a fully geometrical proof of the formulation (1.2) of the evolution equation for the magnetic induction extended compared to that given in [7]. The proof is based on a reparametrization of diffeomorphism $\boldsymbol{\varphi}_{\boldsymbol{B}}$ introduced in [7] and the commutation condition of flows (proposition 4.2.27 in [1]).

Proposition 3.1. If diffeomorphisms $\boldsymbol{\varphi}_{\boldsymbol{w}}$ (2.7) and $\boldsymbol{\varphi}_{\boldsymbol{B}}$ (2.12) commute, the latter being reparametrized the proper way; then the resulted evolution equation for the $\boldsymbol{B}$-field is determined uniquely.

Proof. Let an arbitrary instant t' be the reference one. This means that the Cartesian coordinates $\boldsymbol{x}$ at t' are considered to be the Lagrangian ones: $\boldsymbol{X}=\boldsymbol{x}$. Then take a magnetic line $\Gamma(t')$ and parametrize it due to diffeomorphism $\boldsymbol{\varphi_B}$ as follows

$$\boldsymbol{x} = \boldsymbol{\varphi_B}(t', \boldsymbol{x}_{\mathbf{M}}, \sigma)\,,$$

where $\mathbf{M} \in \Gamma(t')$ is an arbitrary point: $\boldsymbol{x}_{\mathbf{M}} = \boldsymbol{\varphi_B}(t', \boldsymbol{x}_{\mathbf{M}}, \sigma_{\mathbf{M}})$. Choosing an infinitesimal increment $\Delta\sigma$ of the parameter we determine point $\mathbf{N} \in \Gamma(t')$:

$$\boldsymbol{x}_{\mathbf{N}} = \boldsymbol{\varphi_B}(t', \boldsymbol{x}_{\mathbf{M}}, \sigma_{\mathbf{N}})\,, \qquad \sigma_{\mathbf{N}} = \sigma_{\mathbf{M}} + \Delta\sigma\,.$$

At instant $t' + \Delta t$, where Δt is an infinitesimal increment of time, diffeomorphism $\boldsymbol{\varphi_w}$ maps magnetic line $\Gamma(t')$ onto a magnetic line $\Gamma(t' + \Delta t)$. The latter is the image of the former due to diffeomorphism $\boldsymbol{\varphi_u}$, if we assume that the magnetic lines are 'frozen' in the moving continuum, this implies that $\boldsymbol{w}_\perp = \boldsymbol{u}_\perp$ and generally $\boldsymbol{w}_{||} \neq \boldsymbol{u}_{||}$.

Diffeomorphism $\boldsymbol{\varphi_w}$ maps points $\mathbf{M}, \mathbf{N} \in \Gamma(t')$ into points $\mathbf{M}', \mathbf{N}' \in \Gamma(t'+\Delta t)$ as follows

$$\begin{cases} \boldsymbol{x}_{\mathbf{M}'} = \varphi_{\boldsymbol{w}}(t' + \Delta t, \boldsymbol{X}_{\mathbf{M}})\,, \\ \boldsymbol{x}_{\mathbf{N}'} = \varphi_{\boldsymbol{w}}(t' + \Delta t, \boldsymbol{X}_{\mathbf{N}})\,, \end{cases}$$

and finally, at instant $t' + \Delta t$ diffeomorphism $\boldsymbol{\varphi_B}$ maps point $\mathbf{M}'$ into point $\mathbf{N}''$

$$\boldsymbol{x}_{\mathbf{N}''} = \boldsymbol{\varphi_B}(t' + \Delta t, \boldsymbol{x}_{\mathbf{M}'}, \sigma_{\mathbf{M}''})\,,$$

where $\sigma_{\mathbf{M}''} = \sigma_{\mathbf{M}'} + \Delta\sigma$, and $\boldsymbol{x}_{\mathbf{M}'} = \boldsymbol{\varphi_B}(t' + \Delta t, \boldsymbol{x}_{\mathbf{M}'}, \sigma_{\mathbf{M}'})$.

Generally speaking, $\mathbf{N}'' \neq \mathbf{N}'$, that is there is no commutation of diffeomorphisms $\boldsymbol{\varphi_w}$ and $\boldsymbol{\varphi_B}$ (Fig. 1, *a*).

It is clear that for commutation to occur, the parametrization of magnetic lines due to diffeomorphism $\boldsymbol{\varphi_B}$ should be changed as

$$\boldsymbol{x} = \bar{\boldsymbol{\varphi}}_{\boldsymbol{B}}(t, \boldsymbol{x}', \alpha(t, \boldsymbol{x}', \sigma))\,, \qquad t \geqslant t', \tag{3.1}$$

where $\alpha(t', \boldsymbol{x}', \sigma) = \sigma$, and the bar over the symbol of the diffeomorphism denotes the proper reparametrization. The evident restriction to be imposed on function α along magnetic line $\Gamma(t' + \Delta t)$ for diffeomorphisms $\boldsymbol{\varphi_w}$, $\boldsymbol{\varphi_B}$ and $\bar{\boldsymbol{\varphi}}_{\boldsymbol{B}}$ to commute (Fig. 1, *b*) is the following

$$\boldsymbol{x}_{\mathbf{N}'} = \bar{\boldsymbol{\varphi}}_{\boldsymbol{B}}(t' + \Delta t, \boldsymbol{x}_{\mathbf{M}'}, \alpha_{\mathbf{N}'})\,, \tag{3.2}$$

where

$$\alpha_{\mathbf{N}'} = \sigma_{\mathbf{M}'} + \Delta\alpha\,, \qquad \Delta\alpha = \frac{\partial\alpha(t' + \Delta t, \boldsymbol{x}_{\mathbf{M}'}, \sigma)}{\partial\sigma}\Big|_{\sigma=\sigma_{\mathbf{M}'}} \Delta\sigma\,. \tag{3.3}$$

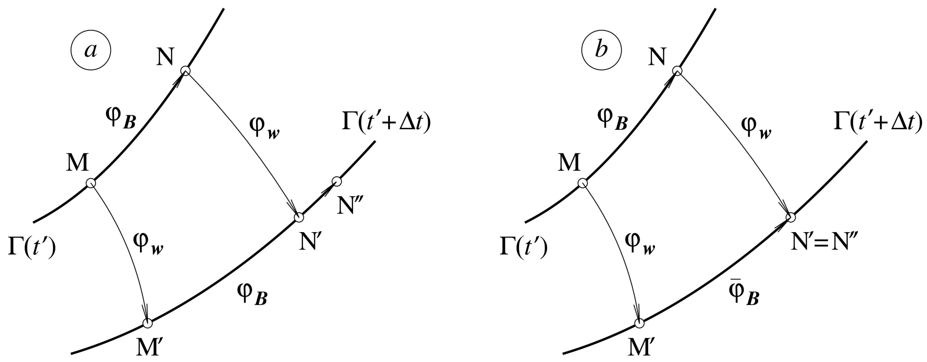

Fig. 1. Evolution of magnetic line Γ between two instants t' and $t' + \Delta t$: infinitesimal quadrilaterals $\mathbf{MM'N'N}$ formed by diffeomorphisms $\boldsymbol{\varphi_w}$ and $\boldsymbol{\varphi_B}$ without commutation (a: $\mathbf{N}' \neq \mathbf{N}''$) and by diffeomorphisms $\boldsymbol{\varphi_w}$, $\boldsymbol{\varphi_B}$ and $\bar{\boldsymbol{\varphi}}_{\boldsymbol{B}}$ with commutation (b: $\mathbf{N}' = \mathbf{N}''$)

But we can easily find $\Delta\alpha$ directly, without finding function α. Indeed, for the reparametrized diffeomorphism $\bar{\boldsymbol{\varphi}}_{\boldsymbol{B}}$ we have (one should refer to (2.11) and replace σ with α in the differential equation for the magnetic lines)

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\alpha} = \boldsymbol{B}\,,$$

hence, the required expression for $\Delta\alpha$ in (3.2), (3.3) is

$$\Delta\alpha = \frac{|\boldsymbol{x}_{\mathbf{N}'} - \boldsymbol{x}_{\mathbf{M}'}|}{|\boldsymbol{B}(t' + \Delta t, \boldsymbol{x}_{\mathbf{M}'})|}\,.$$

Expanding the expressions for the coordinates of points $\mathbf{N}$, $\mathbf{M}'$ and $\mathbf{N}'$ into series with respect to Δt and $\Delta\sigma$ and retaining only the terms of the first order in Δt and $\Delta\sigma$ we obtain

$$\boldsymbol{x}_{\mathbf{N}} = \boldsymbol{\varphi_B}(t', \boldsymbol{x}_{\mathbf{M}}, \sigma_{\mathbf{M}}) + \Delta\sigma\, \frac{\partial \boldsymbol{\varphi_B}(t', \boldsymbol{x}_{\mathbf{M}}, \sigma_{\mathbf{N}})}{\partial\sigma} = \boldsymbol{x}_{\mathbf{M}} + \Delta\sigma\, \boldsymbol{B}(t', \boldsymbol{x}_{\mathbf{M}})\,, \tag{3.4}$$

$$\begin{cases} \boldsymbol{x}_{\mathbf{M}'} = \boldsymbol{\varphi_w}(t', \boldsymbol{X}_{\mathbf{M}}) + \Delta t\, \dfrac{\partial \boldsymbol{\varphi_w}(t', \boldsymbol{X}_{\mathbf{M}})}{\partial t} \overset{(2.8)}{=} \boldsymbol{x}_{\mathbf{M}} + \Delta t\, \boldsymbol{w}(t', \boldsymbol{x}_{\mathbf{M}})\,, \\ \boldsymbol{x}_{\mathbf{N}'} = \boldsymbol{\varphi_w}(t', \boldsymbol{X}_{\mathbf{N}}) + \Delta t\, \dfrac{\partial \boldsymbol{\varphi_w}(t', \boldsymbol{X}_{\mathbf{N}})}{\partial t} \overset{(2.8)}{=} \boldsymbol{x}_{\mathbf{N}} + \Delta t\, \boldsymbol{w}(t', \boldsymbol{x}_{\mathbf{N}})\,. \end{cases} \tag{3.5}$$

Using the obtained expressions for the difference $\boldsymbol{x}_{\mathbf{N}'} - \boldsymbol{x}_{\mathbf{M}'}$ yields to

$$\begin{cases} \boldsymbol{x}_{\mathrm{N}'} - \boldsymbol{x}_{\mathrm{M}'} \overset{(3.5)}{=} \boldsymbol{x}_{\mathrm{N}} - \boldsymbol{x}_{\mathrm{M}} + \Delta t \Big(\boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{N}}) - \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \Big) \\ \qquad\qquad \overset{(3.4)}{=} \Delta\sigma\, \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) + \Delta t \Big(\boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}} + \Delta\sigma\, \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}})) - \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \Big) \\ \qquad\qquad = \Delta\sigma\, \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) + \Delta t\, \Delta\sigma\, \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \cdot \nabla \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \,, \end{cases}$$

and the required expression for the proper value of $\Delta\alpha$ reduces to the following

$$\Delta\alpha = \frac{\Big| \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) + \Delta t\, \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \cdot \nabla \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \Big|}{\Big| \boldsymbol{B}(t, \boldsymbol{x}_{\mathrm{M}}) + \Delta t\, \dfrac{\partial \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}})}{\partial t} + \Delta t\, \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \cdot \nabla \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \Big|} \, \Delta\sigma \,. \qquad (3.6)$$

Now we consider infinitesimal quadrilateral $\mathbf{MM'N'N}$ (Fig. 1, b) and set up the following commutation equality for diffeomorphisms $\boldsymbol{\varphi_w}$, $\boldsymbol{\varphi_B}$ and $\bar{\boldsymbol{\varphi}}_{\boldsymbol{B}}$ acting on $\mathbf{MM'N'N}$

$$\bar{\boldsymbol{\varphi}}_{\boldsymbol{B}} \left(t' + \Delta t, \boldsymbol{\varphi_w}(t' + \Delta t, \boldsymbol{X}_{\mathrm{M}}), \sigma_{\mathrm{N}'} \right) = \boldsymbol{\varphi_w} \left(t' + \Delta t, \boldsymbol{\varphi_B}(t', \boldsymbol{x}_{\mathrm{M}}, \sigma_{\mathrm{N}}) \right) . \qquad (3.7)$$

The necessary and sufficient condition for the above equality to hold is the following commutation condition [1]

$$\left[\frac{\partial^2}{\partial\sigma\, \partial t}\, \bar{\boldsymbol{\varphi}}_{\boldsymbol{B}} \left(t, \boldsymbol{\varphi_w}(t, \boldsymbol{X}), \alpha(t, \boldsymbol{x}, \sigma) \right) \right] = \left[\frac{\partial^2}{\partial t\, \partial\sigma}\, \boldsymbol{\varphi_w} \left(t, \boldsymbol{\varphi_B}(t, \boldsymbol{x}, \sigma) \right) \right], \qquad (3.8)$$

where the brackets herein after denote that a quantity enclosed is calculated at point $(t', \boldsymbol{x}_{\mathrm{M}})$. Using the defined derivatives (2.10) and (2.13) of $\boldsymbol{\varphi_w}$ and $\boldsymbol{\varphi_B}$ in (3.8) yields to

$$\frac{\partial}{\partial t} \left[\boldsymbol{B} \left(t, \boldsymbol{\varphi_w}(t, \boldsymbol{X}) \right) \frac{\partial}{\partial\sigma}\, \alpha(t, \boldsymbol{x}, \sigma) \right] = \frac{\partial}{\partial\sigma} \left[\boldsymbol{w} \left(t, \boldsymbol{\varphi_B}(t, \boldsymbol{x}, \sigma) \right) \right].$$

The derivative of the term in the brackets at the right hand side of the above equality is evident

$$\frac{\partial \boldsymbol{\varphi_B}(t', \boldsymbol{x}_{\mathrm{M}}, \sigma_{\mathrm{M}})}{\partial\sigma} \cdot \nabla \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \equiv \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \cdot \nabla \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \,,$$

and the same is true for the product in the brackets at the left hand side

$$\left(\frac{\partial \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}})}{\partial t} + \frac{\partial \boldsymbol{\varphi_w}(t', \boldsymbol{x}_{\mathrm{M}})}{\partial t} \cdot \nabla \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \right) \left[\frac{\partial\alpha}{\partial\sigma} \right] + \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \left[\frac{\partial^2\alpha}{\partial\sigma\, \partial t} \right]$$
$$\equiv \left(\frac{\partial \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}})}{\partial t} + \boldsymbol{w}(t', \boldsymbol{x}_{\mathrm{M}}) \cdot \nabla \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \right) \left[\frac{\partial\alpha}{\partial\sigma} \right] + \boldsymbol{B}(t', \boldsymbol{x}_{\mathrm{M}}) \left[\frac{\partial^2\alpha}{\partial\sigma\, \partial t} \right].$$

To calculate the first derivative on σ and the mixed derivative of function α we use the divided differences as follows

$$\left[\frac{\partial\alpha}{\partial\sigma}\right] = \lim_{\substack{\Delta t\to 0\\ \Delta\sigma\to 0}} \frac{|\boldsymbol{x}_{\mathbf{N}'} - \boldsymbol{x}_{\mathbf{M}'}|}{|\boldsymbol{B}(t'+\Delta t, \boldsymbol{x}_{\mathbf{M}'})|\,\Delta\sigma} = \lim_{\substack{\Delta t\to 0\\ \Delta\sigma\to 0}} \frac{|\boldsymbol{x}_{\mathbf{N}'} - \boldsymbol{x}_{\mathbf{M}'}||\boldsymbol{B}(t', \boldsymbol{x}_{\mathbf{M}})|}{|\boldsymbol{x}_{\mathbf{N}} - \boldsymbol{x}_{\mathbf{M}}||\boldsymbol{B}(t'+\Delta t, \boldsymbol{x}_{\mathbf{M}'})|} = 1\,,$$

$$= \lim_{\substack{\Delta t\to 0\\ \Delta\sigma\to 0}} \frac{|\boldsymbol{x}_{\mathbf{N}'} - \boldsymbol{x}_{\mathbf{M}'}||\boldsymbol{B}(t', \boldsymbol{x}_{\mathbf{M}})|}{|\boldsymbol{x}_{\mathbf{N}} - \boldsymbol{x}_{\mathbf{M}}||\boldsymbol{B}(t'+\Delta t, \boldsymbol{x}_{\mathbf{M}'})|} = 1\,,$$

$$\left[\frac{\partial^2\alpha}{\partial t\,\partial\sigma}\right] = \lim_{\substack{\Delta t\to 0\\ \Delta\sigma\to 0}} \frac{\dfrac{|\boldsymbol{x}_{\mathbf{N}'} - \boldsymbol{x}_{\mathbf{M}'}|}{|\boldsymbol{B}(t'+\Delta t, \boldsymbol{x}_{\mathbf{M}'})|} - \dfrac{|\boldsymbol{x}_{\mathbf{N}} - \boldsymbol{x}_{\mathbf{M}}|}{|\boldsymbol{B}(t', \boldsymbol{x}_{\mathbf{M}})|}}{\Delta\sigma\Delta t} = \lim_{\Delta t\to 0} \frac{\dfrac{\Delta\alpha}{\Delta\sigma} - 1}{\Delta t}\,.$$

For completing the calculation of the mixed derivative we consider the numerator of the last expression separately as follows

$$\frac{\Delta\alpha}{\Delta\sigma} - 1 \overset{(3.6)}{=} \frac{\left[\left|\boldsymbol{B} + \Delta t\,\boldsymbol{B}\cdot\nabla\boldsymbol{w}\right|\right]}{\left[\left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]} - 1$$

$$= \frac{\left[\left|\boldsymbol{B} + \Delta t\,\boldsymbol{B}\cdot\nabla\boldsymbol{w}\right| - \left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]}{\left[\left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]}$$

$$\times \frac{\left[\left|\boldsymbol{B} + \Delta t\,\boldsymbol{B}\cdot\nabla\boldsymbol{w}\right| + \left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]}{\left[\left|\boldsymbol{B} + \Delta t\,\boldsymbol{B}\cdot\nabla\boldsymbol{w}\right| + \left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]}$$

$$= \frac{\left[\left(\boldsymbol{B} + \Delta t\,\boldsymbol{B}\cdot\nabla\boldsymbol{w}\right)^2 - \left(\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right)^2\right]}{\left[\left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]\left[\left|\boldsymbol{B} + \Delta t\,\boldsymbol{B}\cdot\nabla\boldsymbol{w}\right| + \left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]}$$

$$= \frac{2\Delta t\left[\boldsymbol{B}\cdot\left(\nabla\boldsymbol{w}\right)\cdot\boldsymbol{B} - \boldsymbol{B}\cdot\dfrac{\partial\boldsymbol{B}}{\partial t} - \boldsymbol{w}\cdot\left(\nabla\boldsymbol{B}\right)\cdot\boldsymbol{B} + \mathcal{O}\left(\Delta t\right)\right]}{\left[\left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]\left[\left|\boldsymbol{B} + \Delta t\,\boldsymbol{B}\cdot\nabla\boldsymbol{w}\right| + \left|\boldsymbol{B} + \Delta t\,\dfrac{\partial\boldsymbol{B}}{\partial t} + \Delta t\,\boldsymbol{w}\cdot\nabla\boldsymbol{B}\right|\right]}\,.$$

Replacing the numerator of the expression for the mixed derivative of function α with the above one and taking the limit at $\Delta t \to 0$ we obtain the final expression for the mixed derivative

$$\left[\frac{\partial^2 \alpha}{\partial t\,\partial \sigma}\right] \equiv -\lambda = -\left[\frac{1}{|\boldsymbol{B}|^2}\left(\frac{1}{2}\,\frac{\partial |\boldsymbol{B}|^2}{\partial t} + \boldsymbol{w}\cdot(\nabla \boldsymbol{B})\cdot \boldsymbol{B} - \boldsymbol{B}\cdot(\nabla \boldsymbol{w})\cdot \boldsymbol{B}\right)\right]. \quad (3.9)$$

Gathering all the terms obtained when treating commutation condition (3.8) we conclude that the required evolution equation for the magnetic induction reads

$$\boldsymbol{B}_t + \boldsymbol{w}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{w} = \underbrace{\frac{1}{|\boldsymbol{B}|^2}\left(\frac{1}{2}\,\frac{\partial |\boldsymbol{B}|^2}{\partial t} + \boldsymbol{w}\cdot(\nabla \boldsymbol{B})\cdot \boldsymbol{B} - \boldsymbol{B}\cdot(\nabla \boldsymbol{w})\cdot \boldsymbol{B}\right)}_{\lambda}\boldsymbol{B}\,. \quad (3.10)$$

Arbitrariness of point $(t', \boldsymbol{x}_{\mathbf{M}})$ means that the equation obtained is valid at any point $(t, \boldsymbol{x}) \in \mathcal{D}(t)$ and this is denoted by dropping the brackets referring to point $(t', \boldsymbol{x}_{\mathbf{M}})$ in (3.8). □

4. The Galilean invariance of the formulation

The MHD phenomena in the non-relativistic limit are Galilean invariant [10], but the original magnetic induction equation (1.1) does not obey the Galilean transformation. It was Godunov [6] who showed that (1.1) transforms to formulation (1.3) being Galilean invariant if the solenoidal nature of the $\boldsymbol{B}$-field is accounted for explicitly. And what about evolution equation (3.10)?

Proposition 4.1. Evolution equation (3.10) is Galilean invariant.

Proof. Let $(\tau, \boldsymbol{\xi})$ be an inertial frame of reference such that

$$\begin{cases} \tau = t\,, \\ \boldsymbol{\xi} = \boldsymbol{x} - t\boldsymbol{a}\,, \end{cases} \quad (4.1)$$

where $\boldsymbol{a}$ is a constant velocity, then velocity field $\boldsymbol{w}(t, \boldsymbol{x})$ and magnetic induction $\boldsymbol{B}(t, \boldsymbol{x})$ observed in $(\tau, \boldsymbol{\xi})$ and indicated by an asterisk are

$$\begin{cases} \boldsymbol{w}^*(\tau, \boldsymbol{\xi}) = \boldsymbol{w}(t, \boldsymbol{x}) - \boldsymbol{a}\,, \\ \boldsymbol{B}^*(\tau, \boldsymbol{\xi}) = \boldsymbol{B}(t, \boldsymbol{x})\,. \end{cases}$$

When changing the frame of reference from $(t, \boldsymbol{x})$ to $(\tau, \boldsymbol{\xi})$ the partial derivatives transform as follows

$$\cdot\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\partial \boldsymbol{B}^*}{\partial \tau} + \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) = \frac{\partial \boldsymbol{B}^*}{\partial \tau} - \boldsymbol{a} \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right),$$

$$\nabla_{\boldsymbol{x}} \boldsymbol{B} = \nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*, \qquad \nabla_{\boldsymbol{x}} \boldsymbol{w} = \nabla_{\boldsymbol{\xi}} \boldsymbol{w}^* + \nabla_{\boldsymbol{\xi}} \boldsymbol{a} = \nabla_{\boldsymbol{\xi}} \boldsymbol{w}^*,$$

hence, applying the above transformations to the terms on the left-hand side of evolution equation (3.10) yields to

$$\frac{\partial \boldsymbol{B}^*}{\partial \tau} - \boldsymbol{a} \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) + \boldsymbol{w}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) + \boldsymbol{a} \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) - \boldsymbol{B}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{w}^*\right)$$

$$= \frac{\partial \boldsymbol{B}^*}{\partial \tau} + \boldsymbol{w}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) - \boldsymbol{B}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{w}^*\right).$$

The similar transformations are easily applied to the terms in the parentheses on the right-hand side of evolution equation (3.10) as follows

$$\boldsymbol{B}^* \cdot \frac{\partial \boldsymbol{B}^*}{\partial \tau} - \boldsymbol{a} \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) \cdot \boldsymbol{B}^*$$

$$+ \boldsymbol{w}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) \cdot \boldsymbol{B}^* + \boldsymbol{a} \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) \cdot \boldsymbol{B}^* - \boldsymbol{B}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{w}^*\right) \cdot \boldsymbol{B}^*$$

$$= \boldsymbol{B}^* \cdot \frac{\partial \boldsymbol{B}^*}{\partial \tau} + \boldsymbol{w}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) \cdot \boldsymbol{B}^* - \boldsymbol{B}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{w}^*\right) \cdot \boldsymbol{B}^*.$$

Gathering all the transformed terms we obtain the following evolution equation for the magnetic induction in frame of reference $(\tau, \boldsymbol{\xi})$

$$\boldsymbol{B}^*_\tau + \boldsymbol{w}^* \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{B}^* - \boldsymbol{B}^* \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{w}^*$$

$$= \frac{1}{|\boldsymbol{B}^*|^2} \left(\frac{1}{2} \frac{\partial |\boldsymbol{B}^*|^2}{\partial \tau} + \boldsymbol{w}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{B}^*\right) \cdot \boldsymbol{B}^* - \boldsymbol{B}^* \cdot \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{w}^*\right) \cdot \boldsymbol{B}^* \right) \boldsymbol{B}^*.$$

The resulted equation is seen to be the same as the evolution equation in frame of reference $(t, \boldsymbol{x})$. This completes the proof. □

5. The incompleteness of the formulation

Function λ in (3.10) looks if it were obtained directly from (1.2) by the dot product of the former and the local vector $\boldsymbol{B}(t, \boldsymbol{x})$ as follows

$$\boldsymbol{B} \cdot \boldsymbol{B}_t + \boldsymbol{w} \cdot \left(\nabla \boldsymbol{B}\right) \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \left(\nabla \boldsymbol{w}\right) \cdot \boldsymbol{B} = \lambda \boldsymbol{B} \cdot \boldsymbol{B}, \tag{5.1}$$

nevertheless evolution equation (3.10) is obtainable by direct calculation of mixed derivative (3.9) in commutation condition (3.8). This is an evidence that evolution equation (3.10) is incomplete. Actually, the dot product of (3.10) and the local vector $\boldsymbol{B}(t, \boldsymbol{x})$ produces the trivial identity $\boldsymbol{0} = \boldsymbol{0}$, i. e., the evolution equation being under consideration 'works' only in planes normal to the local vectors $\boldsymbol{B}(t, \boldsymbol{x})$. The situation is clarified by the following

Proposition 5.1. Evolution equation (3.10) for the magnetic induction is incomplete, i. e., it actually includes only two evolution equations for two scalar quantities.

Proof. Let domains $\mathcal{D}(t)$, $t \geqslant t'$, be parametrized using Cartesian orthogonal coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$, hence, $\boldsymbol{B} = (B_1, B_2, B_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$, and evolution equation (3.10) be rewritten in matrix form as the following quasilinear system of the first order

$$\mathsf{A}_0(\mathbf{U})\,\frac{\partial \mathbf{U}}{\partial t} + \sum_{\kappa=1}^{3} \mathsf{A}_\kappa(\mathbf{U}; \boldsymbol{w})\,\frac{\partial \mathbf{U}}{\partial x_\kappa} = \mathbf{G}(\mathbf{U}; \nabla \boldsymbol{w})\,, \tag{5.2}$$

where

$$\mathbf{U} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \qquad \mathsf{A}_0(\mathbf{U}) = \begin{pmatrix} |\boldsymbol{B}|^2 - B_1 B_1 & -B_1 B_2 & -B_1 B_3 \\ -B_2 B_1 & |\boldsymbol{B}|^2 - B_2 B_2 & -B_2 B_3 \\ -B_3 B_1 & -B_3 B_2 & |\boldsymbol{B}|^2 - B_3 B_3 \end{pmatrix},$$

$$\mathsf{A}_\kappa(\mathbf{U}; \boldsymbol{w}) = \mathsf{A}_0(\mathbf{U})\,\mathsf{C}_\kappa(\boldsymbol{w}) = \mathsf{C}_\kappa(\boldsymbol{w})\,\mathsf{A}_0(\mathbf{U})\,, \qquad \mathsf{C}_\kappa(\boldsymbol{w}) = w_\kappa\,\mathrm{diag}\,(1,1,1)\,,$$

$$\mathbf{G}(\mathbf{U}; \nabla \boldsymbol{w}) = \begin{pmatrix} |\boldsymbol{B}|^2\,\phi_1(\boldsymbol{B}; \nabla \boldsymbol{w}) - \phi(\boldsymbol{B}; \nabla \boldsymbol{w})\,B_1 \\ |\boldsymbol{B}|^2\,\phi_2(\boldsymbol{B}; \nabla \boldsymbol{w}) - \phi(\boldsymbol{B}; \nabla \boldsymbol{w})\,B_2 \\ |\boldsymbol{B}|^2\,\phi_3(\boldsymbol{B}; \nabla \boldsymbol{w}) - \phi(\boldsymbol{B}; \nabla \boldsymbol{w})\,B_3 \end{pmatrix},$$

ϕ_1, ϕ_2, and ϕ_3 being linear forms and ϕ being a quadratic form in the components of the $\boldsymbol{B}$-field as follows

$$\phi(\boldsymbol{B}; \nabla \boldsymbol{w}) = \sum_{\kappa=1}^{3} \sum_{\iota=1}^{3} B_\kappa B_\iota\,\frac{\partial w_\kappa}{\partial x_\iota} = \sum_{\iota=1}^{3} B_\iota \left(\sum_{\kappa=1}^{3} B_\kappa\,\frac{\partial w_\kappa}{\partial x_\iota} \right) = \sum_{\iota=1}^{3} B_\iota\,\phi_\iota(\boldsymbol{B}; \nabla \boldsymbol{w})\,.$$

We use matrix notation $\mathbf{U}$ for the dependent variables in matrix formulation (5.2) of evolution equation (3.10) and its constitutive parts and vector notation $\boldsymbol{B}$ in the scalar functions and the entries of the matrices.

Matrix A_0 is singular: $\det \mathsf{A}_0 = 0$, $\operatorname{rank} \mathsf{A}_0 = 2$, but being real symmetric it has real eigenvalues: $\lambda_1 = 0$, $\lambda_2 = |\boldsymbol{B}|^2$, $\lambda_3 = |\boldsymbol{B}|^2$, and complete sets of the left (rows) and the right (columns) normalized real eigenvectors

$$\mathsf{L}(\mathbf{U}) = \mathsf{R}^{-1}(\mathbf{U}) = |\boldsymbol{B}|^{-2} \begin{pmatrix} B_1 B_3 & B_2 B_3 & B_3 B_3 \\ -B_1 B_3 & -B_2 B_3 & B_1 B_1 + B_2 B_2 \\ -B_2 B_1 & B_1 B_1 + B_3 B_3 & -B_2 B_3 \end{pmatrix},$$

$$\mathsf{R}(\mathbf{U}) = \mathsf{L}^{-1}(\mathbf{U}) = \begin{pmatrix} \dfrac{B_1}{B_3} & -\dfrac{B_3}{B_1} & -\dfrac{B_2}{B_1} \\ \dfrac{B_2}{B_3} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The left and the right eigenvectors are normalized as above to diagonalize matrix A_0 as follows

$$\mathsf{L}(\mathbf{U})\,\mathsf{A}_0(\mathbf{U})\,\mathsf{R}(\mathbf{U}) = \operatorname{diag}\left(0, |\boldsymbol{B}|^2, |\boldsymbol{B}|^2\right) \equiv \mathsf{Q}(\mathbf{U})\,, \qquad \mathsf{A}_0(\mathbf{U}) = \mathsf{R}(\mathbf{U})\,\mathsf{Q}(\mathbf{U})\,\mathsf{L}(\mathbf{U})\,.$$

Applying this property to system (5.2) yields to

$$\mathsf{Q}(\mathbf{U}) \left(\mathsf{L}(\mathbf{U})\,\frac{\partial \mathbf{U}}{\partial t} + \sum_{\kappa=1}^{3} \mathsf{C}_\kappa(\boldsymbol{w})\,\mathsf{L}(\mathbf{U})\,\frac{\partial \mathbf{U}}{\partial x_\kappa} \right) = \mathbf{H}(\mathbf{U}; \nabla \boldsymbol{w})\,, \tag{5.3}$$

where the source term on the right-hand side has the following explicit form

$$\mathbf{H}(\mathbf{U}; \nabla \boldsymbol{w}) \equiv \mathsf{L}(\mathbf{U})\,\mathbf{G}(\mathbf{U}; \nabla \boldsymbol{w}) = \begin{pmatrix} 0 \\ |\boldsymbol{B}|^2\,\phi_3(\boldsymbol{B}; \nabla \boldsymbol{w}) - \phi(\boldsymbol{B}; \nabla \boldsymbol{w})\,B_3 \\ |\boldsymbol{B}|^2\,\phi_2(\boldsymbol{B}; \nabla \boldsymbol{w}) - \phi(\boldsymbol{B}; \nabla \boldsymbol{w})\,B_2 \end{pmatrix}, \tag{5.4}$$

whereas the terms on the left-hand side need to be simplified using a proper transformation of the dependent variables $\mathbf{U}$.

The proper choice of the variable transformation is prompted by the differential terms on the the left-hand side of system (5.3) leading to the following matrix-column product

$$|\boldsymbol{B}|^2\,\mathsf{L}(\mathbf{U})\,\mathrm{d}\mathbf{U} = \begin{pmatrix} B_1 B_3\,\mathrm{d}B_1 + & B_2 B_3\,\mathrm{d}B_2 + & B_3 B_3\,\mathrm{d}B_3 \\ -B_1 B_3\,\mathrm{d}B_1 - & B_2 B_3\,\mathrm{d}B_2 + (B_1 B_1 + B_2 B_2)\,\mathrm{d}B_3 \\ -B_2 B_1\,\mathrm{d}B_1 + (B_1 B_1 + B_3 B_3)\,\mathrm{d}B_2 - & B_2 B_3\,\mathrm{d}B_3 \end{pmatrix},$$

where scalar multiplier $|\boldsymbol{B}|^2$ represents the non-zero entries of matrix $\mathsf{Q}(\mathbf{U})$.

The 1-forms given by the entries of the resulted column vector above and denoted below respectively as ω_1, ω_2, and ω_3, are evidently to be integrable, the integrating factors being as follows

$$\theta_1(\boldsymbol{B}) = \frac{1}{B_3\left(B_1^2 + B_2^2 + B_3^2\right)}\,, \quad \theta_2(\boldsymbol{B}) = \frac{1}{B_2^2\sqrt{B_1^2 + B_3^2}}\,, \quad \theta_3(\boldsymbol{B}) = \frac{1}{B_3^2\sqrt{B_1^2 + B_2^2}}\,.$$

Hence, we find that

$$\begin{cases} \theta_1(\boldsymbol{B})\,\omega_1 = \mathrm{d}v_1 = \mathrm{d}\,\sqrt{B_1^2+B_2^2+B_3^2}\,, \\[2ex] \theta_2(\boldsymbol{B})\,\omega_2 = \mathrm{d}v_2 = \mathrm{d}\,\dfrac{\sqrt{B_1^2+B_3^2}}{B_2}\,, \\[2ex] \theta_3(\boldsymbol{B})\,\omega_3 = \mathrm{d}v_3 = \mathrm{d}\,\dfrac{\sqrt{B_1^2+B_2^2}}{B_3}\,, \end{cases} \qquad , \tag{5.5}$$

and the new dependent variables are $\mathbf{V}(\mathbf{U}) = (v_1, v_2, v_3)$.

Eventually, the first differential equation of system (5.3) vanishes, whereas the remaining two ones read

$$\frac{\partial}{\partial t}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} + \sum_{\kappa=1}^{3}\begin{pmatrix} w_\kappa & 0 \\ 0 & w_\kappa \end{pmatrix}\frac{\partial}{\partial x_\kappa}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} p_2(v_1,v_2,v_3;\nabla\boldsymbol{w}) \\ p_3(v_1,v_2,v_3;\nabla\boldsymbol{w}) \end{pmatrix}, \tag{5.6}$$

where

$$\begin{cases} p_2(v_1,v_2,v_3;\nabla\boldsymbol{w}) = \Big[\theta_2(\boldsymbol{B})\,h_2\,(\boldsymbol{B};\nabla\boldsymbol{w})\Big]_{\mathbf{U}\to\mathbf{V}}, \\[2ex] p_3(v_1,v_2,v_3;\nabla\boldsymbol{w}) = \Big[\theta_3(\boldsymbol{B})\,h_3\,(\boldsymbol{B};\nabla\boldsymbol{w})\Big]_{\mathbf{U}\to\mathbf{V}}, \end{cases}$$

and h_2, h_3 are two non-zero entries of column vector $\mathbf{H}$ (5.4). Both equations of system (5.6) are coupled through source terms p_2 and p_3.

Finding the above system completes the proof. □

6. Conclusions of the formulation

1. Evolution equation (1.2) is uniquely determined by diffeomorphisms $\boldsymbol{\varphi}_{\boldsymbol{w}}$, $\boldsymbol{\varphi}_{\boldsymbol{B}}$ and $\bar{\boldsymbol{\varphi}}_{\boldsymbol{B}}$. This means that function λ explicitly depends on the local values of fields $\boldsymbol{B}(t,\boldsymbol{x})$ and $\boldsymbol{w}(t,\boldsymbol{x})$ and the partial derivatives of $\boldsymbol{B}(t,\boldsymbol{x})$ as it is seen from formulation (3.10) of the evolution equation, rather than being 'a free function'.

2. Evolution equation (1.2) is influenced by velocity field $\boldsymbol{w}(t,\boldsymbol{x})$, nevertheless substituting velocity field $\boldsymbol{u}(t,\boldsymbol{x})$ into (1.2) does not convert the former into magnetic induction equation (1.3), since in both formulations (1.1) and (1.3) of the magnetic induction equation and in evolution equation (1.2) for the magnetic induction the velocity fields have quite different meanings.

3. Evolution equation (1.2) is Galilean invariant similarly to formulation (1.3) of the magnetic induction equation.

4. Evolution equation (1.2) is incomplete, since it is reduced to system (5.6) of two partial differential equations for dependent variables v_2, v_3 (5.5). System (5.6) needs to be supplemented with a constraint imposed on variables (v_1, v_2, v_3), either algebraic or differential, to admit the well-posed formulations [4, 8] of IBVP.

5. Variable v_1 (5.5) is introduced similarly to variables v_2, v_3 using the left eigenvectors but variable v_1 turns out to be blind to the sign of component B_1 of the $\boldsymbol{B}$-field, contrary to variables v_2, v_3 accounting for the signs of components B_2 and B_3. Therefore, an other choice for v_1 may be more appropriate.

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For notes

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