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STEPS TO THE FUTURE

Irina G. Balanenko*

Abstract. Historical milestones and directions of scientific research. To the 100th anniversary of the Oles Honchar Dnipro National University and the 52nd anniversary of the Department of Differential Equations

University is a special concept in the memory of many generations of people who have somehow connected their destiny with it. These people have showed the samples of diligence and honor, a high sense of duty and dignity, and devotion to the chosen case with their work, professional accomplishments and scientific discoveries, honest daily work. They have created the centenary chronicle of the alma mater, one of the most prestigious higher educational institutions in Ukraine.

Oles Honchar Dnipro National University today combines the best traditions of classical higher education, powerful potential of the world-famous research schools and modern trends in the introduction of innovative technologies.

The multi-volume history of the university created of human lives, personalities who began to form university traditions, determined the ways of further development, prepared a new generation of scholars that was so necessary for the development of the state has been made for 100 glorious, though difficult, years of our university.

Its glorious history goes back to 1918, when under the rule of Hetman Skoropadsky the Statute of Katerynoslav University was approved. The formation and growth of the university are closely connected with the names of such outstanding scholars as academicians L.V. Pisarzhevsky, A.N. Dynnyk, F.V. Taranovsky and D.I. Yavornytsky [1]. It is symbolic that the first educational building of the university was Prince Potiomkin's palace because the idea to set up an institution of higher learning in the city emerged when he was the governor of the region in the late 18th century. At that time Tsarina Catherine II issued an edict about the foundation of a university in Katerynoslav, "in which not only sciences but also arts are obliged to be taught".

Glimpses of DNU's glorious past reflect the dramatic history of the country in which the university has grown and gained firm standing and international recognition. About 30 research schools have sprung up here making the institution a leader in the university and academic science. Creation of a powerful trend in space engineering design in post-war years made DNU a unique classical university.

Pride and joy of the university are its alumni. Among those who emerged from the lecture halls of DNU are well-known writers Oles Honchar, Pavio Zahrebelny, Valerian Pidmohylny, academicians Serhiy Nikolsky, Volodymyr Mossakovsky,

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Oleh Trubachov, Viktor Pylypenko, prominent political and public figures. Five of our alumni have been given the highest award of the country - the title of Hero of Ukraine. They are the renowned humanist of today Pavio Zahrebelny and famous space engineering designers who have created modern space image of our country: Stanislav Koniukhov, Yuri Alexeyev, Volodymyr Komanov and Volodymyr Sichovy [3].

Traditions of the past generations are being upheld and developed today by the researchers, teachers and students of the university. By the results of the polls among experts and students DNU hold a firm position in the annual "Sophia of Kyiv" rating competition among institutions of higher education of Ukraine and International exhibition "Modern Education in Ukraine" [6].

University's compliance with world educational standards has been recognized by the foreign colleagues. As of today there are about 50 bilateral cooperation agreements between DNU and leading educational and research institutions of Germany, France, Italy, Spain, Great Britain, Poland, Russia, Turkey, China, Brazil and the United States. Our university is an officially recognized regional leader by the quantity and quality of successfully realized international TEMPUS/ TASIS projects .

DNU is also a member of several international university associations, the most known of which are UNINET and "Eurasian Forum". Last year alone 47 foreign delegations, both educational and governmental, visited DNU. The university has become alma mater for alumni from 49 countries.

The history of the Dnipro National University would be incomplete without the history of the Department of Differential Equations.

A special fortune of the university, transforming it in September 1966 into one of the leading (under the terminology of that time - the basic) institution of higher education of Ukraine and the former Soviet Union, is quite rightly associated with the name of the academician, the Hero of the Socialist Labor, rector V. I. Mossakovskyi, who consistently headed the team for 22 years (1964–1986) [4].

As a basic institution, the university was assigned the role of a coordinating scientific and methodological center in the region. It became a kind of ground for innovations in the system of higher education. The new research laboratories and departments were created.

Department of differential equations was created in 1966 on the Mechanics and Mathematics Faculty of the Dnipropetrovsk State University. Associate Professor B. I. Kryukov headed by of the Department. Staff of the Department was as follows in 1966: Associate Professor B. I. Kryukov, Associate Professor V. A. Ostapenko, senior lecturer O. V. Zenkin, Assistant I. N. Shields, Assistant L. I. Shelest, senior laboratory L.V. Lybina.

The Laboratory of "Vibration Machines" (Head Assoc.Prof. E. A. Logvinenko) was organized at the Department in 1969. The study of the dynamics of vibration non-linear systems which are generating asymmetric vibrations were conducted in this Laboratory. The methods of calculation and design principles of new resonance vibration machines have been developed as a result. Here were designed, manufactured and introduced asymmetric vibration platforms, cluster plants, screen machines which allow to not only intensify the processes but the reduce the consumption of energy and materials and the fall of noise level up to sanitary standards. Division "Vibrational Technics" at the Department of Differential Equations was created with the aim of the implementation acceleration of the research results in 1971 [5].

A number of scientific and engineering organizations with priority of vibrational Technics were created in capital of USSR given the high economic efficiency of the new vibrational equipment, which was created the department staff, and the need the wide implementation into constructional industry of country. The government had decided to transfer of Prof. B. I. Kryukov, Assoc.Prof. E. A. Logvinenko (Head of Department), Ph.D. L. M. Litvin (Chief designer Department) as the heads of these organizations. At the end of the work (in 1979) these scientists were awarded the State Prize for the development of the theory of nonlinear systems, which are generating asymmetric vibrations, calculation methods and design principles of resonant vibration of machines and widespread adoption of scientific achievements in industry. All the theoretical and experimental studies were carried out at the Department of Differential Equations at the Dnepropetrovsk University.

Fig. 1. Asymmetric vibration platform.

In time the workload and the number of employees of the Department were increased. The skills of teachers are constantly improved. Professor V. A. Ostapenko (1975–1995), Professor N. V. Poliakov (1995–2014), Professor P. I. Kogut from 2014 (to the present) were headed the Department after Prof. B. I. Kryukov.

The present time of the native educational institution, its destiny, fame, achieve ments, outstanding scientific schools, well-known names in science and education, are closely connected with the people who studied and worked in it.

Prof. I. N. Shitov, Prof. V. A. Tychinin, Prof. M. V. Dmitriev, Assoc. Prof. V. I. Perechrest, Assoc. Prof. O. V. Zenkin, Assoc. Prof. G. I. Skorokhod, Assoc. Prof. V. B. Spivakovsky (Israel), Assoc. Prof. V. B. Kamen (USA), Assoc. Prof. T. V. Ridvanskaya, Assoc. Prof. A. A. Busurulov, Senior Lecture S. M. Ilyina, Assistant E. S. Mnouchkina, Assistant V. Z. Kachan, Assistant N. D. Pashkovskaya, Assistant S. A. Tyr worked at the Department at various times.

Fig. 2. Department of Differential Equations.

Since 1968 the scientific seminar "Differential equations and their applications" has worked at the department and a collection of scientific works with the same name has been published.

Among the research areas are the followings: the theory of diffraction and wave resonators, investigation of inverse problems for differential equations, investigation of gas-air tract of internal combustion engines, investigation of nonlocal and conditional symmetries of nonlinear equations of mathematical physics, setting and solving of inverse problems for differential equations, construction of exact solutions and asymptotic of solutions of nonlinear differential equations, solution of dynamic problems of the theory of elasticity, related problems of nonlinear thermoplasticity (two-phase problem), qualitative theory of differential equations, asymptotic methods in the theory of differential equations, mathematical modeling of dynamic systems, mathematical bases of the method of boundary integral equations, methods for solving incorrect problems.

Fundamental results were obtained in the directions:

– solving nonlinear boundary value problems of immersion of bodies in a liquid;

– asymptotic methods for solving nonlinear differential equations with partial derivatives;

– construction of new solutions of the Euler and Navier-Stokes hydrodynamic equations;

– integral images of problem solving for elliptic equations;

– solving inverse problems for differential equations with approximated coefficients;

– analytical solutions of boundary value problems for growing bodies, taking into account phase transitions.

Over the years the range of tasks was extended. The new problems were determined by the new needs of the national economy and the state, new scientific interests, the topics of contractual and state budget subjects of scientific research.

Fig. 3. The scientific seminar.

In 1997-1998 Department participated in scientific and applied research on the program "Sea Launch" together with the design department "Southern".

Five doctoral theses (V.A. Ostapenko, I.N. Shitov, M.V. Dmitriev, V.A. Tychinin, N.V. Poliakov) and 21 master's theses were protected since the establishment of the Department to the currently.

Many graduates of the Department of Differential Equations became later teachers of the Dnipropetrovsk and other universities of Ukraine and other countries, and succeeded in various spheres of scientific and educational, economic and industrial, public and political life. More than 700 mathematicians have been trained by the Department for 50 years.

The university school of mathematical theory of controlled systems, aerohydromechanics and energy-mass transfer is well-known in Ukraine and abroad. The newest theories of differential equations are directly used for the creation of nanomaterials, solving problems of optimization of structures, calculating the dynamics of complex mechanical systems, solving environmental problems and filtration of systems with numerous microstructures, calculating the destruction of materials, and the creation of novel materials with predefined properties [5].

The Department has close educational and scientific links with the Institute of Mathematics of the National Academy of Sciences of Ukraine, the Kyiv National University, the Donetsk State University, the universities of Erlangen (Germany), Salerno (Italy), Naples (Italy), Bilbao (Spain), Moscow State University (Russia), Technical University of Cottbus (Germany), Lodz Polytechnic University (Poland) Technical University of Koblitz (Germany), as well as with universities in the USA, France, Israel and Russia.

Main scientific interests of the department now:

– Mathematical Modeling, Optimization and Control of Dynamical Systems and Processes in Science and Engineering;

– Optimal Control Problems for Partial Differential Equations with Control and State Constraints;

– Optimization of Traffic Flows on Networks;

– Optimal Control Problems for Nonlinear Hyperbolic Conservation Lows;

– Asymptotic Analysis of Optimization Problems on Reticulated Structures (perforated domains, thick multi-structures, thick junctions);

– Homogenization and Variational Convergence of Optimal Control Problems.

– Optimal Control Problems on Thick Periodic Graphs;

– Optimal Control Problems in Coefficients for Nonlinear PDE;

– Optimization Theory in Partially Ordered Normed Spaces;

– Inverse Problem Theory and Methods for Model Parameter Estimation;

– Exact and Approximate Methods of the Theory of Differential and Integral-Differential Equations;

– Mathematical Modeling and Simulation of the Heat and Mass Transfer Processes with Nonequilibrium Relaxation.

On a regular base the department conducts the scientific seminar on "Modern Problems of Optimization and Mathematical Modeling" for research fellows, postgraduates, and senior students.

In 2013–2016 the research on the state budget subject "Modeling and optimization of nonlinear evolution systems" was conducted, since 2017 the research on the subject "Optimization of nonlinear systems with distributed parameters: qualitative analysis, approximation of solutions, necessary optimality conditions" has been carried out.

Staff of the Department now: Peter I. Kogut — Prof. Dr., Head of the Department, Mykola V. Poliakov — Corresponding Member of National Academy of Sciences of Ukraine, Prof. Dr., Professor, the Rector of Oles Honchar Dnipro National University, Andrii V. Siasiev — Ph.Dr., Associate Professor, Deputy Dean for Research of Faculty of Mechanics and Mathematics. Ph.Dr., Associate Professors: Vladimir L. Borsch, Marina V. Matyash, Yurii L. Menshikov, Eugene V. Turchin, Svitlana A. Gorbonos, Yurii P.Sovit, Olga S. Filippova, Eugene A. Makarenkov, Tamara A. Bozhanova, Irina G. Balanenko.

"The Bulletin of Dnipropetrovsk University. Series: Communications in Mathematical Modeling and Differential Equations Theory" is concerned with the theory and the application of partial differential equations, dynamical systems, optimal control theory and other related topics. The journal had been founded in 2009. The goal is to provide a complete and reliable source of mathematical methods and results in this field. The journal will also accept papers from some related fields such as functional analysis, probability theory and stochastic analysis, inverse problems for differential equations, optimization, numerical computation, mathematical finance, game theory, system theory, etc., provided that they have some intrinsic connections with control theory and differential equations [6].

Since 2016 a collection of scientific works for students and postgraduates "Differential equations and their applications" is published.

We hope that "Journal of Optimization Differential Equations and Their Applications" will not only become the descendant of previous editions, but also provide an opportunity to broaden the range of issues to be solved and the geography of its authors and readers.

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ON EXISTENCE OF BOUNDED FEASIBLE SOLUTIONS TO NEUMANN BOUNDARY CONTROL PROBLEM FOR p-LAPLACE EQUATION WITH EXPONENTIAL TYPE OF NONLINEARITY

Peter I. Kogut*, Rosanna Manzo†, Mykola V. Poliakov‡

Abstract. We study an optimal control problem for mixed Dirichlet-Neumann boundary value problem for the strongly non-linear elliptic equation with p-Laplace operator and L^1-nonlinearity in its right-hand side. A distribution u acting on a part of boundary of open domain is taken as a boundary control. The optimal control problem is to minimize the discrepancy between a given distribution $y_d \in L^2(\Omega)$ and the current system state. We deal with such case of nonlinearity when we cannot expect to have a solution of the state equation for any admissible control. After defining a suitable functional class in which we look for solutions and assuming that this problem admits at least one feasible solution, we prove the existence of optimal pairs. We derive also conditions when the set of feasible solutions has a nonempty intersection with the space of bounded distributions $L^\infty(\Omega)$.

Key words: existence result, optimal control, p-Laplace operator, elliptic equation, bounded solutions.

2010 Mathematics Subject Classification: 49J20, 35J20, 35B45, 35B65.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$. We assume that its boundary $\partial\Omega$ is of the class C^1. So, the unit outward normal $\nu = \nu(x)$ is well-defined for $\mathcal{H}^{N-1}$-a.a. $x \in \partial\Omega$, where a.a. means here with respect to the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. We also assume that the boundary $\partial\Omega$ consists of two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_N$, where the sets Γ_D and Γ_N have positive $(N-1)$-dimensional measures. Let $F\colon \mathbb{R} \to [0, +\infty)$ be a mapping such that $F \in C^1_{loc}(\mathbb{R})$, F is a non-decreasing positive function, and there exists a constant $C_F > 0$ satisfying

$$F'(z) \geq C_F F(z), \quad \forall\, z \in \mathbb{R} \ \text{ and } \ \left| \int_{-\infty}^{0} z F'(z)\, dz \right| < +\infty. \tag{1.1}$$

Further we define the function $f \in C_{loc}(\mathbb{R})$ as follows: $f(z) = F'(z)$.

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Let p, r, and q be real numbers such that $p \geq 2$, $q \geq \frac{pN}{pN-N+p}$, and $r \geq p'$, where $p' = \frac{p}{p-1}$ is the conjugate exponent to p.

We are concerned with the following optimal control problem for a nonlinear elliptic equation with p-Laplace operator:

$$\text{Minimize } J(u,y) = \frac{1}{2}\int_\Omega |y-y_d|^2\,dx + \frac{1}{p'}\int_{\Gamma_N} |u|^{p'}\,dx + \frac{\alpha}{r}\int_\Omega |f(y)|^r\,dx, \quad (1.2)$$

subject to constraints

$$-\mathrm{div}\left(|\nabla y|^{p-2}\nabla y\right) = f(y) + g \quad \text{in } \Omega, \quad (1.3)$$

$$y = 0 \quad \text{on } \Gamma_D, \quad |\nabla y|^{p-2}\partial_\nu y = u \quad \text{on } \Gamma_N, \quad (1.4)$$

$$u \in \mathfrak{A}_{ad} \subset L^{p'}(\Gamma_N),\ y \in W_0^{1,p}(\Omega;\Gamma_D), \quad (1.5)$$

where $\alpha > 0$ is a given weight which is assumed to be small enough, $\mathfrak{A}_{ad}$ is a closed convex subset of $L^{p'}(\Gamma_N)$, $g \in L^q(\Omega)$ and $y_d \in L^2(\Omega)$ are given distributions.

Let $C_0^\infty(\mathbb{R}^N;\Gamma_D) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N) \ : \ \varphi = 0 \text{ on } \Gamma_D\right\}$. In what follows we associate with the optimal control problem (1.2)–(1.5) the Banach space $W_0^{1,p}(\Omega;\Gamma_D)$ which we define as the closure of $C_0^\infty(\mathbb{R}^N;\Gamma_D)$ with respect to the norm

$$\|y\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla y|^p\,dx\right)^{1/p}.$$

So, we can suppose that each element of the space $W_0^{1,p}(\Omega;\Gamma_D)$ has zero trace at the Γ_D-part of boundary $\partial\Omega$. Let $W^{-1,p'}(\Omega;\Gamma_D) := \left(W_0^{1,p}(\Omega;\Gamma_D)\right)^*$ be the dual space to $W^{1,p}(\Omega;\Gamma_D)$.

Definition 1.1. We say that $(u,y) \in L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega;\Gamma_D)$ is a feasible solution to the problem (1.2)–(1.5) if

- u is an admissible control, i.e. $u \in \mathfrak{A}_{ad}$;
- $J(u,y) < +\infty$;
- the function $y = y(u)$ is a weak solution to the boundary value problem (BVP) (1.3)–(1.4) for a given control u, i.e. $y \in W_0^{1,p}(\Omega;\Gamma_D)$ and the integral identity

$$\int_\Omega |\nabla y|^{p-2}\left(\nabla y, \nabla\varphi\right)\,dx = \int_\Omega f(y)\varphi\,dx + \int_{\Gamma_N} u\varphi\,d\mathcal{H}^{N-1} + \int_\Omega g\varphi\,dx \quad (1.6)$$

holds for every test function $\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_D)$.

We denote by Ξ the set of all feasible solutions to the problem (1.2)–(1.5).

Equations like (1.3) appear in a number of applications. In particular, it has been applied for the description of a ball of isothermal gas in gravitational equilibrium, proposed by lord Kelvin [7] in the study of stellar structures [7]. It has been also actively investigated in connection with combustion theory (see, for instance, [9, 12, 14]). However, it is well known that the indicated BVP is ill-posed, in general. It means that there is no reason to assert the existence of weak solutions to (1.3)–(1.4) for given $g \in L^q(\Omega)$ and $u \in L^{p'}(\Gamma_N)$, or to suppose that such solution, even if it exists, is unique (see, for instance, I.M. Gelfand [12], H. Brezis and J.L. Vázquez [3], M.G. Crandall and P.H. Rabinowitz [13], F. Mignot and J.P. Puel [22], T. Gallouët, F. Mignot and J.P. Puel [11], H. Fujita [10], R.G. Pinsky [24], R. Ferreira, A. De Pablo, J.L. Vazquez [8]). In view of this it is worth to emphasize the following result (see [2]): there exists a finite positive number λ^*, called the extremal value, such that the boundary value problem

$$-\Delta y = \lambda e^y + v \quad \text{in} \quad \Omega, \qquad y = 0 \quad \text{on} \quad \partial\Omega \tag{1.7}$$

has at least a classical positive solution $y \in C^2(\overline{\Omega})$ provided $0 < \lambda < \lambda^*$ and $v = 0$, while no solution exists, even in the weak sense, for $\lambda > \lambda^*$. In the case $\lambda = \lambda^*$ and $v = 0$, this problem admits the existence of the so-called singular solutions $u \in H^1_0(\Omega)$ that do not belong to $L^\infty(\Omega)$. Thus, in the context of the optimal control problem that we deal with in this paper, there is no reason to suppose that a weak solution to (1.3)–(1.4) for given $u \in L^{p'}(\Gamma_N)$, and $g \in L^q(\Omega)$, even if it exists, is unique and bounded. Moreover, to the best knowledge of authors, the existence and uniqueness of the weak solutions to the original BVP is an open question for nowadays. In view of this, we adopt the so-called non-triviality assumption:

Hypothesis A. For given $f \in C_{loc}(\mathbb{R})$, $g \in L^q(\Omega)$, $y_d \in L^2(\Omega)$, and $\mathfrak{A}_{ad}$, the set of feasible solutions Ξ is nonempty.

Before proceeding further, it is worth to note here that some optimal control problems, related with the Dirichlet problem (1.7), was first discussed in detail by Casas, Kavian, and Puel [5]. The questions of existence and uniqueness of optimal solutions were treated and optimality systems have been derived and analyzed in [5]. At the same time, analogous results for the case of nonlinear elliptic equations (1.3) with mixed boundary conditions (1.4) remain arguably open. Some related questions in this field can be found in the recent papers [15,16] (see also [6, 18, 19]).

We also emphasize that the corresponding strongly nonlinear differential operator $-\operatorname{div}(|\nabla y|^{p-2}\nabla y) - f(y)$ is not monotone and, in principle, has degeneracy as ∇y tends to zero. Moreover, when the term $|\nabla y|^{p-2}$ is regarded as the coefficient of the Laplace operator, we have also the case of unbounded coefficients. Because of this and specific properties of the function $f(y)$, there are serious hurdles to deduce an a priori estimate for the weak solutions of BVP (1.3)–(1.4) even in the standard Sobolev space $W^{1,p}_0(\Omega)$. On the other hand, the existence of bounded feasible solutions to the problem (1.2)–(1.5) is a crucial characteristic for the wide spectrum of investigations related with this problem: differentiability of the state $y(u)$ with respect to the boundary control u, deriving and substantiation of optimality conditions, and many others (see, for instance, [4]). In view of this, our main concern in this paper is to discuss the existence of bounded feasible solutions to the optimal control problem (1.2)–(1.5). In particular, we are focused on the following question: Let (u, y) be a feasible solution to the problem (1.2)–(1.5). Which conditions should be imposed on p, r, q, Ω, Γ_N, $u \in L^{p'}(\Gamma_N)$, and $g \in L^q(\Omega)$ in order to guarantee the inclusions $y \in L^\infty(\Omega)$ and/or $y \in L^\infty(\partial\Omega)$? As was shown in the recent paper of the first author [17], the existence of at least one feasible pair (u, y) with the extra property $y \in W_0^{1,p}(\Omega; \Gamma_D) \cap L^\infty(\Omega)$ plays a crucial role for the substantiation of attainability of optimal pairs to the problem (1.2)–(1.5) by optimal solutions of some fictitious optimization problem for quasi-linear elliptic equations with coercive and monotone operators.

The plan of the paper is as follows. In Section 2 we give some preliminaries concerning the original problem (1.2)–(1.5). In particular, we give the formal statement of the boundary value problem and establish the necessary background to its study. We also study in this section some auxiliary properties of the feasible solutions to the Dirichlet-Neumann boundary value problem (1.3)–(1.4). In particular, we show that an a priori estimate for the weak solutions in $W_0^{1,p}(\Omega)$ can be derived if only such solutions are feasible to the original optimal control problem. The key result of this section is Proposition 2.2, which gives the grounds to suppose that the set of feasible solutions with $L^{p'}(\Omega)$-bounded nonlinearity $f(y)$ is weakly closed in $W_0^{1,p}(\Omega)$. The existence of optimal boundary controls is discussed in Theorem 2.2. We give the proof of our main results in Section 3 and they can be stated as follows.

Theorem 1.1. *Let p, q, r be exponents such that*

$$1 \le p < N, \quad q > \max\left\{\frac{N}{p}; \frac{p}{p-1}\right\} \quad \text{and} \quad r > \max\left\{\frac{N}{p}; \frac{p}{p-1}\right\}. \tag{1.8}$$

Let (u, y) be a feasible solution to the problem (1.2)–(1.5) and let $u \in L^t(\Gamma_N)$ for some

$$t > \max\left\{\frac{N-1}{p-1}; \frac{p}{p-1}\right\}. \tag{1.9}$$

Then

$$y \in W_0^{1,p}(\Omega; \Gamma_D) \cap L^\infty(\Omega) \quad \text{and} \quad \gamma_0(y) \in W^{1/p',p}(\Gamma_N) \cap L^\infty(\Gamma_N),$$

where $\gamma_0 : W^{1,p}(\Omega; \Gamma_D) \to W^{1/p',p}(\Gamma_N)$ stands for the trace operator.

Theorem 1.2. *Let p, q, r be exponents such that*

$$p > N, \quad q \ge \frac{p}{p-1} \quad \text{and} \quad r \ge \frac{p}{p-1}. \tag{1.10}$$

Let (u, y) be a feasible solution to the problem (1.2)–(1.5). Then

$$y \in W_0^{1,p}(\Omega; \Gamma_D) \cap L^\infty(\Omega).$$

2. On Consistency of Optimal Control Problem (1.2)–(1.5)

As we mentioned before, it is unknown whether the original BVP admits at least one weak solution for any admissible control $u \in \mathfrak{A}_{ad} \subset L^{p'}(\Gamma_N)$ and a given distribution $g \in L^q(\Omega)$. Hence, it is not an easy matter to touch directly on the set of feasible solutions Ξ to the original optimal control problem because its structure and the main topological properties are unknown in general. To lighten this problem, we make use of the following observation. Let $(u, y) \in L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega; \Gamma_D)$ be an arbitrary feasible solution to the problem (1.2)–(1.5) in the sense of Definition 1.1. Then $f(y) \in L^{p'}(\Omega)$ and, therefore, the form $[y, \varphi]_f := \int_\Omega f(y)\varphi\, dx$ is continuous onto the set

$$Y = \left\{ y \in W_0^{1,p}(\Omega; \Gamma_D) \mid (u, y) \in \Xi \right\}. \tag{2.1}$$

Indeed, in this case, for each $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$, we have

$$\begin{aligned}\left| \int_\Omega f(y)\varphi\, dx \right| &\le \left(\int_\Omega |f(y)|^{p'}\, dx \right)^{1/p'} \left(\int_\Omega |\nabla \varphi|^p\, dx \right)^{1/p} \\ &\le |\Omega|^{\frac{1}{p'} - \frac{1}{r}} \left(\int_\Omega |f(y)|^r\, dx \right)^{1/r} \left(\int_\Omega |\nabla \varphi|^p\, dx \right)^{1/p} \\ &\le |\Omega|^{\frac{1}{p'} - \frac{1}{r}} \left(\frac{r}{\alpha} J(u, y) \right)^{1/r} \|\varphi\|_{W_0^{1,p}(\Omega;\Gamma_D)}. \end{aligned} \tag{2.2}$$

Thus, it is easy to show by continuity that the integral identity (1.6) remains valid for all $\varphi \in W^{1,p}(\Omega; \Gamma_D)$. Hence, if $(u, y) \in \Xi$ then

$$\begin{aligned}\int_\Omega |\nabla y|^{p-2} (\nabla y, \nabla \varphi)\, dx = \int_\Omega f(y)\varphi\, dx &+ \int_{\Gamma_N} \gamma_0(\varphi) u\, d\mathcal{H}^{N-1} \\ &+ \langle g, \varphi \rangle_{W^{-1,p'}(\Omega;\Gamma_D); W^{1,p}(\Omega;\Gamma_D)} \end{aligned} \tag{2.3}$$

holds true for all $\varphi \in W^{1,p}(\Omega; \Gamma_D)$, where

$$\langle \cdot, \cdot \rangle_{W^{-1,p'}(\Omega;\Gamma_D); W^{1,p}(\Omega;\Gamma_D)} : W^{-1,p'}(\Omega; \Gamma_D) \times W^{1,p}(\Omega; \Gamma_D) \to R$$

denotes the duality pairing between $W^{-1,p'}(\Omega; \Gamma_D)$ and $W^{1,p}(\Omega; \Gamma_D)$, and

$$\gamma_0 : W^{1,p}(\Omega; \Gamma_D) \to W^{1/p',p}(\Gamma_N)$$

stands for the trace operator (see [21, Theorem 8.3]), i.e.

$$\gamma_0(\varphi) = \varphi|_{\Gamma_D}, \quad \forall\, \varphi \in W^{1,p}(\Omega; \Gamma_D) \cap C(\overline{\Omega}).$$

We note that the duality pairing $\langle g, \varphi \rangle_{W^{-1,p'}(\Omega;\Gamma_D); W^{1,p}(\Omega;\Gamma_D)}$ is well defined for each $\varphi \in W^{1,p}(\Omega; \Gamma_D)$ provided $g \in L^q(\Omega)$ with $q \ge \frac{pN}{pN - N + p}$. Indeed, by Sobolev embedding theorem, the space $W^{1,p}(\Omega; \Gamma_D)$ is continuously embedded in

$L^{p^*}(\Omega)$ with $p^* = \frac{pN}{N-p}$. Hence, by duality arguments, $\big(L^{p^*}(\Omega)\big)^*$ is continuously embedded in $W^{-1,p'}(\Omega;\Gamma_D)$. So, if we define

$$p_* = (p^*)' = \frac{pN}{pN - N + p}, \tag{2.4}$$

then we have $L^q(\Omega) \subset L^{p_*}(\Omega) \subset W^{-1,p'}(\Omega;\Gamma_D), \quad \forall\, q \geq \frac{pN}{pN-N+p}$. Hence,

$$\begin{aligned} \Big| \langle g, \varphi \rangle_{W^{-1,p'}(\Omega;\Gamma_D);W^{1,p}(\Omega;\Gamma_D)} \Big| &\leq \|g\|_{W^{-1,p'}(\Omega;\Gamma_D)} \|\varphi\|_{W^{1,p}(\Omega;\Gamma_D)} \\ &\leq C_{em} \|g\|_{L^q(\Omega)} \|\varphi\|_{W^{1,p}(\Omega;\Gamma_D)}, \ \forall\, \varphi \in W^{1,p}(\Omega;\Gamma_D). \end{aligned} \tag{2.5}$$

We also note that, in view of the compactness of the injection $W^{1/p',p}(\Gamma_N) \hookrightarrow L^p(\Gamma_N)$ and continuity of the trace operator $\gamma_0 : W^{1,p}(\Omega;\Gamma_D) \to W^{1/p',p}(\Gamma_N)$,

$$\|\gamma_0(\varphi)\|_{L^p(\Gamma_N)} \leq C_{\gamma_0} \|\varphi\|_{W^{1,p}(\Omega;\Gamma_D)}, \quad \forall\, \varphi \in W^{1,p}(\Omega;\Gamma_D), \tag{2.6}$$

we have

$$\begin{aligned} \left| \int_{\Gamma_N} u\gamma_0(\varphi)\, d\mathcal{H}^{N-1} \right| &\leq \|u\|_{L^{p'}(\Gamma_N)} \|\varphi\|_{L^p(\Gamma_N)} \\ &\leq C_{\gamma_0} \|u\|_{L^{p'}(\Gamma_N)} \|\varphi\|_{W_0^{1,p}(\Omega;\Gamma_D)} < +\infty. \end{aligned} \tag{2.7}$$

Taking into account these observations, we immediately arrive at the following conclusion.

Lemma 2.1. *Let $(u, y) \in L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega;\Gamma_D)$ be an arbitrary feasible solution to the problem (1.2)–(1.5) in the sense of Definition 1.1. Then this pair is related by the energy equality*

$$\begin{aligned} \int_\Omega |\nabla y|^p\, dx = \int_\Omega y\, f(y)\, dx + \int_\Omega \gamma_0(y) u\, d\mathcal{H}^{N-1} \\ + \langle g, y \rangle_{W^{-1,p'}(\Omega;\Gamma_D);W^{1,p}(\Omega;\Gamma_D)}. \end{aligned} \tag{2.8}$$

It is worth to emphasize that energy equality (2.8) makes sense if only the pair (u, y) is feasible and it is unknown whether we can guarantee the fulfilment of this relation for an arbitrary weak solution $(u, y(u))$ to BVP (1.3)–(1.4). Nevertheless, taking into account the inequalities (2.2), (2.5), and (2.7), we can deduce from (2.8) the following result.

Theorem 2.1. *For fixed $p \geq 2$, $r \geq p'$, and $q \geq \frac{pN}{pN-N+p}$, let $u \in L^{p'}(\Gamma_N)$ and $g \in L^q(\Omega)$ be given distributions. Let $y = y(u) \in W_0^{1,p}(\Omega;\Gamma_D)$ be a weak solution to BVP (1.3)–(1.4) such that (u, y) is a feasible pair to optimal control problem*

(1.2)–(1.5). *Then*

$$\left|\int_\Omega y\,f(y)\,dx\right| \le \left(3^{p'-1}\frac{(p+1)}{p-1}\left[|\Omega|^{1-\frac{p'}{r}}\left(\frac{r}{\alpha}\right)^{\frac{p'}{r}} + C_{\gamma_0}^{p'}p'\right] + 2^{p'-1}C_{\gamma_0}^{p'}\right)$$
$$\times \max\{1, J(u,y)\} + \left(\frac{(p+1)}{p}3^{p'-1} + \frac{1}{p'}2^{p'-1}\right)C_{em}^{p'}\|g\|_{L^q(\Omega)}^{p'}, \tag{2.9}$$

$$\|y\|_{W_0^{1,p}(\Omega;\Gamma_D)}^p \le 3^{p'-1}\left[|\Omega|^{1-\frac{p'}{r}}\left(\frac{r}{\alpha}\right)^{\frac{p'}{r}} + C_{\gamma_0}^{p'}p'\right]\max\{1, J(u,y)\}$$
$$+ 3^{p'-1}C_{em}^{p'}\|g\|_{L^q(\Omega)}^{p'}. \tag{2.10}$$

Proof. Let (u,y) be a given feasible solution. Then relation (2.8) and inequalities (2.2), (2.5), and (2.7), immediately lead to the following estimate

$$\|y\|_{W_0^{1,p}(\Omega;\Gamma_D)}^{p-1} \le \|f(y)\|_{L^{p'}(\Omega)} + C_{\gamma_0}\|u\|_{L^{p'}(\Gamma_N)} + C_{em}\|g\|_{L^q(\Omega)}, \tag{2.11}$$

where $\|f(y)\|_{L^{p'}(\Omega)} \le |\Omega|^{\frac{1}{p'}-\frac{1}{r}}\left(\frac{r}{\alpha}J(u,y)\right)^{1/r} < +\infty$ by the feasibility property of the pair (u,y). Since $p-1 = p/p'$ and $\|u\|_{L^{p'}(\Gamma_N)}^{p'} \le p'J(u,y)$, the a priori estimate (2.10) is a direct consequence of (2.11).

In order to establish the estimate (2.9), we make use of the energy equality (2.8) and the standard form of Young's inequality. As a result, we obtain

$$\begin{aligned}
\left|\int_\Omega y\,f(y)\,dx\right| &\le \|y\|_{W_0^{1,p}(\Omega;\Gamma_D)}^p + \Big(C_{\gamma_0}\|u\|_{L^{p'}(\Gamma_N)} + C_{em}\|g\|_{L^q(\Omega)}\Big)\|y\|_{W_0^{1,p}(\Omega;\Gamma_D)} \\
&\le \left(1+\frac{1}{p}\right)\|y\|_{W_0^{1,p}(\Omega;\Gamma_D)}^p + \frac{1}{p'}\Big(C_{\gamma_0}\|u\|_{L^{p'}(\Gamma_N)} + C_{em}\|g\|_{L^q(\Omega)}\Big)^{p'} \\
&\le \frac{p+1}{p}\|y\|_{W_0^{1,p}(\Omega;\Gamma_D)}^p + 2^{p'-1}\left[C_{\gamma_0}^{p'}J(u,y) + \frac{C_{em}^{p'}}{p'}\|g\|_{L^q(\Omega)}^{p'}\right] \\
&\le \frac{(p+1)3^{p'-1}}{p}\left[|\Omega|^{1-\frac{p'}{r}}\left(\frac{r}{\alpha}\right)^{\frac{p'}{r}} + C_{\gamma_0}^{p'}p'\right]\max\{1, J(u,y)\} \\
&+ \frac{(p+1)3^{p'-1}}{p}C_{em}^{p'}\|g\|_{L^q(\Omega)}^{p'} + 2^{p'-1}\left[C_{\gamma_0}^{p'}J(u,y) + \frac{C_{em}^{p'}}{p'}\|g\|_{L^q(\Omega)}^{p'}\right].
\end{aligned}$$

After simplification, we arrive at the expected estimate (2.9). □

The following Propositions reflect some interesting properties of feasible solutions. In particular, Proposition 2.1 can be interpreted as some specification of the well-known Boccardo–Murat Theorem (see L. Boccardo and F. Murat [1, Theorem 2.1]).

Proposition 2.1. Assume that $q \geq p' = p/(p-1)$. Let

$$\{(u_k, g_k, y_k)\}_{k\in\mathbb{N}} \subset L^{p'}(\Gamma_N) \times L^q(\Omega) \times W_0^{1,p}(\Omega;\Gamma_D)$$

be a sequence such that

$$f(y_k) \in L^{p'}(\Omega) \quad \text{for all} \quad k \in \mathbb{N}, \tag{2.12}$$

$$u_k \rightharpoonup u \quad \text{weakly in} \quad L^{p'}(\Gamma_N), \tag{2.13}$$

$$g_k \rightharpoonup g \quad \text{weakly in} \quad L^q(\Omega), \tag{2.14}$$

$$y_k \to y \quad \text{weakly in } W_0^{1,p}(\Omega;\Gamma_D) \text{ and a.e. in } \Omega, \tag{2.15}$$

$$f(y_k) \to f(y) \text{ strongly in } \; L^1(\Omega), \tag{2.16}$$

$$-\operatorname{div}\left(|\nabla y_k|^{p-2}\nabla y_k\right) = f(y_k) + g_k \quad \text{in} \quad \left(C_0^\infty(\mathbb{R}^N;\Gamma_D)\right)^*, \quad \forall\, k \in \mathbb{N}, \tag{2.17}$$

$$\gamma_0(y_k) = 0 \quad \text{and} \quad |\gamma_1(y_k)|^{p-2}\gamma_1(y_k) = u_k, \quad \forall\, k \in \mathbb{N}, \tag{2.18}$$

where $\gamma_1(y) = \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_N}$ for all $y \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega;\Gamma_D)$. Then

$$\nabla y_k \to \nabla y \;\text{ strongly in } L^r(\Omega)^N \text{ for any } 1 \leq r < p. \tag{2.19}$$

Proof. As follows from (2.17)–(2.18), the functions y_k are the weak solutions to the boundary value problem (1.3)–(1.4) for the corresponding controls $u_k \in L^{p'}(\Gamma_N)$. For every $\varepsilon > 0$, let $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the truncation operator defined by

$$T_\varepsilon(s) = \max\left\{\min\left\{s, \varepsilon^{-1}\right\}, -\varepsilon^{-1}\right\}. \tag{2.20}$$

Since $T_{\varepsilon^{-1}}(y_k - y) \in W_0^{1,p}(\Omega;\Gamma_D)$, it follows from (2.12) that

$$\varphi = T_{\varepsilon^{-1}}(y_k - y) \in W_0^{1,p}(\Omega;\Gamma_D)$$

can be used as the test function in integral identity (2.3). Hence, for every $k \in \mathbb{N}$, we have the relation

$$\begin{aligned}\int_\Omega \left(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y, \nabla T_{\varepsilon^{-1}}(y_k - y)\right) dx = \int_\Omega f(y_k) T_{\varepsilon^{-1}}(y_k - y)\, dx \\ + \int_{\Gamma_N} u_k \gamma_0(T_{\varepsilon^{-1}}(y_k - y))\, d\mathcal{H}^{N-1} + \langle g_k, T_{\varepsilon^{-1}}(y_k - y)\rangle_{W^{-1,p'}(\Omega;\Gamma_D); W^{1,p}(\Omega;\Gamma_D)} \\ - \int_\Omega \left(|\nabla y|^{p-2}\nabla y, \nabla T_{\varepsilon^{-1}}(y_k - y)\right) dx = J_1 + J_2 + J_3 - J_4.\end{aligned} \tag{2.21}$$

Taking into account the fact that $p' > \frac{pN}{pN-N+p}$ and $q \geq p'$, we can deduce compactness of the embedding $L^q(\Omega) \hookrightarrow W^{-1,p'}(\Omega;\Gamma_D)$. Then (2.14) and (2.15) imply that

$$g_k \to g \;\text{ strongly in } \; W^{-1,p'}(\Omega;\Gamma_D),$$

$$T_{\varepsilon^{-1}}(y_k - y) \to 0 \;\text{ weakly in } W_0^{1,p}(\Omega;\Gamma_D) \text{ and strongly in } L^p(\Omega).$$

Thus, $J_3 - J_4$ tends to zero as $k \to \infty$. As for the term J_2, we see that, by Sobolev embedding theorem, the injection $W^{1/p',p}(\Gamma_N) \hookrightarrow L^r(\Gamma_N)$ is compact for all $1 \le r < p\frac{N-1}{N-p}$. Hence, by duality arguments, $(L^r(\Gamma_N))^*$ is compactly embedded in $\left(W^{1/p',p}(\Gamma_N)\right)^*$. So, if we define

$$r_* = \left(\frac{N-1}{N-p}p\right)' = \frac{N-1}{N}p'$$

then we have $p' > r_*$ and, therefore, the injection $L^{p'}(\Gamma_N) \hookrightarrow \left(W^{1/p',p}(\Gamma_N)\right)^*$ is compact as well. Thus, due to (2.13)–(2.15), we have

$$u_k \to u \ \text{ strongly in } \ \left(W^{1/p',p}(\Gamma_N)\right)^* \ \text{ and}$$
$$\gamma_0(y_k) \rightharpoonup \gamma_0(y) \ \text{ weakly in } \ W^{1/p',p}(\Gamma_N).$$

As a result, we obtain

$$J_2 = \int_{\Gamma_N} u_k \gamma_0(T_{\varepsilon^{-1}}(y_k - y))\, d\mathcal{H}^{N-1} \to 0 \text{ as } k \to \infty.$$

It remains to note that condition (2.16) leads to the inequality

$$J_1 \le C\|T_{\varepsilon^{-1}}(y_k - y)\|_{L^\infty(\Omega)}, \quad \forall\, k \in \mathbb{N}.$$

Hence, mollifying $T_\varepsilon(y_k - y)$ and the poinwise convergence $y_k(x) \to y(x)$ a.e. in Ω imply that

$$|J_1| \le C\|T_{\varepsilon^{-1}}(y_k - y)\|_{L^\infty(\Omega)} \le C_1\varepsilon, \quad \forall\, k \in \mathbb{N}. \tag{2.22}$$

Combining all issues given above, we can finally deduce that, for a fixed $\varepsilon > 0$,

$$\limsup_{k\to\infty} \int_\Omega \left(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y, \nabla T_{\varepsilon^{-1}}(y_k - y)\right) dx \le C_1\varepsilon. \tag{2.23}$$

Let us define now the following functions

$$d_k(x) = \left(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y, \nabla y_k - \nabla y\right), \quad k \in \mathbb{N}$$

and fix θ with $0 < \theta < 1$. In view of the initial assumptions, it is clear that $\{d_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^1(\Omega)$ and

$$\left(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y, \nabla y_k - \nabla y\right) \ge 2^{2-p}|\nabla y_k - \nabla y|^p \tag{2.24}$$

by the strict monotonicity property of the p-Laplace operator. Splitting the set Ω into

$$S_\varepsilon^k = \{x \in \Omega \ : \ |y_k(x) - y(x)| \le \varepsilon\}, \quad G_\varepsilon^k = \{x \in \Omega \ : \ |y_k(x) - y(x)| > \varepsilon\}$$

and using Hölder inequality, we get

$$\begin{aligned}\int_\Omega d_k^\theta\,dx &= \int_{S_\varepsilon^k} d_k^\theta\,dx + \int_{G_\varepsilon^k} d_k^\theta\,dx\\ &\le \left(\int_{S_\varepsilon^k} d_k\,dx\right)^\theta |S_\varepsilon^k|^{1-\theta} + \left(\int_{G_\varepsilon^k} d_k\,dx\right)^\theta |G_\varepsilon^k|^{1-\theta}\\ &\le \left(\int_\Omega \left(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y, \nabla T_{\varepsilon^{-1}}\left(y_k - y\right)\right)\,dx\right)^\theta |S_\varepsilon^k|^{1-\theta}\\ &\quad + \left(\int_\Omega d_k\,dx\right)^\theta |G_\varepsilon^k|^{1-\theta}. \end{aligned} \tag{2.25}$$

Since, for a fixed ε, $|G_\varepsilon^k|$ tends to zero as $k\to\infty$, it follows from (2.23), (2.24), and (2.25) that

$$\limsup_{k\to\infty}\int_\Omega \left(|\nabla y_k - \nabla y|^p\right)^\theta\,dx \le 2^{\theta(p-2)}\limsup_{k\to\infty}\int_\Omega d_k^\theta\,dx \le 2^{\theta(p-2)}\left(C_1\varepsilon\right)^\theta|\Omega|^{1-\theta}.$$

Letting ε tend to 0 and θ tend to 1 this implies that $|\nabla y_k - \nabla y|^p$ tends strongly to 0 in $L^1(\Omega)$ and thus, there exists a subsequence $\{k_n\}_{n\in\mathbb{N}}$ such that

$$\nabla y_{k_n}(x) \to \nabla y(x) \quad \text{a.e. in } \ \Omega \ \text{ as } k_n\to\infty. \tag{2.26}$$

Since $\{\nabla y_{k_n}\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega)^N$, it follows from Vitali's theorem that

$$\nabla y_{k_n}\to\nabla y \ \text{ strongly in } L^r(\Omega)^N \text{ for any } 1\le r<p. \tag{2.27}$$

It remains to note that, in fact, we have the convergence in (2.27) for the whole sequence $\{\nabla y_k\}_{k\in\mathbb{N}}$ because the limit ∇y in (2.27) is independent of the subsequence $\{k_n\}_{n\in\mathbb{N}}$. □

Proposition 2.2. Assume that $q\ge p'$ and $r\ge p'$. Let $\{(u_k,y_k)\}_{k\in\mathbb{N}}\subset\Xi$ be a sequence of feasible solutions such that

$$\sup_{k\in\mathbb{N}} J(u_k,y_k) < +\infty, \tag{2.28}$$

$$(u_k,y_k)\rightharpoonup(u,y) \ \text{ weakly in } \ L^{p'}(\Gamma_N)\times W_0^{1,p}(\Omega;\Gamma_D) \ \text{ as } k\to\infty. \tag{2.29}$$

Then $(u,y)\in\Xi$ and

$$f(y_k)\to f(y) \ \text{ strongly in } L^1(\Omega) \text{ and weakly in } L^r(\Omega) \text{ as } k\to\infty. \tag{2.30}$$

Proof. By the Sobolev Embedding Theorem, the injection $W_0^{1,p}(\Omega;\Gamma_D)\hookrightarrow L^p(\Omega)$ is compact. Hence, the weak convergence $y_k\rightharpoonup y$ in $W_0^{1,p}(\Omega;\Gamma_D)$ implies the strong convergence in $L^p(\Omega)$. Therefore, up to a subsequence, we can suppose that $y_k(x)\to y(x)$ for almost every point $x\in\Omega$. As a result, we have the pointwise convergence: $f(y_k) \to f(y)$ almost everywhere in Ω. Let us show that this fact implies the strong convergence (2.30).

With that in mind we recall that a sequence $\{f_k\}_{k\in\mathbb{N}}$ is called equi-integrable on Ω if for any $\delta > 0$, there is a $\tau = \tau(\delta)$ such that $\int_S |f_k|\,dx < \delta$ for every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$. Let us show that the sequence $\{f(y_k)\}_{k\in\mathbb{N}}$ is equi-integrable on Ω. To do so, we take $m > 0$ such that

$$m > 2L\delta^{-1}, \tag{2.31}$$

where

$$\begin{aligned} L := &\left(3^{p'-1}\frac{(p+1)}{p-1}\left[|\Omega|^{1-\frac{p'}{r}}\left(\frac{r}{\alpha}\right)^{\frac{p'}{r}} + C_{\gamma_0}^{p'}p'\right] + 2^{p'-1}C_{\gamma_0}^{p'}\right)\\ &\quad\times \max\left\{1, \sup_{k\in\mathbb{N}} J(u_k,y_k)\right\} + \left(\frac{(p+1)}{p}3^{p'-1} + \frac{1}{p'}2^{p'-1}\right)C_{em}^{p'}\|g\|_{L^q(\Omega)}^{p'}. \end{aligned}$$

We also set $\tau = \delta/(2f(m))$. Then for every measurable set $S \subset \Omega$ with $|S| < \tau$, we have

$$\begin{aligned} \int_S f(y_k)\,dx &= \int_{\{x\in S\,:\,y_k(x)>m\}} f(y_k)\,dx + \int_{\{x\in S\,:\,y_k(x)\le m\}} f(y_k)\,dx\\ &\le \frac{1}{m}\int_{\{x\in S\,:\,y_k(x)>m\}} y_k f(y_k)\,dx + \int_{\{x\in S\,:\,y_k(x)\le m\}} f(m)\,dx\\ &\overset{\text{by (2.9)}}{\le} \frac{L}{m} + f(m)|S| \overset{\text{by (2.31)}}{\le} \frac{\delta}{2} + \frac{\delta}{2}. \end{aligned}$$

As a result, the assertion (2.30) is a direct consequence of Lebesgue's Convergence Theorem.

Let us show now that the limit pair (u,y) is a feasible pair to optimal control problem (1.2)–(1.5). Indeed, in view of the initial assumptions and property (2.30), the limit passage in the right-hand side of the equality

$$\begin{aligned} \int_\Omega |\nabla y_k|^{p-2}\left(\nabla y_k, \nabla\varphi\right)\,dx = \int_\Omega f(y_k)\varphi\,dx + \int_{\Gamma_N} u_k\varphi\,d\mathcal{H}^{N-1}\\ + \int_\Omega g\varphi\,dx,\ \forall\,\varphi\in C_0^\infty(\mathbb{R}^N;\Gamma_D) \end{aligned} \tag{2.32}$$

becomes trivial. Taking into account Proposition 2.1, we have, up to a subsequence, the pointwise convergence (2.26). Since the sequence $\left\{|\nabla y_k|^{p-2}\nabla y_k\right\}_{k\in\mathbb{N}}$ is bounded in $L^{p'}(\Omega)^N$, it follows from (2.26) that

$$\begin{aligned} &|\nabla y_{k_n}|^{p-2}\nabla y_{k_n} \to |\nabla y|^{p-2}\nabla y \text{ almost everywhere in } \Omega,\\ &|\nabla y_{k_n}|^{p-2}\nabla y_{k_n} \rightharpoonup |\nabla y|^{p-2}\nabla y \text{ weakly in } L^{p'}(\Omega)^N. \end{aligned}$$

This allows us to pass to the limit as $k_n \to \infty$ in the left hand side of the equality (2.32). Thus, y is a weak solution to BVP (1.3)–(1.4) for the given $u \in L^{p'}(\Gamma_N)$.

Since the set $\mathfrak{A}_{ad}$ is convex and closed in $L^{p'}(\Gamma_N)$, it follows that this set is sequentially weakly closed in $L^{p'}(\Gamma_N)$ by the Mazur theorem. Therefore, the weak convergence (2.29) implies that $u \in \mathfrak{A}_{ad}$.

It remains to prove that the limit pair (u, y) satisfies the condition $J(u, y) < +\infty$. With that in mind we take into account the lower semi-continuity of the norm in $L^{p'}(\Gamma_N) \times L^2(\Omega)$ with respect to the weak convergence in $L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega; \Gamma_D)$ and property (2.30). This yields

$$\lim_{k\to\infty} \int_\Omega |y_k - y_d|^2\, dx \overset{\text{by (2.29)}}{=} \int_\Omega |y - y_d|^2\, dx, \tag{2.33}$$

$$\liminf_{k\to\infty} \int_\Omega |u_k|^{p'}\, d\mathcal{H}^{N-1} \overset{\text{by (2.29)}}{\geq} \int_\Omega |u|^{p'}\, d\mathcal{H}^{N-1}. \tag{2.34}$$

In view of condition (2.28), we have

$$\sup_{k\in\mathbb{N}} \|f(y_k)\|_{L^r(\Omega)} < +\infty.$$

Utilizing this fact together with the pointwise convergence

$$f(y_k) \to f(y) \text{ a.e. in } \Omega$$

that is a consequence of the property (2.30), we get $f(y_k) \rightharpoonup f(y)$ in $L^r(\Omega)$. Hence,

$$\liminf_{k\to\infty} \int_\Omega |f(y_k)|^r\, dx \geq \int_\Omega |f(y)|^r\, dx. \tag{2.35}$$

As a result, we deduce from (2.33), (2.34), and (2.35) that

$$J(u, y) \leq \liminf_{k\to\infty} J(u_k, y_k) < \sup_{k\in\mathbb{N}} J(u_k, y_k) < +\infty.$$

Thus, (u, y) is a feasible solution to the problem (1.2)–(1.5) in the sense of Definition 1.1. The proof is complete. □

Now it is easy to show that, in contrast to the BVP (1.3)–(1.4), the corresponding optimal control problem (1.2)–(1.5) is well-posed and consistent.

Theorem 2.2. *Let $p \geq 2$, $r \geq p'$, and $q \geq p'$ be given exponents. Assume that for a given distribution $g \in L^q(\Omega)$ Hypothesis A is fulfilled. Then, for any $y_d \in L^2(\Omega)$, optimal control problem* (1.2)–(1.5) *has at least one solution.*

Proof. Since $J(u, y) \geq 0$ for all $(u, y) \in \Xi$, it follows that there exists a non-negative value $\mu \geq 0$ such that $\mu = \inf_{(u,y)\in\Xi} J(u, y)$. Let $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ be a minimizing sequence to the problem (1.2)–(1.5), i.e.

$$(u_k, y_k) \in \Xi \quad \forall\, k \in \mathbb{N} \quad \text{and} \quad \lim_{k\to\infty} J(u_k, y_k) = \mu.$$

So, we can suppose that

$$J(u_k, y_k) \le \mu + 1 \quad \text{for all } k \in \mathbb{N}. \tag{2.36}$$

Then taking into account the implicit form of the cost functional (1.2), Theorem 2.1, and the fact that $q \ge p' > \frac{pN}{pN-N+p}$, we deduce the following estimates

$$\begin{aligned}
\sup_{k\in\mathbb{N}} \|y_k\|^p_{W_0^{1,p}(\Omega;\Gamma_D)} &\le 3^{p'-1}\left[|\Omega|^{1-\frac{p'}{r}}\left(\frac{r}{\alpha}\right)^{\frac{p'}{r}} + C^{p'}_{\gamma_0}p'\right]\max\left\{1, \sup_{k\in\mathbb{N}} J(u_k,y_k)\right\}\\
&\quad + 3^{p'-1}C^{p'}_{em}\|g\|^{p'}_{L^q(\Omega)} \overset{\text{by (2.36)}}{\le} 3^{p'-1}C^{p'}_{em}\|g\|^{p'}_{L^q(\Omega)}\\
&\quad + 3^{p'-1}\left[|\Omega|^{1-\frac{p'}{r}}\left(\frac{r}{\alpha}\right)^{\frac{p'}{r}} + C^{p'}_{\gamma_0}p'\right](\mu+1),
\end{aligned} \tag{2.37}$$

$$\|u_k\|^{p'}_{L^{p'}(\Gamma_N)} \le p' \sup_{k\in\mathbb{N}} J(u_k,y_k) \le p'(\mu+1), \tag{2.38}$$

$$\|f(y_k)\|^r_{L^r(\Omega)} \le \frac{r}{\alpha}\sup_{k\in\mathbb{N}} J(u_k,y_k) \le \frac{r}{\alpha}(\mu+1). \tag{2.39}$$

Thus, without loss of generality, we can suppose that there exists a subsequence of the minimizing sequence $\{(u_k,y_k)\}_{k\in\mathbb{N}}$ (still denoted by the same index) and a pair $(u^0,y^0) \in L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega;\Gamma_D)$ such that

$$(u_k,y_k) \rightharpoonup (u^0,y^0) \quad \text{weakly in} \quad L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega;\Gamma_D) \quad \text{as } k \to \infty, \tag{2.40}$$

$$y_k(x) \rightharpoonup y^0(x) \quad \text{a.e. in} \quad \Omega. \tag{2.41}$$

Utilizing properties (2.36), (2.40), and (2.41), we deduce from Proposition 2.2 that $(u^0,y^0) \in \Xi$. To conclude the proof, it remains to take into account the lower semi-continuity of the cost functional $J : L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega;\Gamma_D) \to \mathbb{R}$ with respect to the weak convergence in $L^{p'}(\Gamma_N) \times W_0^{1,p}(\Omega;\Gamma_D)$ and property (2.30). This yields

$$\mu = \inf_{(u,y)\in\Xi} J(u,y) = \lim_{k\to\infty} J(u_k,y_k) \ge J(u^0,y^0).$$

Thus, $(u^0,y^0) \in \Xi$ is an optimal pair to the problem (1.2)–(1.5). □

3. On bounded feasible solutions

Before proceeding with the proof of the main result of this paper, we begin with some preliminaries.

Lemma 3.1. *Let $1 \le p < N$ and let $s^* = \frac{(N-1)p}{N-p}$. Then the following norms*

$$\|y\|_{W_0^{1,p}(\Omega;\Gamma_D)} := \left(\int_\Omega |\nabla y|^p\,dx\right)^{1/p},$$

$$\|y\|_* := \left(\int_\Omega |\nabla y|^p\,dx\right)^{1/p} + \left(\int_{\Gamma_N} |\gamma_0(y)|^{s^*}\,d\mathcal{H}^{N-1}\right)^{1/s^*}$$

are equivalent for $W_0^{1,p}(\Omega;\Gamma_D)$.

Proof. Since the inequality $\|y\|_{W_0^{1,p}(\Omega;\Gamma_D)} \le \|y\|_*$ is obvious, we focus on the reverse one. With that in mind we remind that by continuity of the trace operator $\gamma_0 : W^{1,p}(\Omega;\Gamma_D) \to W^{1/p',p}(\Gamma_N)$, we have

$$\|\gamma_0(y)\|_{W^{1/p',p}(\Gamma_N)} \le C_{\gamma_0}\|y\|_{W^{1,p}(\Omega;\Gamma_D)}, \quad \forall\, y \in W^{1,p}(\Omega;\Gamma_D).$$

Since, for $p < N$, the Sobolev space $W^{1/p',p}(\Gamma_N)$ is continuously embedded in $L^s(\Gamma_N)$ for all $s \in [1, s^*]$, it follows existence of a constant $C_s > 0$ such that

$$\|\gamma_0(y)\|_{L^{s^*}(\Gamma_N)} \le C_s\|\gamma_0(y)\|_{W^{1/p',p}(\Gamma_N)} \le C_s C_{\gamma_0}\|y\|_{W^{1,p}(\Omega;\Gamma_D)}, \tag{3.1}$$

for all $y \in W^{1,p}(\Omega;\Gamma_D)$. Hence,

$$\frac{1}{1 + C_s C_{\gamma_0}}\Big(\|\gamma_0(y)\|_{L^{s^*}(\Gamma_N)} + \|y\|_{W_0^{1,p}(\Omega;\Gamma_D)}\Big) \le \|y\|_{W_0^{1,p}(\Omega;\Gamma_D)}.$$

Thus, the indicated norms are equivalent on $W_0^{1,p}(\Omega;\Gamma_D)$. For our further analysis, we make use of another representation for the last estimate. As immediately follows from (3.1), we have

$$\int_\Omega |\nabla y|^p\, dx \ge \frac{1}{2}\left[\frac{1}{C_s^p C_{\gamma_0}^p}\|\gamma_0(y)\|^p_{L^{s^*}(\Gamma_N)} + \int_\Omega |\nabla y|^p\, dx\right]. \tag{3.2}$$

□

The next result reflexes some special properties of composition of $W_0^{1,p}(\Omega;\Gamma_D)$-functions with regular functions and is a direct consequence of the well-know Stampacchia Lemma.

Lemma 3.2 ([20]). *Let* $G : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz continuous function such that* $G(0) = 0$. *Then for every function* $y \in W_0^{1,p}(\Omega;\Gamma_D)$ *we have:*

(i) $G(y) \in W_0^{1,p}(\Omega;\Gamma_D)$*;*

(ii) $\nabla G(y) = G'(y)\nabla y$ *almost everywhere in* Ω.

We note that at the first glance the equality in (ii) is not valid because a Lipschitz continuous function $G : \mathbb{R} \to \mathbb{R}$ is only almost everywhere differentiable, so that the right-hand side in (ii) may not be defined. On the other hand, we have two possible cases: if $k \in \mathbb{R}$ is a value such that $G'(k)$ does not exist, then either the set $\{x \in \Omega \;:\; y(x) = k\}$ has zero measure or the set $\{x \in \Omega \;:\; y(x) = k\}$ has positive measure. In the first case, since the identity $\nabla G(y) = G'(y)\nabla y$ only holds almost everywhere, this value does not give any problems. In this latter case, however, we have both $\nabla y = 0$ and $\nabla G(y) = 0$ almost everywhere, so that the identity $\nabla G(y) = G'(y)\nabla y$ still holds.

In what follows, we will use the composition of functions of Sobolev space $W_0^{1,p}(\Omega;\Gamma_D)$ with the following Lipschitz continuous function

$$G_k(z) = z - T_{k^{-1}}(z) = (|z| - k|)_+ \operatorname{sign}(z), \tag{3.3}$$

where $k > 0$ is a given value. Here, $T_{k^{-1}}(z)$ stands for the truncation operator (see (2.20)). Then Lemma 3.2 implies the following equality for $W_0^{1,p}(\Omega;\Gamma_D)$-functions

$$\nabla G_k(y) = \nabla y \ \chi_{\{x\in\Omega \,:\, |y(x)|\geq k\}} \quad \text{almost everywhere in } \Omega, \tag{3.4}$$

where χ_A denotes the characteristic function of the set A (for the details we refer to L. Orsina [23]).

The first result concerning the boundedness of the weak solutions of Dirichlet boundary value problem for elliptic equations comes from Stampacchia classical work [25].

Theorem 3.1. *Let $y \in W_0^{1,p}(\Omega)$ be the weak solution of the following BVP*

$$-\operatorname{div}\left(|\nabla y|^{p-2}\nabla y\right) = g \quad in \quad \Omega,$$
$$y = 0 \quad on \quad \partial\Omega,$$

where $g \in W^{-1,q}(\Omega)$ and $q > \frac{N}{p-1}$. Then $y \in L^\infty(\Omega)$.

The proof of this result essentially based on the following technical lemma.

Lemma 3.3 ([25])**.** *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that*

$$\psi(h) \leq \frac{M\psi^\delta(k)}{(h-k)^\gamma}, \quad \forall\, h > k > 0, \tag{3.5}$$

where $M > 0$, $\delta > 1$, and $\gamma > 0$. Then $\psi(d) = 0$, where

$$d^\gamma = M\psi^{\delta-1}(0)2^{\frac{\delta\gamma}{\delta-1}}.$$

For the reader's convenience, we cite the proof of this lemma.

Proof. We define the numerical sequence $\{d_k\}_{k\in\mathbb{N}}$ as follows $d_k = d(1-2^{-k})$ for each $k \in \mathbb{N}$. Let us show that

$$\psi(d_k) \leq \psi(0)2^{-\frac{k\gamma}{\delta-1}}, \tag{3.6}$$

where ψ possesses the property (3.5). Indeed, inequality (3.6) is clearly true if $k = 0$. If we suppose, by the induction, that it is true for some k, then (3.5) implies

$$\psi(d_{k+1}) \leq \frac{M\psi^\delta(d_k)}{(d_{k+1}-d_k)^\gamma} \leq M\psi^\delta(0)2^{-\frac{k\gamma\delta}{\delta-1}}2^{(k+1)\gamma}d^{-\gamma} = \psi(0)2^{-\frac{(k+1)\gamma}{\delta-1}}.$$

Since (3.6) holds for every k, and since ψ is a non-increasing function, it follows that

$$0 \leq \psi(d) \leq \liminf_{k\to\infty} \psi(d_k) \leq \lim_{k\to\infty} \psi(0)2^{-\frac{k\gamma}{\delta-1}} = 0.$$

The proof is complete. □

We are now in a position to prove the main result of our paper that has been announced in Theorem 1.1.

Proof. Let $k > 0$ and let $(u, y) \in \Xi$ be a feasible solution to the original optimal control problem. We define the set Ω_k as the bigest closed subset of Ω such that

$$\Omega_k \subseteq \{x \in \Omega \ : \ |\nabla y| \le k\}.$$

Hereinafter, we suppose that the parameter k varies within a strictly increasing sequence of positive real numbers tending to ∞ and such that

$$A_k := \Omega \setminus \Omega_k \tag{3.7}$$

is an open set with Lipschitz boundary for each k and $\{A_k\}_{k>0}$ form a strictly monotone by inclusion (i.e. $A_h \subset A_k$ for $h > k$) sequence such that $\lim_{k\to\infty} |A_k| = 0$. We also set

$$\Gamma_{N,k} := \{\sigma \in \Gamma_N \ : \ |\gamma_0(y)(\sigma)| \ge k\}. \tag{3.8}$$

By definition of the trace operator $\gamma_0 : W^{1,p}(\Omega; \Gamma_D) \to W^{1/p',p}(\Gamma_N)$, we can suppose that $\Gamma_{N,k} \subset \partial A_k$ for each $k \in \mathbb{N}$ within a subset of $\Gamma_{N,k}$ with zero Hausdorff surface $(N-1)$-dimensional measure.

Since the integral identity (2.3) is valid for each function $\varphi \in W^{1,p}(\Omega; \Gamma_D)$, we chose $\varphi = G_k(y)$ as the test function in (2.3). Here, $G_k(z)$ is defined in (3.3). Then $G_k(y) = G_k(y)\chi_{A_k}$ a.e. in Ω, and, by Lemma 3.2, $\nabla G_k(y) = \nabla y \chi_{A_k}$ for almost all $x \in \Omega$. Moreover, the inclusion $\Gamma_{N,k} \subset \partial A_k$ implies the following relations

$$\gamma_0(G_k(y)) = G_k(\gamma_0(y)) \ \text{ and } \ G_k(\gamma_0(y)) = G_k(\gamma_0(y))\chi_{\Gamma_{N,k}} \text{ a.e. on } \Gamma_N.$$

Using the fact that $g \in L^q(\Omega)$ and $q > p'$ (see (1.8)), we deduce from (2.3) that

$$\langle g, G_k(y)\rangle_{W^{-1,p'}(\Omega;\Gamma_D);W^{1,p}(\Omega;\Gamma_D)} = \int_\Omega g G_k(y)\, dx$$

and, therefore,

$$\begin{aligned} \int_{A_k} |\nabla G_k(y)|^p\, dx &= \int_\Omega |\nabla y|^{p-2}(\nabla y, \nabla y)\chi_{A_k}\, dx = \int_\Omega f(y) G_k(y)\, dx \\ &+ \int_{\Gamma_N} \gamma_0(G_k(y)) u \, d\mathcal{H}^{N-1} + \langle g, G_k(y)\rangle_{W^{-1,p'}(\Omega;\Gamma_D);W^{1,p}(\Omega;\Gamma_D)} \\ &= \int_{A_k} f(y) G_k(y)\, dx + \int_{\Gamma_{N,k}} \gamma_0(G_k(y)) u \, d\mathcal{H}^{N-1} + \int_{A_k} g G_k(y)\, dx \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.9}$$

In order to estimate the terms I_i, we make use of the Hölder inequality and the following facts: $W_0^{1,p}(\Omega; \Gamma_D) \hookrightarrow L^{p^*}(\Omega)$ and $W^{1/p',p}(\Gamma_N) \hookrightarrow L^{s^*}(\Gamma_N)$ with continuous embedding for $p^* = \frac{Np}{N-p}$ and $s^* = \frac{(N-1)p}{N-p}$, respectively. As a result, we have

$$I_1 \le \left(\int_{A_k} |f(y)|^{p_*}\,dx\right)^{\frac{1}{p_*}} \left(\int_{A_k} |G_k(y)|^{p^*}\,dx\right)^{\frac{1}{p^*}} \tag{3.10}$$

$$I_2 \le \left(\int_{\Gamma_{N,k}} |u|^{s_*}\,dx\right)^{\frac{1}{s_*}} \left(\int_{\Gamma_{N,k}} |\gamma_0(G_k(y))|^{s^*}\,dx\right)^{\frac{1}{s^*}}, \tag{3.11}$$

$$I_3 \le \left(\int_{A_k} |g|^{p_*}\,dx\right)^{\frac{1}{p_*}} \left(\int_{A_k} |G_k(y)|^{p^*}\,dx\right)^{\frac{1}{p^*}}, \tag{3.12}$$

where $s_* = (s^*)' = \frac{p}{p-1}\frac{N-1}{N}$ and p_* is defined by (2.4).

To estimate the left-hand side of (3.9), we make use of the well-known Sobolev inequality. Namely, in view of the Sobolev embedding theorem there exists a constant S_p (depending only on N and p) such that

$$\|G_k(y)\|_{L^{p^*}(A_k)} \le S_p \left(\int_{A_k} |\nabla G_k(y)|^p\,dx\right)^{\frac{1}{p}} \quad \text{provided } 1 \le p < N. \tag{3.13}$$

Then utilizing (3.13), Lemma 3.1 (see (3.2)), and our assumptions with respect to the set A_k and its boundary, we obtain

$$\begin{aligned}\int_{A_k} |\nabla G_k(y)|^p\,dx &\ge \frac{1}{2}\left[\frac{1}{C_s^p C_{\gamma_0}^p}\|\gamma_0(G_k(y))\|^p_{L^{s^*}(\Gamma_{N,k})} + \int_{A_k} |\nabla G_k(y)|^p\,dx\right]\\ &\ge \frac{1}{2}\left[\frac{1}{C_s^p C_{\gamma_0}^p}\|\gamma_0(G_k(y))\|^p_{L^{s^*}(\Gamma_{N,k})} + \frac{1}{S_p^p}\|G_k(y)\|^p_{L^{p^*}(A_k)}\right]\\ &\ge \frac{1}{2^p}\min\left\{\frac{1}{C_s^p C_{\gamma_0}^p}, \frac{1}{S_p^p}\right\}\left[\|\gamma_0(G_k(y))\|_{L^{s^*}(\Gamma_{N,k})} + \|G_k(y)\|_{L^{p^*}(A_k)}\right]^p\\ &= \widehat{C}\left[\|\gamma_0(G_k(y))\|_{L^{s^*}(\Gamma_{N,k})} + \|G_k(y)\|_{L^{p^*}(A_k)}\right]^p. \end{aligned} \tag{3.14}$$

Combining this issue with estimates (3.10)–(3.12), we see from (3.9) that

$$\begin{aligned}\widehat{C}&\left[\|\gamma_0(G_k(y))\|_{L^{s^*}(\Gamma_{N,k})} + \|G_k(y)\|_{L^{p^*}(A_k)}\right]^{p-1}\\ &\le \|f(y)\|_{L^{p_*}(A_k)} + \|g\|_{L^{p_*}(A_k)} + \|u\|_{L^{s_*}(\Gamma_{N,k})}.\end{aligned} \tag{3.15}$$

We now take $h > k$ so that

$$A_h \subseteq A_k \quad \text{and} \quad G_k(y) \ge h-k \quad \text{on} \quad A_h,$$
$$\Gamma_{N,h} \subseteq \Gamma_{N,k} \quad \text{and} \quad \gamma_0(G_k(y)) \ge h-k \quad \text{on} \quad \Gamma_{N,h}.$$

Then we have

$$\begin{aligned}\|G_k(y)\|_{L^{p^*}(A_k)} = \left(\int_{A_k} |G_k(y)|^{p^*}\,dx\right)^{1/p^*} &\ge \left(\int_{A_h} |G_k(y)|^{p^*}\,dx\right)^{1/p^*}\\ &\ge (h-k)|A_h|^{1/p^*},\end{aligned} \tag{3.16}$$

$$\begin{aligned}\|\gamma_0(G_k(y))\|_{L^{s^*}(\Gamma_{N,k})} &= \left(\int_{\Gamma_{N,k}} |G_k(y)|^{s^*}\, d\mathcal{H}^{N-1}\right)^{1/s^*} \\ &\geq \left(\int_{\Gamma_{N,h}} |G_k(y)|^{s^*}\, d\mathcal{H}^{N-1}\right)^{1/s^*} \geq (h-k)|\Gamma_{N,h}|^{1/s^*}. \quad (3.17)\end{aligned}$$

Since

$$\frac{1}{s^*} = \frac{N-p}{(N-1)p} = \frac{N}{N-1}\frac{1}{p^*},$$

it follows that

$$\begin{aligned}&\widehat{C}\Big[\|\gamma_0(G_k(y))\|_{L^{s^*}(\Gamma_{N,k})} + \|G_k(y)\|_{L^{p^*}(A_k)}\Big]^{p-1} \\ &\overset{\text{by (3.16)–(3.17)}}{\geq} \widehat{C}(h-k)^{p-1}\left[|A_h|^{1/p^*} + |\Gamma_{N,h}|^{\frac{N}{N-1}\frac{1}{p^*}}\right]^{p-1} \\ &\geq \widehat{C}(h-k)^{p-1}\left[\psi(h)\right]^{\frac{p-1}{p^*}}, \quad (3.18)\end{aligned}$$

where

$$\psi(h) := |A_h| + |\Gamma_{N,h}|^{\frac{N}{N-1}}. \quad (3.19)$$

For our further analysis, we make use of the following observations. Since, by the initial assumptions, we have

$$p' \geq p_* = \frac{Np}{Np-N+p} \quad \text{and} \quad q, r \geq p', \quad (3.20)$$

it follows by the Hölder inequality that

$$\|g\|_{L^{p_*}(A_k)} = \left(\int_{A_k} |g|^{p_*}\, dx\right)^{1/p_*} \leq \|g\|_{L^q(\Omega)}|A_k|^{\frac{1}{p_*}\frac{q-p_*}{q}}, \quad (3.21)$$

$$\|f(y)\|_{L^{p_*}(A_k)} = \left(\int_{A_k} |f(y)|^{p_*}\, dx\right)^{1/p_*} \leq \|f(y)\|_{L^r(\Omega)}|A_k|^{\frac{1}{p_*}\frac{r-p_*}{r}}. \quad (3.22)$$

As for the term $\|u\|_{L^{s_*}(\Gamma_{N,k})}$ in (3.15), following the similar arguments and taking into account the inclusion $u \in L^t(\Gamma_N)$ for t satisfying condition (1.9), we get

$$\|u\|_{L^{s_*}(\Gamma_{N,k})} = \left(\int_{\Gamma_{N,k}} |u|^{s_*}\, dx\right)^{\frac{1}{s_*}} \leq |\Gamma_{N,k}|^{\frac{t-s_*}{ts_*}}\|u\|_{L^t(\Gamma_N)}. \quad (3.23)$$

Since $p^*/(p-1) > 1$, it follows from (3.15), (3.18), and (3.21)–(3.23) that

$$\begin{aligned}(h-k)^{p^*}\psi(h) &\leq \underbrace{\left[\widehat{C}^{-1}\left(\|f(y)\|_{L^r(\Omega)} + \|g\|_{L^q(\Omega)} + \|u\|_{L^t(\Gamma_N)}\right)\right]^{\frac{p^*}{p-1}}}_{D} \\ &\times 3^{\frac{p^*-p+1}{p-1}}\left[|A_k|^{\frac{1}{p_*}\left(1-\frac{p_*}{r}\right)\frac{p^*}{p-1}} + |A_k|^{\frac{1}{p_*}\left(1-\frac{p_*}{q}\right)\frac{p^*}{p-1}} + |\Gamma_{N,k}|^{\frac{1}{s_*}\left(1-\frac{s_*}{t}\right)\frac{p^*}{p-1}}\right]. \quad (3.24)\end{aligned}$$

We also see that

$$p^* \left(r - p_*\right) p_* r(p-1) = \frac{Np\left[r(Np - N + p - Np + p^2 + N - p) - Np\right]}{(N-p)(Np-N+p)} = \frac{Np}{(N-p)(Np-N+p)}\left[p^2 r - Np\right] \overset{\text{by }(1.8)}{>} 0.$$

Hence,

$$\delta_1 := \frac{1}{p_*}\left(1 - \frac{p_*}{r}\right)\frac{p^*}{p-1} > 1. \tag{3.25}$$

By analogy it can be shown that

$$\delta_2 := \frac{1}{p_*}\left(1 - \frac{p_*}{q}\right)\frac{p^*}{p-1} > 1 \ \text{ provided inequality } (1.8)_1 \text{ holds true.} \tag{3.26}$$

As for the third exponent in (3.24), we see that

$$\frac{1}{s_*}\left(1 - \frac{s_*}{t}\right)\frac{p^*}{p-1} = \frac{N}{N-1}\delta_3,$$

where

$$\delta_3 = \frac{N-1}{N}\frac{1}{s_*}\left(1 - \frac{s_*}{t}\right)\frac{p^*}{p-1} = \frac{\left[(N-1)p - N + p\right]t - (N-1)p}{(N-p)(p-1)t} > 1 \tag{3.27}$$

provided the parameter t satisfies inequality (1.9).

Since $|\Gamma_{N,k}| < 1$ and $|A_k| < 1$ for k large enough, it follows from (3.24) that

$$\psi(h) \le 3^{\frac{p^*-p+1}{p-1}} \frac{D}{(h-k)^{p^*}}\left[2\left(|A_k| + |\Gamma_{N,k}|^{\frac{N}{N-1}}\right)\right]^{\min\{\delta_1;\delta_2;\delta_3\}} = \frac{M\psi^{\delta}(k)}{(h-k)^{p^*}}, \tag{3.28}$$

where

$$\delta = \min\{\delta_1;\delta_2;\delta_3\} \overset{\text{by }(3.25)\text{–}(3.27)}{>} 1,$$

$$M = 3^{\frac{p^*-p+1}{p-1}} 2^{\delta}\left[\widehat{C}^{-1}\left(\|f(y)\|_{L^r(\Omega)} + \|g\|_{L^q(\Omega)} + \|u\|_{L^t(\Gamma_N)}\right)\right]^{\frac{p^*}{p-1}}.$$

Therefore, by Lemma 3.3 we finally deduce that

$$\psi(d) := |A_d| + |\Gamma_{N,d}|^{\frac{N}{N-1}} = 0$$

for

$$d = M\left[|\Omega| + |\Gamma_N|^{\frac{N}{N-1}}\right]^{\delta-1} 2^{\frac{\delta p^*}{\delta-1}}.$$

Thus, for the given feasible pair $(u, y) \in \Xi$, the following inference is valid: conditions (1.8)–(1.9) imply that $y \in L^\infty(\Omega)$ and $\gamma_0(y) \in L^\infty(\partial\Omega)$. The proof of Theorem 1.1 is complete. □

As for the proof of Theorem 1.1, its validity immediately follows from Theorem 2.2 and Sobolev embedding theorem saying that the injection $W_0^{1,p}(\Omega;\Gamma_D) \hookrightarrow C(\overline{\Omega})$ is compact if $p > N$.

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A CONTRIBUTION TO MAGNETIC RECONNECTION: A BOLTZMANN CORRECTION TO THE MAGNETIC INDUCTION EQUATION FOR FARADAY VORTEX TUBES

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Abstract. The Boltzmann correction to the Maxwell induction law for a moving medium filled with vortex tubes of Faraday has been implemented.

Key words: the magnetic induction law, the theory of molecular vortices.

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...It is true that at one time those who speculated as to the causes of physical phenomena, were in the habit of accounting for each kind of action at a distance by means of a special æthereal fluid, whose function and property it was to produce these actions. They filled all space three and four times over with æthers of different kinds, the properties of which were invented merely to 'save appearances,' so that more rational enquirers were willing rather to accept not only Newton's definite law of attraction at a distance, but even the dogma of Cotes, that action at a distance is one of the primary properties of matter, and that no explanation can be more intelligible than this fact...

<...>

But in all of these theories the question naturally occurs: — If something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other?...Hence all these theories lead to the conception of a medium in which the propagation takes place, and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavour to construct a mental representation of all the details of its action, and this has been my constant aim in this treatise. [36]

1. Introduction

Maxwell's equations are foundational to electromagnetic theory. They are the cornerstone of a myriad of technologies and are basic to the understanding of innumerable effects. Yet there are a few effects or phenomena that cannot be explained by the conventional Maxwell theory. [2]

The governing equations of ideal magnetohydrodynamics (IMHD) are resulted from coupling the Maxwell equations for ideal conductive medium and the Euler equations for ideal fluid. A constitutive part of the IMHD governing equations is the (magnetic) induction law

$$\boldsymbol{B}_t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{1.1}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the fluid velocity, $\boldsymbol{B}(\boldsymbol{x}, t)$ is the magnetic induction, $(\boldsymbol{x}, t)$ is an inertial Cartesian orthogonal frame of reference, and the lower index t indicates the partial derivative with respect to t.

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The Alfvén theorem [14, 16, 21, 37] of IMHD implies that the following two important properties of the magnetic induction $\boldsymbol{B}$ hold. The *first* property is the conservation of the magnetic flux

$$\Psi\left[\mathcal{L}(t)\right] = \iint_{\mathcal{S}(t)} \boldsymbol{\nu} \cdot \boldsymbol{B}\, \delta\mathcal{S}\,, \tag{1.2}$$

where $\mathcal{L}(t)$ is an arbitrary closed material (co-moving) contour, $\mathcal{S}(t)$ is a surface bounded by the contour $\mathcal{L}(t)$, $\boldsymbol{\nu}(\boldsymbol{x},t)$ is the unit vector normal to $\mathcal{S}(t)$, δ is the (purely) 'spatial' differential (at time t being constant). The property is proved directly

$$\dot{\Psi}\left[\mathcal{L}(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathcal{S}(t)} \boldsymbol{\nu} \cdot \boldsymbol{B}\, \delta\mathcal{S} = \iint_{\mathcal{S}(t)} \boldsymbol{\nu} \cdot \left[\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B})\right] \delta\mathcal{S} \overset{(1.1)}{=} 0\,, \tag{1.3}$$

where dot over a symbol here and below indicates the material ('total') derivative with respect to t.

An other proof follows from the first Zorawski criterion [53]. It says that the necessary and sufficient condition for the flux of an arbitrary vector field $\boldsymbol{a}(\boldsymbol{x},t)$ through the material surface $\mathcal{S}(t)$ to conserve reads

$$\dot{\boldsymbol{a}} - \boldsymbol{a} \cdot \nabla \boldsymbol{u} + (\nabla \cdot \boldsymbol{u})\, \boldsymbol{a} = \boldsymbol{0}\,, \tag{1.4}$$

or equivalently

$$\boldsymbol{a}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \boldsymbol{u} + (\nabla \cdot \boldsymbol{u})\, \boldsymbol{a} = \boldsymbol{0}\,. \tag{1.5}$$

Using the following well known vector identity

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \boldsymbol{a} - (\nabla \cdot \boldsymbol{a})\, \boldsymbol{b} - \boldsymbol{a} \cdot \nabla \boldsymbol{b} + (\nabla \cdot \boldsymbol{b})\, \boldsymbol{a}\,, \tag{1.6}$$

where $\boldsymbol{b}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t)$, we obtain the condition (1.5) to be

$$\boldsymbol{a}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{a}) + (\nabla \cdot \boldsymbol{a})\, \boldsymbol{u} = \boldsymbol{0}\,. \tag{1.7}$$

When $\boldsymbol{a} = \boldsymbol{B}$, $\nabla \cdot \boldsymbol{B} = 0$ (see the fourth equation of the system (2.1) or the second equation of the system (2.3)), the condition (1.7) transforms into the induction law (1.1).

The *second* property is the magnetic field line conservation, that is the magnetic lines co-move with the fluid, or they are 'frozen' into the fluid. The second Zorawski criterion [53] says that the necessary and sufficient condition for the vector field $\boldsymbol{a}(\boldsymbol{x},t)$ to be material reads

$$\boldsymbol{a} \times \left[\boldsymbol{a}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{a}) + (\nabla \cdot \boldsymbol{a})\, \boldsymbol{u}\right] = \boldsymbol{0}\,. \tag{1.8}$$

Again, if $\boldsymbol{a} = \boldsymbol{B}$, the above condition holds due to $\nabla \cdot \boldsymbol{B} = 0$ and the induction law (1.1).

The both Zorawski criteria are thoroughly discussed in [42, 48].

The second property of the magnetic induction is usually used to introduce the magnetic field line velocity $\boldsymbol{w}(\boldsymbol{x}, t)$ and to consider the induction equation (1.1) in the following formulation

$$\boldsymbol{B}_t = \nabla \times (\boldsymbol{w} \times \boldsymbol{B}) \ . \tag{1.9}$$

The component $\boldsymbol{w}_{\|}$ of $\boldsymbol{w}$ in the direction of $\boldsymbol{B}$ is actually not determined because a one-to-one correspon dence between field lines does not require one-to-one correspondence between the individual points lying on them [41], whereas the component of $\boldsymbol{w}$ in the direction normal to $\boldsymbol{B}$ is $\boldsymbol{w}_{\perp} = \boldsymbol{u}_{\perp}$. However, the usual convention is to assume that $\boldsymbol{w}_{\|} = \boldsymbol{u}_{\|}$, that is $\boldsymbol{w} = \boldsymbol{u}$.

The above two properties of the magnetic induction $\boldsymbol{B}$ are exactly those known in ideal hydrodynamics (IHD) for the vorticity

$$\boldsymbol{\Omega} = \nabla \times \boldsymbol{u} \tag{1.10}$$

and derived from the Kelvin theorem [25] or the Helmholtz equation [19, 25]

$$\boldsymbol{\Omega}_t = \nabla \times (\boldsymbol{u} \times \boldsymbol{\Omega})\,. \tag{1.11}$$

In IHD phenomena when all the hypotheses of the Kelvin theorem meet then the property of the conservation of the vorticity lines holds. In contrast to IHD, in IMHD the magnetic topology may change even when all the hypotheses of the Alfvén theorem meet. A well known example of such a change is magnetic reconnection. The phenomenon occurs in the solar corona, the Earth's magnetosphere, and laboratory plasmas. Detailed surveys on the subject are presented in [17, 22, 41, 43, 51, 52]. We note, just in case, that Barrett [2] set up a list of electromagnetic phenomena not explained by the Maxwell equations.

Since in most theories of magnetic reconnection the induction law (whether in ideal or non-ideal cases) plays an important role, our concern is the origin of the induction law in the IMHD limit, rather then magnetic reconnection itself.

The article is arranged as follows.

In section 2 we consider the Minkowski approach currently adopted as a 'standard' in most of the existing textbooks on MHD for deriving the Maxwell equations in moving media.

In section 3 we consider the Maxwell approach based on the theory of molecular vortices and some mechanical analogies to derive the induction law. One should refer to [46] to learn more about the theory of molecular vortices and to [10, 26, 39] to know out much interesting on the Maxwell way of reasoning. We show that in contrast to the well known common opinion Maxwell himself derived the induction law not only for media at rest but for moving ones as well. In the IMHD limit his induction law is nothing but the induction law (1.1) of the IMHD. The history of electrodynamics of moving media is fundamentally surveyed by Darrigol [11–13].

In section 4 we consider some observations of Boltzmann concerning the Maxwell study on the subject. Boltzmann thoroughly studied the Maxwell legacy on electromagnetism, namely three articles [29], [30–33], [34], and the two-volume book [35, 36]. He translated the first and the second articles in German [4, 5] and supplemented both translations with his own very detailed and insight comments. He also published in English [6] the list of faults found by him in the first article. We implement some comments of Boltzmann to the second article to derive the corrected induction law.

In section 5 we consider the induction law corrected by Hornig [20] to preserve the magnetic line topology and not to preserve the magnetic flux. The induction law after Hornig happens to include the induction law after Boltzmann as a particular case provided some conditions meet.

In section 6 we consider the analogy between the Kelvin and the Alfvèn theorems and their consequences once again. Some of quite recently published results of other authors, for example, by Tsinober [50], prove that the analogy is imperfect or even does not hold. We show that the induction law after Boltzmann does not actually obey the analogy.

In section 7 we list in brief our observations on the subject.

2. The induction law after Minkowski

> Because of our incomplete knowledge of the structure of matter, however, we are entitled to ask ourselves what statements the relativity principle allows us to make concerning (macroscopic) processes in moving bodies, assuming processes in bodies at rest to be experimentally known. This question was answered by Minkowski... He showed that the equations for moving bodies follow unambiguously from the relativity principle and from Maxwell's equations for bodies at rest... [40]

> We shall not use these formulae in the rather complicated form in which they can be found in Maxwell's treatise, but in the clearer and more condensed form that has been given them by Heaviside and Hertz. [28]

> The "Maxwell's equations" of today are due to Heaviside's "redressing" of Maxwell's work, and should, more accurately, be known as the "Maxwell – Heaviside equations." Essentially, Heaviside took the twenty equations of Maxwell and reduced them to the four now known as "Maxwell's equations." [2]

The governing equations of electromagnetism being actually the Hertz – Heaviside ones but usually attributed to Maxwell in the proper inertial Cartesian orthogonal frame of reference $(\boldsymbol{x}', t')$, where an undeformable conductive medium is at rest, read [14, 16, 21, 40]

$$\begin{cases} \nabla' \times \boldsymbol{H}' = +\boldsymbol{D}'_{t'} + \boldsymbol{j}' , \\ \nabla' \times \boldsymbol{E}' = -\boldsymbol{B}'_{t'} , \\ \nabla' \cdot \boldsymbol{D}' = q , \\ \nabla' \cdot \boldsymbol{B}' = 0 , \end{cases} \tag{2.1}$$

where the following constitutive equations: $\boldsymbol{B}' = \mu\,\boldsymbol{H}'$, $\boldsymbol{D}' = \varepsilon\,\boldsymbol{E}'$, and the Ohm law $\boldsymbol{j}' = \sigma\,\boldsymbol{E}'$ are used; $\boldsymbol{E}$ and $\boldsymbol{H}$ being the electric and magnetic fields, $\boldsymbol{D}$ being the electric induction (displacement), ε and μ being the electric and magnetic impermeabilities of the medium, q being the volume density of the free electric charges, $\boldsymbol{j}$ being the surface density of the electric current, σ being the conductivity of the medium.

For an undeformable moving medium in an inertial frame of reference $(\boldsymbol{x}, t)$ moving with the constant velocity $\boldsymbol{v}$: $|\boldsymbol{v}| \ll c$, where c is the speed of light, with respect to the frame of reference $(\boldsymbol{x}', t')$: $t' = t$, $\boldsymbol{x}' = \boldsymbol{x} - t\boldsymbol{v}$, the following non-relativistic transformations of the dependent variables [40]

$$\begin{cases} \boldsymbol{H}' = \boldsymbol{H} - \boldsymbol{v} \times \boldsymbol{D}\,, \\ \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\,, \\ \boldsymbol{j}' = \boldsymbol{j} \ - \rho\,\boldsymbol{v}\,, \\ \rho' = \rho \ + \varepsilon\,\nabla \cdot (\boldsymbol{v} \times \boldsymbol{B})\,, \end{cases}$$

are used for the system of equations (2.1) to hold.

In the IMHD limit the above transformations simplify to the following ones

$$\begin{cases} \boldsymbol{H}' = \boldsymbol{H}\,, \\ \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{0}\,, \end{cases} \tag{2.2}$$

and the remaining part of the system (2.1) reads

$$\begin{cases} \boldsymbol{B}_t - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0\,, \\ \nabla \cdot \boldsymbol{B} = 0\,. \end{cases} \tag{2.3}$$

Accounting for the vector identity (1.6), where $\boldsymbol{a} = \boldsymbol{v}$, $\boldsymbol{b} = \boldsymbol{B}$, the magnetic induction law (the first equation of the system (2.3)) simplifies as follows

$$\boldsymbol{B}_t - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{B}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{B} \equiv \dot{\boldsymbol{B}} = \boldsymbol{0}\,. \tag{2.4}$$

The above equation means that if an undeformable conductive medium moves with constant velocity $\boldsymbol{v}$ the magnetic field $\boldsymbol{B}$ remains unaltered.

In case of a deformable medium it is usually assumed that there is a unique continuously differentiable transformation between laboratory $(\boldsymbol{x}, t)$ (or Eulerean) and material $(\boldsymbol{X}, t)$ (or Lagrangean) frames of reference

$$\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{X}, t)\,, \tag{2.5}$$

referred to the law of motion. Actually, the transformation (2.5) is rarely known, and the solution to the following Cauchy problem

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{u}(\boldsymbol{x}, t)\,, \\ \boldsymbol{x}(0) = \boldsymbol{X}, \end{cases} \tag{2.6}$$

where $\boldsymbol{u}$ is the medium velocity, is implied by the law of motion.

Point-wise application of the transformations (2.2) at $\boldsymbol{v} = \boldsymbol{u}$ leads to the induction equation for the moving deformable medium as follows

$$\boldsymbol{B}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0}\,. \tag{2.7}$$

These formulae are rigorously valid only for uniformly moving bodies and, because of the additivity of the fields, also when several bodies are present which move uniformly with different velocities and are separated by vacuum regions. The approximation to which... are correct will generally be the better, the smaller the acceleration of the substance. [40]

Hence, obtaining the induction law after Minkowski implies supplementary assumptions not referred to by most of the textbooks. Sedov [44,45] studied applicability of these assumptions to moving media at large deformations.

3. The induction law after Maxwell

The consideration of the action of magnetism on polarized light leads, as we have seen, to the conclusion that in a medium under the action of magnetic force something belonging to the same mathematical class as an angular velocity, whose axis is in the direction of the magnetic force, forms a part of the phenomenon.

This angular velocity cannot be that of any portion of the medium of sensible dimensions rotating as a whole. We must therefore conceive the rotation to be that of very small portions of the medium, each rotating on its own axis. This is the hypothesis of molecular vortices. [36]

We shall suppose at present that all the vortices in any one part of the field are revolving in the same direction about axes nearly parallel, but that in passing from one part of the field to another, the direction of the axes, the velocity of rotation, and the density of the substance of the vortices are subject to change. [30]

Auch die Gleichungen, welche *Maxwell* hier für die electromagnetische Wirkung in bewegten medien aufstellt, hat *Hertz* anfangs übersehen. [5]

To derive the induction law (as a constitutive part of his set of the governing equations for the electromagnetic phenomena) Maxwell, *firstly*, introduced an inviscid continuum (or a medium, referred to the *microfluid* below) consisting of cylindrical vortices rotating as quasi-rigid bodies (prop. I [30], pp. 165 – 167), as shown in Fig. 1.

Secondly, Maxwell interpreted quantities used in electrodynamics as follows: μ (magnetic impermeability) being a value depending on the density of the microfluid and the position of the vortices (prop. I [30], pp. 165 – 167, prop. III [30], pp. 167 – 175), $\boldsymbol{E}$ (the electric field induced by free electric charges) being the force with what intermediate particles treated as free electric charges act on the vortices (prop. VII [31], pp. 288 – 289), and $\boldsymbol{H}$ (the magnetic field, the magnetic induction $\boldsymbol{B} = \mu\boldsymbol{H}$) being the following vector (prop. I [30], pp. 165 – 167, prop. III [30], pp. 167 – 175)

$$\boldsymbol{H} = w\,\boldsymbol{\tau} = r\omega\,\boldsymbol{\tau} = r\,\boldsymbol{\omega}\,, \tag{3.1}$$

where r, w, and $\boldsymbol{\omega}$ are denoted in Fig. 1, *a*.

In prop. VIII [31], pp. 289 – 291, Maxwell derived the induction law for the microfluid at rest (the second equation of the system (2.1)), i. e. for the case when

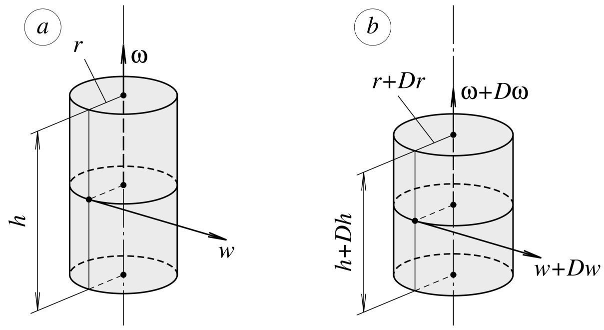

Fig. 1. Micromotion of the element of a vortex tube: before deformation at instant t (a) and after deformation at instant t' (b)

magnetic induction changes being influenced by the only field $\boldsymbol{E}$. In props. IX and X he considered the change of magnetic field being influenced by only small deformation, treating separately the strain and the rigid rotation between two instants t and $t' = t + \Delta t$, $\Delta t = \mathrm{d}t = \mathrm{D}t$.

In prop. IX [31], p. 340, Maxwell considered the infinitesimal 'parallelopiped' (not the infinitesimal cylinder, as the element of a vortex tube!) with its three edges being parallel to the axes x_1, x_2, x_3 of a fixed orthogonal frame of reference (or a Cartesian laboratory frame, see Fig. 2) and equaled to h_1, h_2, h_3. From the continuity property of the medium (this means that the volume of the parallelepiped remains unaltered) and the conservation of energy Maxwell concluded that due to the strain the following relations hold

$$\frac{\mathrm{D}_{str} H_\kappa}{H_\kappa} = \frac{\mathrm{D}_{str} h_\kappa}{h_\kappa} \equiv \lambda_\kappa \,, \qquad \kappa = 1, 2, 3, \tag{3.2}$$

where D stands for the 'deformational' differential (one should not confuse the differential D with the differential δ in the magnetic flux definition (1.2) and the magnetic flux conservation property (1.3), since the differential δ is used only for the spatial integration, as the increment for spatial variables at t being constant), H_κ are the Cartesian components of the magnetic field $\boldsymbol{H}$, and λ_κ are the extensions of the corresponding edges.

In the modern notation the above relations read

$$\mathrm{D}_{str} \boldsymbol{H} = \boldsymbol{H} \cdot \hat{\boldsymbol{S}} \, \Delta t \,, \tag{3.3}$$

where $\hat{\boldsymbol{S}} \, \Delta t$ is the symmetric tensor of small deformation, $\hat{\boldsymbol{S}}$ being the Euler stretching tensor [49].

In prop. X [31], pp. 340 – 341, Maxwell considered the rigid rotation of the 'parallelopiped' and derived the following equation (in the modern notation)

$$\mathrm{D}_{rot}\boldsymbol{H} = \boldsymbol{H}\cdot\hat{\boldsymbol{W}}\,\Delta t\,, \tag{3.4}$$

where $\hat{\boldsymbol{W}}\,\Delta t$ is the skew-symmetric tensor of small rigid rotation, $\hat{\boldsymbol{W}}$ being the Cauchy spin tensor [49].

It is evident that the equations (3.3) and (3.4) are valid for the magnetic induction $\boldsymbol{B}$ being rewritten as follows

$$\frac{\mathrm{D}_{str}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B}\cdot\hat{\boldsymbol{S}}, \qquad \frac{\mathrm{D}_{rot}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B}\cdot\hat{\boldsymbol{W}}, \qquad \frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B}\cdot\left(\hat{\boldsymbol{S}}+\hat{\boldsymbol{W}}\right).$$

In prop. XI [31], pp. 341 – 348, Maxwell collected all the results obtained in props. VIII, IX, and X for the rates of change of $\boldsymbol{B}$ and equated the substantial derivative of $\boldsymbol{B}$ to the sum of the rates of change of $\boldsymbol{B}$ due to: 1) the action of the electric field $\boldsymbol{E}$, given by the second equation of the system (2.1); 2) the strain, given by the first of the above equations; and 3) the rigid rotation, given by the second of the above equations, to obtain

$$\dot{\boldsymbol{B}} \equiv \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \equiv \boldsymbol{B}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} - \nabla\times\boldsymbol{E}\,, \tag{3.5}$$

where the unique decomposition $\nabla\boldsymbol{u} = \hat{\boldsymbol{S}}+\hat{\boldsymbol{W}}$ [49] is applied.

Then Maxwell used the microfluid incompressibility condition $\nabla\cdot\boldsymbol{u}=0$ once again, the condition of 'the absence of free magnetism' $\nabla\cdot\boldsymbol{B} = 0$ (Maxwell formulated this condition in terms of magnetic field, i. e. as $\nabla\cdot\boldsymbol{H}=0$, μ being constant), and the vector identity (1.6) to derive from the equation (3.5) the induction law for the moving microfluid

$$\boldsymbol{B}_t = \nabla\times(\boldsymbol{B}\times\boldsymbol{u}) - \nabla\times\boldsymbol{E}\,. \tag{3.6}$$

The above equation was not aimed to be principal or final in the Maxwell theory and happened to be hidden in his calculations. Actually, Maxwell tried to account for the notion of *electrotonic* state introduced by Faraday [15].

> The conception of such a quantity, on the changes of which, and not on its absolute magnitude, the induction current depends, occurred to Faraday at an early stage of his researches (*Exp. Res.*, series I, 60)... He therefore recognised... a 'peculiar electrical condition of matter,' to which he gave the name of the Electrotonic State. He afterwards found that he could dispense with this idea by means of considerations founded on the lines of magnetic force (*Exp. Res.*, series II, 242), but even in his latest researches (*Exp. Res.*, series II, 3269), he says, 'Again and again the idea of an *electrotonic* state (*Exp. Res.*, 60, 1114, 1661, 1729, 1733) has been forced upon my mind.' [36]

> Central to the Maxwell formulation of electromagnetism was the Faraday concept of the *electrotonic state* (from the new Latin *tonicus*, "of tension or tone"; from the Greek *tonos*, "a stretching"). [2]

Hence, following the idea of electrotonic state, Maxwell introduced the vector potential $\boldsymbol{A}$: $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, and derived from equation (3.6) the following one for the electric field

$$\boldsymbol{E} = \boldsymbol{A}_t + \boldsymbol{u}\times\boldsymbol{B} + \nabla\varphi\,,$$

where φ is a scalar potential.

Subsequently, the $\boldsymbol{A}$ field was *banished from playing the central role in Maxwell's theory and relegated to being a mathematical* (*but not physical*) *auxiliary.* This banishment took place during the interpretation of Maxwell's theory by the Maxwellians, i. e. chiefly by Heaviside, Fitzgerald, Lodge and Hertz. The "Maxwell theory" and "Maxwell's equations" we know today are really the interpretation of Maxwell by these Maxwellians. It was Heaviside who "murdered the $\boldsymbol{A}$ field" (Heaviside's description) and whose work influenced the crucial discussion which took place at the 1888 Bath meeting of the British Association (although Heaviside was not present). [2]

In the IMFD limit the induction law (3.6) of Maxwell reads

$$\dot{\boldsymbol{B}} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} \equiv \boldsymbol{B}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} \equiv \boldsymbol{B}_t + \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{0}\,. \qquad (3.7)$$

4. The induction law after Boltzmann

Boltzmann, being an inquisitive and shrewd researcher of the Maxwell legacy, noticed (comment 39 to prop. IX [5], p. 114) that Maxwell had derived the induction equation (3.6) not accounting for the definition of the magnetic field (3.1) of his own. And it was Boltzmann who supplemented both the conditions used by Maxwell, the incompressibility one and the conservation of energy for the medium, with the condition of preserving the cylindrical shape of the vortex tubes to obtain the following correct constraints for the deformation of any element of the vortex tubes

$$\frac{1}{2}\,\frac{\mathrm{D}\omega}{\omega} = \frac{\mathrm{D}w}{w} = -\frac{\mathrm{D}r}{r} = \frac{1}{2}\,\frac{\mathrm{D}h}{h}\,\cdot \qquad (4.1)$$

Boltzmann showed that the corresponding Maxwell constraints were as follows

$$\frac{\mathrm{D}\omega}{\omega} = \frac{\mathrm{D}w}{w} = -\frac{\mathrm{D}r}{r} = \frac{\mathrm{D}h}{h} \qquad (4.2)$$

and did not agree with preserving the cylindrical shape of the vortex tubes.

Unfortunately Boltzmann himself did not implement the constraints (4.1) and the Maxwell definition of the magnetic field (3.1) to derive the correct induction law. Hence, in what follows, we implement the Boltzmann correction.

For this we consider the material vector $\boldsymbol{h} = h\boldsymbol{\tau} = \boldsymbol{x}_{\mathrm{N}} - \boldsymbol{x}_{\mathrm{M}}$, determining the position of the vortex tube element MN at instant t (Fig. 2, a). At instant t' the material vector transforms into $\boldsymbol{h}' = h'\boldsymbol{\tau}' = \boldsymbol{x}_{\mathrm{N}'} - \boldsymbol{x}_{\mathrm{M}'}$ (Fig. 2, b), where $\boldsymbol{\tau}$ is the unit vector tangent to the axis of the element: $|\boldsymbol{\tau}| = |\boldsymbol{\tau}'| = 1$.

Using the law of motion of the medium (2.5) we represent the change of $\boldsymbol{h}$ through the material variables $\boldsymbol{X}$ and the time increment Δt as follows

$$\boldsymbol{h}' - \boldsymbol{h} = \Big[\boldsymbol{\Phi}(\boldsymbol{X}_{\mathrm{N}}, t') - \boldsymbol{\Phi}(\boldsymbol{X}_{\mathrm{M}}, t')\Big] - \Big[\boldsymbol{\Phi}(\boldsymbol{X}_{\mathrm{N}}, t) - \boldsymbol{\Phi}(\boldsymbol{X}_{\mathrm{M}}, t)\Big]\,,$$

and consequently find that [49]

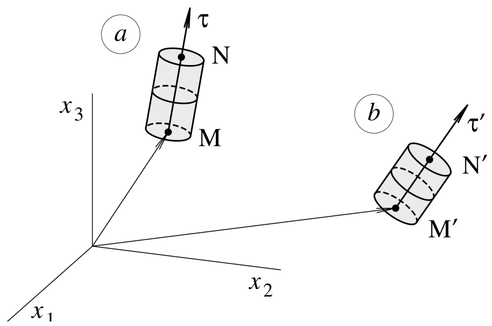

Fig. 2. Macromotion of the element MN of a vortex tube: before deformation at instant t (a) and after deformation at instant t' (b)

$$\frac{\mathrm{D}\boldsymbol{h}}{\mathrm{D}t} = \lim_{\Delta\to 0} \frac{\boldsymbol{h}' - \boldsymbol{h}}{\Delta t} = \boldsymbol{h}\cdot\nabla\boldsymbol{u}\,, \tag{4.3}$$

where the material variables $\boldsymbol{X}$ are assumed to coincide with the laboratory ones $\boldsymbol{x}$ at instant t (see the formulation of the Cauchy problem (2.6)).

From the above relation for the 'deformational' time derivative of the material vector $\boldsymbol{h}$ we find for the squared length of $\boldsymbol{h}$

$$\boldsymbol{h}\cdot\frac{\mathrm{D}\boldsymbol{h}}{\mathrm{D}t} = \frac{1}{2}\frac{\mathrm{D}\boldsymbol{h}^2}{\mathrm{D}t} = \frac{1}{2}\frac{\mathrm{D}h^2}{\mathrm{D}t} = h\,\frac{\mathrm{d}h}{\mathrm{d}t} = \boldsymbol{h}\cdot\left(\hat{\boldsymbol{S}} + \hat{\boldsymbol{W}}\right)\cdot\boldsymbol{h} = \boldsymbol{h}\cdot\hat{\boldsymbol{S}}\cdot\boldsymbol{h} = h^2\,\theta\,, \tag{4.4}$$

where the scalar function

$$\theta\left(\hat{\boldsymbol{S}},\boldsymbol{h}\right) = |\boldsymbol{h}|^{-2}\,\boldsymbol{h}\cdot\hat{\boldsymbol{S}}\cdot\boldsymbol{h} = \boldsymbol{\tau}\cdot\hat{\boldsymbol{S}}\cdot\boldsymbol{\tau} \tag{4.5}$$

is the normal component of $\hat{\boldsymbol{S}}$ in the direction of the axis of the element.

Then, differentiating the definition of the magnetic field (3.1), we obtain

$$\frac{\mathrm{D}\boldsymbol{H}}{\mathrm{D}t} = \frac{\mathrm{D}w}{\mathrm{D}t}\,\boldsymbol{\tau} + w\,\frac{\mathrm{D}\boldsymbol{\tau}}{\mathrm{D}t} \overset{(4.1)}{=} \frac{1}{2}\,\frac{w}{h}\,\frac{\mathrm{D}h}{\mathrm{D}t}\,\boldsymbol{\tau} + w\,\frac{\mathrm{D}\boldsymbol{\tau}}{\mathrm{D}t}\,, \tag{4.6}$$

where the logarithmic 'deformational' derivative of the length h of the material vector $\boldsymbol{h}$ is already known from the equation (4.4) to be

$$\frac{1}{h}\,\frac{\mathrm{D}h}{\mathrm{D}t} = \theta\,, \tag{4.7}$$

and the only derivative is needed to be find is the following one

$$\frac{\mathrm{D}\boldsymbol{\tau}}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{h}}{h}\right) = \frac{1}{h^2}\left(h\,\frac{\mathrm{D}\boldsymbol{h}}{\mathrm{D}t} - \frac{\mathrm{D}h}{\mathrm{D}t}\,\boldsymbol{h}\right) \overset{(4.3)}{=} \boldsymbol{\tau}\cdot\nabla\boldsymbol{u} - \theta\,\boldsymbol{\tau}\,. \tag{4.8}$$

Substituting the 'deformational' derivatives of $\ln h$ (4.7) and $\boldsymbol{\tau}$ (4.8) into the right hand side of the equation (4.6) we find for the 'deformational' derivative of the magnetic field

$$\frac{\mathrm{D}\boldsymbol{H}}{\mathrm{D}t} = \frac{1}{2}\, w\theta\, \boldsymbol{\tau} + w\, \boldsymbol{\tau}\cdot\nabla\boldsymbol{u} - w\theta\, \boldsymbol{\tau} = -\frac{1}{2}\,\theta\boldsymbol{H} + \boldsymbol{H}\cdot\nabla\boldsymbol{u}\,.$$

The same equation is evident to hold for the magnetic induction

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = -\frac{1}{2}\,\theta\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\boldsymbol{u}\,. \tag{4.9}$$

Combining the 'deformational' derivative of $\boldsymbol{B}$ (4.9) with the time derivative of $\boldsymbol{B}$ due to the action of the electric field (the second equation of the system (2.1)) we obtain the induction law after Boltzmann

$$\dot{\boldsymbol{B}} \equiv \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} - \frac{1}{2}\,\theta\boldsymbol{B} - \nabla\times\boldsymbol{E}\,,$$

and in the IMFD limit it reads

$$\dot{\boldsymbol{B}} \equiv \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} - \frac{1}{2}\,\theta\boldsymbol{B}\,.$$

Representing the total time derivative (material) at the left hand side of the above equation as the sum of the local and the convective derivatives we obtain the induction law in more usual formulation

$$\boldsymbol{B}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \tfrac{1}{2}\,\theta\boldsymbol{B} \equiv \boldsymbol{B}_t + \nabla\times(\boldsymbol{B}\times\boldsymbol{u}) + \tfrac{1}{2}\,\theta\boldsymbol{B} = \boldsymbol{0}\,. \tag{4.10}$$

5. The induction law after Hornig

> Besides, some models are based not on a solution of the corresponding MHD equations but on some geometrical consideration and on ideas about the motion of frozen-in magnetic field lines. This concept of magnetic field line motion has often led to some confusion; because of that, some models based on that concept were accurately criticized by Alfvén (1976, 1977). We also believe that physical models cannot be based on the qualitative and to some degree speculative ideas on magnetic field line motion (the more so because in some regions the frozen-in conditions are surely violated); physical models must be constructed on the basis of meaningful solutions of the problems of magnetic hydrodynamics (or even better, kinetics). [43]

It is known [17,41] that in non-ideal conductive media (plasmas) the magnetic flux conservation and the magnetic field line conservation properties are no longer equivalent, and the field line velocity $\boldsymbol{w}$ is not determined uniquely.

Hornig [20] considered this case in a purely geometric way and proved that the most general form of the induction equation preserving the magnetic field lines (magnetic topology) and not preserving the magnetic flux is as follows

$$\boldsymbol{B}_t + \nabla\times(\boldsymbol{B}\times\boldsymbol{w}) = \lambda\boldsymbol{B}\,, \tag{5.1}$$

$\boldsymbol{w}$ being the field line velocity, the component $\boldsymbol{w}_\perp$ of the field $\boldsymbol{w}$ not being uniquely determined, λ being a scalar function of the fields $\boldsymbol{w}$ and $\boldsymbol{B}$.

We note that the equation (5.1) at $\boldsymbol{w} = \boldsymbol{u}$ directly follows from the second Zorawski criterion (1.8) applied to the vector field $\boldsymbol{a}(\boldsymbol{x},t) = \boldsymbol{B}(\boldsymbol{x},t)$ when accounting for the second equation of the system (2.3).

Kozlov [24] used even more general form of the condition for the vector field $\boldsymbol{a}(\boldsymbol{x},t)$ to be material, $\boldsymbol{a}(\boldsymbol{x},t), \boldsymbol{u}(\boldsymbol{x},t)\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, as follows

$$\boldsymbol{a}_t + [\boldsymbol{a},\boldsymbol{u}] = \lambda \boldsymbol{a}\,, \tag{5.2}$$

where $[\boldsymbol{a},\boldsymbol{u}]$ is the commutator of the vector fields $\boldsymbol{a}(\boldsymbol{x},t)$ and $\boldsymbol{u}(\boldsymbol{x},t)$.

Preserving the magnetic topology means that the corresponding topological invariants of the filed lines, for example, knottedness, linkage etc., remain unaltered. Topological invariants of the field lines are explained in [1, 37].

6. Magnetohydrodynamic Analogy

The MHD analogy was originated by Batchelor [3] whose reasoning had been based on the well known fact that the equations for the vorticity in non-ideal fluids and for the magnetic induction (or for magnetic field) in non-ideal conductive media

$$\dot{\boldsymbol{\Omega}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\Omega}) + \nu\, \nabla^2 \boldsymbol{\Omega}\,,$$

$$\dot{\boldsymbol{B}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \epsilon\, \nabla^2 \boldsymbol{B}\,,$$

where ν and ϵ are the kinematic and the magnetic viscosities, are identical in form.

> There is thus a formal analogy between the two solenoidal vectors $\boldsymbol{\Omega}$ and $\boldsymbol{H}$, provided $\boldsymbol{\Omega}$ refers to the motion of non-conducting fluid and $\boldsymbol{H}$ to the motion of conducting liquid.
>
> Many of the results concerning vorticity in classical hydrodynamics can now be interpreted in terms of magnetic field in the electromagnetic hydrodynamic problem. [3]

The MHD analogy "is, in fact, an extension of the popular analogy between vorticity $\boldsymbol{\Omega}$ and material line elements $\boldsymbol{h}$ (proposed by Taylor 1938 [47], and which goes back to Helmholtz 1858 [19] and Kelvin 1880 [23]), equations for which in the absence of viscosity are identical in form as well" [19] (see the above equations (1.11), (4.3)):

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{\Omega}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{\Omega} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u}\,,$$

$$\dot{\boldsymbol{h}} = \boldsymbol{h}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{h} = \boldsymbol{h} \cdot \nabla \boldsymbol{u}\,.$$

We note that it was surely Maxwell who first proposed the IMHD analogy. In the footnote at the last page of [31] he remarked the following.

> Since the first part of this paper was written, I have seen in Crelle's *Journal* for 1859, a paper by Prof. Helmholtz on Fluid Motion, in which he has pointed out that the lines of fluid motion are arranged according to the same laws as the lines of magnetic force, the path of an electric current corresponding to a line of axes of those particles of the fluid which are in a state of rotation. This is an additional

instance of a *physical analogy*, the investigation of which may illustrate both electro-magnetism and hydrodynamics. [30]

Later on he referred to the IMHD analogy, but as an assumption.

It is impossible, in our present state of ignorance as to the nature of the vortices, to assign the form of the law which connects the displacement of the medium with the variation of the vortices. We shall therefore assume that the variation of the vortices caused by the displacement of the medium is subject to the same conditions which Helmholtz, in his great memoir on Vortex-motion [19], has shewn to regulate the variation of the vortices of a perfect liquid. [36]

Nowadays these analogies are utilized in most of textbooks on HD and MHD, for example, the analogy between $\boldsymbol{\Omega}$ and $\boldsymbol{h}$ is considered to be valid in [27], though "the above analogies have since been realized to be flawed" [50].

Indeed, at the kinematic level, $\boldsymbol{\Omega} = \nabla \times \boldsymbol{u}$, whereas $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, but the vector potential $\boldsymbol{A}$ is not present in the induction law for $\boldsymbol{B}$ (for both cases, the ideal and the non-ideal ones). At the dynamic level the differences between $\boldsymbol{\Omega}$ and $\boldsymbol{H}$ (or $\boldsymbol{B}$) are even more evident. One should address directly to the article of Tsinober [50] to find much more on the subject, including experimental evidence.

The current study explains the absence of the MHD analogy between $\boldsymbol{\Omega}$ and $\boldsymbol{H}$ (or $\boldsymbol{B}$) and some known flaws of the analogy, since the definition of the magnetic field $\boldsymbol{H}$ (3.1) given by Maxwell has nothing in common with the definition of the vorticity $\boldsymbol{\Omega}$ of a medium or the angular velocity $\boldsymbol{\omega}$ of quasi-rigid rotation of the vortex tubes of Faraday. And it is the Boltzmann correction to the magnetic induction law that explicitly accounts for the difference between $\boldsymbol{H}$ (or $\boldsymbol{B}$) and $\boldsymbol{\Omega}$.

7. Conclusions

1. The induction law after Minkowski is based on relativistic geometrical approach involving no physics of deformable media.
2. The induction law after Maxwell is fully based on the evident theory of molecular vortices but contradicts the definition of magnetic field by Maxwell.
3. The induction law after Boltzmann fixes faults of the Maxwell approach but implies the tubular foliation of the space filled with a deformable medium.
4. The induction law after Hornig involves an undetermined scalar function and looks as it were a more general case compared to the induction law after Boltzmann, nevertheless the former does not imply tubular foliation of the space.

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ON APPROXIMATION OF STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM IN COEFFICIENTS FOR p-BIHARMONIC EQUATION

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Abstract. We study a Dirichlet-Navier optimal design problem for a quasi-linear monotone p-biharmonic equation with control and state constraints. The coefficient of the p-biharmonic operator we take as a design variable in $BV(\Omega)\cap L^\infty(\Omega)$. In order to handle the inherent degeneracy of the p-Laplacian and the pointwise state constraints, we use regularization and relaxation approaches. We derive existence and uniqueness of solutions to the underlying boundary value problem and the optimal control problem. In fact, we introduce a two-parameter model for the weighted p-biharmonic operator and Henig approximation of the ordering cone. Further we discuss the asymptotic behaviour of the solutions to regularized problem on each (ε, k)-level as the parameters tend to zero and infinity, respectively.

Key words: p-biharmonic problem, optimal control, control in coefficients, approximation, existence result.

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1. Introduction

The aim of this article is to study a state constrained Dirichlet-Navier optimal control problem (OCP) for a quasi-linear monotone elliptic equation, the so-called weighted p-biharmonic problem. The coefficient of the p-biharmonic operator, the weight u, we take as a control in $BV(\Omega)\cap L^\infty(\Omega)$. Since an important matter for applications is to obtain a solution to a given boundary value problem with desired properties, it leads to the reasonable questions: can we define an appropriate coefficient of p-biharmonic operator to minimize the discrepancy between a given displacement y_d and an expected solution to such problem. More precisely, we analyse the following optimal design problem, which can be regarded as an optimal control problem, for quasi-linear partial differential equation (PDE) with mixed boundary conditions

$$\text{Minimize } \left\{ I(u,y) = \int_\Omega |y-y_d|^p\,dx + \int_\Omega |Du| \right\} \tag{1.1}$$

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subject to the quasi-linear equation

$$\Delta(u|\Delta y|^{p-2}\Delta y) = f \quad \text{in} \quad \Omega, \tag{1.2}$$

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on} \ \ \Gamma_D, \quad y = \Delta y = 0 \quad \text{on} \ \Gamma_S, \tag{1.3}$$

the pointwise state constraints

$$0 \leqslant \frac{\partial y(s)}{\partial \nu} \leqslant \zeta^{max}(s) \quad \text{a.e. on} \ \ \Gamma_S, \tag{1.4}$$

and the design (control) constraints

$$u \in BV(\Omega) \ \text{ and } \ 0 < \alpha \leqslant \xi_1(x) \leqslant u(x) \leqslant \xi_2(x) \text{ a.e. in } \Omega. \tag{1.5}$$

Here, Γ_D and Γ_S are the disjoint part of the boundary $\partial\Omega$ ($\partial\Omega = \Gamma_D \cup \Gamma_S$), $BV(\Omega) \cap L^\infty(\Omega)$ stands for the control space, $y_d \in L^p(\Omega)$, $\xi_1, \xi_2 \in L^\infty(\Omega)$, $f \in L^{p'}(\Omega)$, and $\zeta^{max} \in L^p(\Gamma_S)$ are given distributions. Problems of this type appear for $p-$power-like elastic isotropic flat plates of uniform thickness, where the design variable u is to be chosen such that the deflection of the plate matches a given profile. The model extends the classical weighted biharmonic equation, where the weight $u = a^3$ involves the thickness a of the plate, see e.g. [8, 21, 25, 26], or u can be regarded as a rigidity parameter. The OCP (1.1)–(1.4) can be considered as a prototype of design problems for quasilinear state equations. For an interesting exposure to this subject we can refer to the monographs [8, 16, 17].

A particular feature of OCP (1.1)–(1.4) is the restriction by the pointwise constraints (1.4) in $L^p(\Gamma_S)$-space. In fact, the ordering cone of positive elements in L^p-spaces is typically non-solid, i.e. it has an empty topological interior. Following the standard multiplier rule, which gives a necessary optimality condition for local solutions to state constrained OCPs, the constraint qualifications such as the Slater condition or the Robinson condition should be applied in this case. However, these conditions cannot be verified for cones such as

$$L^p_+(\Gamma_S) = \{v \in L^p(\Gamma_S) \mid v \geqslant 0 \quad \text{a.e. in} \ \ \Omega\}$$

due to the fact that $\operatorname{int}\left(L^p_+(\Gamma_S)\right) = \emptyset$, where $\operatorname{int}(A)$ stands for the topological interior of the set A. Therefore, it would be reasonable to propose a suitable relaxation of the pointwise state constraints in the form of some inequality conditions involving a so-called Henig approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ of the ordering cone of positive elements $L^p_+(\Gamma_S)$. Here, B is a fixed closed base of $L^p_+(\Gamma_S)$. As it was shown in our recent publication [12], due to fact that $L^p_+(\Gamma_S) \subset \left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ for all $\varepsilon > 0$, we can replace the cone $L^p_+(\Gamma_S)$ by its approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$. As a result, it leads to some relaxation of the inequality constraints of the considered problem, and, hence, to the approximation of the feasible set to the original OCP. Hence, the solvability of a given class of OCPs can be characterized by solving the corresponding Henig relaxed problems in the limit $\varepsilon \to 0$ (for the details, we refer to [12, 13]).

The ones more characteristic feature of the OCP (1.1)–(1.4) is related with degeneracy of quasilinear differential operator $\Delta(u|\Delta y|^{p-2}\Delta y)$ as Δy tends to zero and also if u approaches zero. Moreover, when the term $u|\Delta y|^{p-2}$ is regarded as the coefficient of the harmonic operator, we also have the case of unbounded coefficients. In spite of the fact that the Control in the coefficients of elliptic problems has a long history of its own starting with work of Murat [19, 20] and Tartar [27] (see also Casas [4], where the constrained optimal control problem in the coefficients of the leading order differential expressions was first discussed in details), analogous results for the case of weighted p-biharmonic equations of the type $\Delta(u|\Delta y|^{p-2}\Delta y)$ remained open. In this paper, in order to avoid degeneracy with respect to the control u, we assume that u is bounded away from zero. For the precise statements see the next section. We leave the case of potentially degenerating controls to a future contribution. Instead, in this article, we focus on the degeneracies related to the nonlinearity. A number of regularizations have been suggested in the literature. See [22] for a discussion for what has come to be known as ε-p-Laplace problem, such as $\Delta_{u,\varepsilon,p}y := \operatorname{div}(u(\varepsilon + |\nabla y|^2)^{\frac{p-2}{2}})\nabla y$. While the ε-p-Laplacean regularizes the degeneracy as the gradients tend to zero, the term $u|\nabla y|^{p-2}$, viewed again as a coefficient for the otherwise linear problem, may grow large. Therefore, we introduce yet another regularization that leads to a sequence of monotone and bounded approximation $\mathcal{F}_k(|\Delta y|^2)$ of $|\Delta y|^2$ (see our recent publication [6], where this approach was developed for p-Laplace problem). For fixed parameter $p \in [2, \infty)$, and control u, we arrive at a two-parameter problem governed by

$$\Delta^2_{\varepsilon,k,p}y := \Delta(u(\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}})\Delta y.$$

Finally, we have to deal with a two-parameter family of optimal control problems in the coefficients for monotone nonlinear differential equations and Henig relaxation of the the inequality state constraints. We consequently provide the well-posedness analysis for the underlying partial differential equations as well as for the optimal control problems. After that we pass to the limits as $k \to \infty$ and $\varepsilon \to 0$. The approximations and Henig relaxation are not only considered to be useful for the mathematical analysis, but also for the purpose of numerical simulations.

2. Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \geqslant 2)$. We assume that the boundary $\partial\Omega$ is Lipschitzian so that the unit outward normal $\nu = \nu(x)$ is well-defined for a.e. $x \in \partial\Omega$, where a.e. means here with respect to the $(N-1)$-dimensional Hausdorff measure. We also assume that the boundary $\partial\Omega$ consists of two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_S$, where the sets Γ_D and Γ_S have positive $(N-1)$-dimensional measures, and Γ_S is now C^2. Let p be a real number such that $2 \leqslant p < \infty$.

By $W^{2,p}(\Omega)$ we denote the Sobolev space as the subspace of $L^p(\Omega)$ of functions y having generalized derivatives $D^k y$ up to order $k = 2$ in $L^p(\Omega)$. We note that due to the interpolation theory, see [1, Theorem 4.14], $W^{2,p}(\Omega)$ is a Banach space with respect to the norm

$$\|y\|_{W^{2,p}(\Omega)} = \left(\|y\|^p_{L^p(\Omega)} + \|D^2 y\|^p_{L^p(\Omega)}\right)^{1/p} = \left(\int_\Omega \left(|y|^p + |D^2 y|^p\right)\,dx\right)^{1/p},$$

where

$$D^2 y \cdot D^2 v = \left(\sum_{i_1,i_2=1}^{N} \frac{\partial^2 y}{\partial x_{i_1}\partial x_{i_2}}\frac{\partial^2 v}{\partial x_{i_1}\partial x_{i_2}}\right)^{1/2}, \quad \text{and} \quad |D^2 y| = \left(D^2 y \cdot D^2 y\right)^{1/2}.$$

Since $\partial\Omega$ is Lipschitzian and Γ_S is of C^2, it follows that a function $y \in W^{2,p}(\Omega)$ admits some traces on $\partial\Omega$. In particular, if ν denotes the unit outer normal to $\partial\Omega$, then for any $y \in C^2(\overline{\Omega})$ we can define the traces

$$\gamma_0(y) = y\,|_{\partial\Omega}\,,\ \ \gamma_1(y) = \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_D} \quad \text{and} \quad \gamma_2(y) = \left.\frac{\partial^2 y}{\partial \nu^2}\right|_{\Gamma_S},$$

where $\partial y/\partial\nu$ denotes the outer normal derivative of y on Γ_D defined by $\partial y/\partial\nu = (\nabla y, \nu)$. By [15, Theorem 8.3], these linear operators can be extended continuously to the space $W^{2,p}(\Omega)$. We set

$$W^{2-1/p,p}(\partial\Omega) := \gamma_0\left[W^{2,p}(\Omega)\right], \quad W^{1-1/p,p}(\Gamma_D) := \gamma_1\left[W^{2,p}(\Omega)\right]$$

as closed subspaces of $W^{1,p}(\partial\Omega)$ and $L^p(\Gamma_D)$, respectively. Since $1 - 1/p = 1/p'$, where p' stands for the conjugate of p (that is $p + p' = pp'$), we have $\gamma_1\left[W^{2,p}(\Omega)\right] = W^{1/p',p}(\Gamma_D)$. Moreover, the injections

$$W^{2-1/p,p}(\partial\Omega) \hookrightarrow W^{1,p}(\partial\Omega) \quad \text{and} \quad W^{1/p',p}(\Gamma_D) \hookrightarrow L^p(\Gamma_D) \tag{2.1}$$

are compact by the Sobolev embedding theorem. We also put

$$\begin{aligned}\gamma_2\left[W^{2,p}(\Omega)\right] &= W^{-1/p,p}(\Gamma_S) := \left[W^{1/p,p'}(\Gamma_S)\right]^* \\ &= \text{ the dual space of } \ W^{1/p,p'}(\Gamma_S).\end{aligned}$$

Let

$$C_0^\infty(\mathbb{R}^N;\Gamma_D) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N)\ :\ \begin{array}{c}\varphi = 0 \ \text{ on } \ \partial\Omega,\ \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Gamma_D, \\ \text{and } \ \Delta\varphi = 0 \text{ on } \partial\Omega\setminus\Gamma_D.\end{array}\right\}$$

We define the Banach space $W_0^{2,p}(\Omega;\Gamma_D)$ as the closure of $C_0^\infty(\mathbb{R}^N;\Gamma_D)$ with respect to the norm $\|y\|_{W^{2,p}(\Omega)}$. Let $W^{-2,p'}(\Omega;\Gamma_D)$ be the dual space to $W_0^{2,p}(\Omega;\Gamma_D)$. We also define the space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|y\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega \|\nabla y\|^p\,dx\right)^{1/p}$.

Throughout this paper, we use the notation $\mathbb{W}_p(\Omega) := W_0^{2,p}(\Omega;\Gamma_D)$. Let us notice that $\mathbb{W}_p(\Omega)$ equipped with the norm

$$\|y\|_{p,\Delta} := \|\Delta y\|_{L^p(\Omega)} = \left(\int_\Omega |\Delta y|^p\,dx\right)^{1/p} = \left(\int_\Omega \left|\sum_{i=1}^N \frac{\partial^2 y}{\partial x_i^2}\right|^p dx\right)^{1/p} \tag{2.2}$$

is a uniformly convex Banach space [3]. Moreover, the norm $\|\cdot\|_{p,\Delta}$ is equivalent on $\mathbb{W}_p(\Omega)$ to the usual norm $\|\cdot\|_{W^{2,p}(\Omega)}$ of $W^{2,p}(\Omega)$. For reader's convenience, we give below the proof of the equivalence between the standard Sobolev space norm $\|\cdot\|_{W^{2,p}(\Omega)}$ and the norm $\|\cdot\|_{p,\Delta}$. For that, let us consider the classical Dirichlet problem for the famous Poisson's equation

$$\Delta y = f \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Since the Laplace operator $-\Delta$ acts from $\mathbb{W}_p(\Omega)$ in $L^p(\Omega)$, it is well-known that this problem is uniquely solvable in $\mathbb{W}_p(\Omega)$ for all $f \in L^p(\Omega)$. Hence, the inverse operator $T := (-\Delta)^{-1} : L^p(\Omega) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is well defined and satisfies the following elliptic regularity estimate [9]

$$\|Tf\|_{W^{2,p}(\Omega)} \leqslant C_p \|f\|_{L^p(\Omega)}.$$

This allows us to conclude the following. If $f \in L^p(\Omega)$ and $y \in W_0^{1,p}(\Omega)$ are such that $\frac{\partial y}{\partial \nu} = 0$ on Γ_D, $\Delta y = 0$ on Γ_S, and y is a solution of (2.3), then $-\Delta y \in L^p(\Omega)$, $y = 0$ on the boundary $\partial\Omega$, and, therefore, $y \in \mathbb{W}_p(\Omega)$. Hence,

$$\|y\|_{W^{2,p}(\Omega)} = \|T(-\Delta y)\|_{W^{2,p}(\Omega)} \leqslant C_p \|\Delta y\|_{L^p(\Omega)} = C_p \|y\|_{p,\Delta}, \tag{2.4}$$

for a suitable positive constant C_p independent of f. On the other hand, it is easy to observe that

$$\|y\|_{p,\Delta} \leqslant \|y\|_{W^{2,p}(\Omega)}.$$

Thus, by the Closed Graph Theorem, we can conclude that $y \mapsto \|y\|_{p,\Delta} = \left(\int_\Omega |\Delta y|^p\,dx\right)^{1/p}$ is equivalent to the norm induced by $W^{2,p}(\Omega)$ (for the details we refer to [7,18]).

Remark 2.1. Observe that $J : \mathbb{W}_p(\Omega) \to (\mathbb{W}_p(\Omega))^*$ defined by

$$J(y) = \begin{cases} \|\Delta y\|_{L^p(\Omega)}^{2-p} |\Delta y|^{p-2} \Delta y, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0 \end{cases}$$

is the duality mapping of $\mathbb{W}_p(\Omega)$ assicuated with the norm $\|\cdot\|_{p,\Delta}$ (see [23]).

By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ for which the norm

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_\Omega |Df| = \|f\|_{L^1(\Omega)} + \sup\left\{ \int_\Omega f \operatorname{div}\varphi\,dx \;:\; \varphi \in C_0^1(\Omega;\mathbb{R}^N),\ |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega \right\}$$

is finite.

We recall that a sequence $\{f_k\}_{k=1}^{\infty}$ converges weakly-$*$ to f in $BV(\Omega)$ if and only if the two following conditions hold (see [10]): $f_k \to f$ strongly in $L^1(\Omega)$ and $Df_k \rightharpoonup Df$ weakly-* in the space of Radon measures $\mathcal{M}(\Omega$, i.e.

$$\lim_{k\to\infty} \int_\Omega \varphi\, Df_k = \int_\Omega \varphi\, Df \quad \forall\, \varphi \in C_0(\Omega).$$

It is well-known also the following compactness result for BV-spaces (Helly's selection theorem, see [2]).

Theorem 2.1. *If $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{BV(\Omega)} < +\infty$, then there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ strongly converging in $L^1(\Omega)$ to some $f \in BV(\Omega)$ such that $Df_k \overset{*}{\rightharpoonup} Df$ weakly-$*$ in the space of Radon measures $\mathcal{M}(\Omega)$. Moreover, if $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$ strongly converges to some f in $L^1(\Omega)$ and satisfies $\sup_{k\in\mathbb{N}} \int_\Omega |Df_k| < +\infty$, then*

$$\begin{aligned} &(i)\ f \in BV(\Omega) \quad \text{and} \quad \int_\Omega |Df| \leqslant \liminf_{k\to\infty} \int_\Omega |Df_k|; \\ &(ii)\ f_k \overset{*}{\rightharpoonup} f \quad \text{in} \quad BV(\Omega). \end{aligned} \tag{2.5}$$

3. Setting of the Optimal Control Problem

Let ξ_1, ξ_2 be fixed elements of $L^\infty(\Omega) \cap BV(\Omega)$ satisfying the conditions

$$0 < \alpha \leqslant \xi_1(x) \leqslant \xi_2(x) \text{ a.e. in } \Omega, \tag{3.1}$$

where α is a given positive value.

Let $f \in W^{-2,p'}(\Omega;\Gamma_D)$, $y_d \in L^p(\Omega)$, and $\zeta^{max} \in L^p(\Gamma_S)$ be given distributions. The optimal control problem, we consider in this paper, is to minimize the discrepancy between y_d and the solutions of the following homogeneous Dirichlet-Navier boundary valued problem

$$\Delta_p^2(u,y) = f \quad \text{in} \quad \Omega, \tag{3.2}$$

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_D, \quad y = \Delta y = 0 \quad \text{on} \ \Gamma_S, \tag{3.3}$$

$$0 \leqslant \frac{\partial y(s)}{\partial \nu} \leqslant \zeta^{max}(s) \quad \text{a.e. on} \quad \Gamma_S \tag{3.4}$$

by choosing an appropriate weight function $u \in \mathfrak{A}_{ad}$ as control. Here, $\Delta_p^2(u,\cdot)$ is the generalized p-biharmonic operator, i.e.

$$\Delta_p^2(u,y) := \Delta(u|\Delta y|^{p-2}\Delta y), \quad \Delta y = \sum_{i=1}^N \frac{\partial^2 y}{\partial x_i^2}$$

and the class of admissible controls $\mathfrak{A}_{ad}$ we define as follows

$$\mathfrak{A}_{ad} = \left\{ u \in L^1(\Omega) \;\middle|\; \xi_1(x) \leqslant u(x) \leqslant \xi_2(x) \text{ a.e. in } \Omega \right\}. \tag{3.5}$$

It is clear that $\mathfrak{A}_{ad}$ is a nonempty convex subset of $L^1(\Omega)$ with an empty topological interior.

More precisely, we are concerned with the following optimal control problem

$$\begin{gathered} \text{Minimize } \left\{ I(u,y) = \int_\Omega |y - y_d|^p \, dx + \int_\Omega |Du| \right\} \\ \text{subject to the constraints (3.2)–(3.5).} \end{gathered} \tag{3.6}$$

Definition 3.1. We say that an element $y \in \mathbb{W}_p(\Omega)$ is the weak solution (in the sense of Minty) to the boundary value problem (3.2)–(3.3), if

$$\int_\Omega u\Delta\varphi\,(\Delta\varphi - \Delta y)\; dx \geqslant \langle f, \varphi - y\rangle\,,\ \forall\, \varphi \in C_0^\infty(\Omega;\Gamma_D). \tag{3.7}$$

Here, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)}$ stands for the duality pairing between $(\mathbb{W}_p(\Omega))^*$ and $\mathbb{W}_p(\Omega)$ and, in the sequel, we will omit this index when it is from the context.

The existence of a unique solution to the boundary value problem (3.2)–(3.3) follows from an abstract theorem on monotone operators; see, for instance, [14] or [24, §II.2].

Theorem 3.1. *Let V be a reflexive separable Banach space. Let V^* be the dual space, and let $A : V \to V^*$ be a bounded, semicontinuous, coercive and strictly monotone operator. Then the equation $Ay = f$ has a unique solution for each $f \in V^*$. Moreover, $Ay = f$ if and only if $\langle A\varphi, \varphi - y\rangle \geqslant \langle f, \varphi - y\rangle$ for all $\varphi \in V^*$.*

Here, the above mentioned properties of the strict monotonicity, semicontinuity, and coercivity of the operator A have respectively the following meaning:

$$\langle Ay - Av, y - v\rangle_{V^*;V} \geqslant 0, \quad \forall\, y, v \in V; \tag{3.8}$$

$$\langle Ay - Av, y - v\rangle_{V^*;V} = 0 \implies y = v; \tag{3.9}$$

$$\text{the function } \ t \mapsto \langle A(y + tv), w\rangle_{V^*;V} \ \text{ is continuous for all } \ y, v, w \in V; \tag{3.10}$$

$$\lim_{\|y\|_V \to \infty} \frac{\langle Ay, y\rangle_{V^*;V}}{\|y\|_V} = +\infty. \tag{3.11}$$

In our case, we can define the operator A as a mapping $\mathbb{W}_p(\Omega) \to (\mathbb{W}_p(\Omega))^*$ by

$$\langle A\varphi, v\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} := \int_\Omega u|\Delta\varphi|^{p-2}\Delta\varphi\Delta v\, dx.$$

Remark 3.1. The reason of such representation comes from the following observation: having applied Green's formula twice to the operator $\Delta(u|\Delta y|^{p-2}\Delta y)$

tested by $v \in C_0^\infty(\Omega;\Gamma_D)$, where y is an element of $\mathbb{W}_p(\Omega)$, we arrive at the identity

$$\begin{aligned}\int_\Omega \Delta(u|\Delta y|^{p-2}\Delta y)v\,dx &= -\int_\Omega \left(\nabla(u|\Delta y|^{p-2}\Delta y),\nabla v\right)\,dx\\ &+\int_{\partial\Omega}\frac{\partial}{\partial\nu}(u|\Delta y|^{p-2}\Delta y)v\,d\mathcal{H}^{N-1} = \int_\Omega u|\Delta y|^{p-2}\Delta y\Delta v\,dx\\ &-\int_{\Gamma_D} u|\Delta y|^{p-2}\Delta y\frac{\partial v}{\partial\nu}\,d\mathcal{H}^{N-1} - \int_{\Gamma_S} u|\Delta y|^{p-2}\Delta y\frac{\partial v}{\partial\nu}\,d\mathcal{H}^{N-1}\\ &= \int_\Omega u|\Delta y|^{p-2}\Delta y\Delta v\,dx \quad \forall\, v\in C_0^\infty(\Omega;\Gamma_D).\end{aligned}$$

Then it is easy to show that A satisfies all assumptions of Theorem 3.1 (for the details we refer to [14, 22]). As a consequence of this theorem, we also know that $y \in \mathbb{W}_p(\Omega)$ satisfies (3.7) if and only if the relations (3.2)–(3.3) are fulfilled as follows (for the details, we refer to [22, Section 2.4.4] and [8, Section 2.4.2])

$$\left.\begin{array}{c}\Delta^2(u,y) = f \quad \text{in} \quad \left(C_0^\infty(\Omega;\Gamma_D)\right)^*,\\ \gamma_0(y) = 0 \ \text{ in } \ W^{2-1/p,p}(\partial\Omega),\\ \gamma_1(y) = 0 \ \text{ in } \ W^{1/p',p}(\Gamma_D),\\ \gamma_0(\Delta y) = 0 \ \text{ in } \ W^{-1/p,p}(\Gamma_S) := \left(W^{1/p,p'}(\Gamma_S)\right)^*,\end{array}\right\}$$

that is, the integral identity holds

$$\int_\Omega u|\Delta y|^{p-2}\Delta y\Delta\varphi\,dx = \int_\Omega f\varphi\,dx \quad \forall\varphi\in\mathbb{W}_p(\Omega). \tag{3.12}$$

In particular, taking $\varphi = y$ in (3.12), this yields the relation

$$\int_\Omega u|\Delta y|^p\,dx = \int_\Omega fy\,dx, \tag{3.13}$$

which is usually referred to as the energy equality. As a result, conditions (3.1), (3.5), Friedrich's inequality, and identity (3.13) lead us to the following a priori estimate

$$\|y\|_{p,\Delta} := \left(\int_\Omega |\Delta y|^p\,dx\right)^{1/p} \leqslant C_\Omega\left(\alpha^{-1}\|f\|_{L^{p'}(\Omega)}\right)^{p'/p} \quad \forall\, u\in\mathfrak{A}_{ad}. \tag{3.14}$$

Taking this fact into account, we adopt the following notion.

Definition 3.2. We say that (u, y) is a feasible pair to the OCP (3.6) if $u \in \mathfrak{A}_{ad} \subset L^1(\Omega)$, $y \in \mathbb{W}_p(\Omega)$, the pair (u, y) is related by the integral identity (3.12), and

$$\frac{\partial y}{\partial\nu} \in L^p_+(\Gamma_S), \quad \zeta^{max} - \frac{\partial y}{\partial\nu} \in L^p_+(\Gamma_S), \tag{3.15}$$

where $L^p_+(\Gamma_S)$ stands for the natural ordering cone of positive elements in $L^p(\Gamma_S)$, i.e.

$$L^p_+(\Gamma_S) := \left\{v \in L^p(\Gamma_S) \mid v \geqslant 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \ \Gamma_S\right\}.$$

We denote by Ξ the set of all feasible pairs for the OCP (3.6).

Remark 3.2. Before we proceed further, we need to make sure that minimization problem (3.6) is consistent, i.e. there exists at least one pair (u, y) such that (u, y) satisfying the control and state constraints (3.3)–(3.5), and (u, y) would be a physically relevant solution to the boundary value problem (3.2)–(3.3). In fact, one needs the set of feasible solutions to be nonempty. But even if we are aware that $\Xi \neq \emptyset$, this set must be sufficiently rich in some sense, otherwise the OCP (3.6) becomes trivial. From a mathematical point of view, to deal directly with the control and especially state constraints is typically very difficult [4, 11, 23]. Thus, the consistency of OCPs with control and state constraints is an open question even for the simplest situation.

In view of this remark, it is reasonably now to make use of the following Hypothesis.

(H_1) OCP (3.6) is regular in the following sense — there exists at least one pair $(u, y) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ such that $(u, y) \in \Xi$.

Let τ be the topology on the set $\Xi \subset L^1(\Omega) \times \mathbb{W}_p(\Omega)$ which we define as the product of the norm topology of $L^1(\Omega)$ and the weak topology of $W_0^{2,p}(\Omega; \Gamma_D)$. We say that a pair $(u^0, y^0) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ is an optimal solution to problem (3.6) if

$$(u^0, y^0) \in \Xi \quad \text{and} \quad I(u^0, y^0) = \inf_{(u,y)\in\Xi} I(u, y).$$

With this notation, the control problem (3.6) can be written as follows

$$(\mathbb{P}) \qquad \min_{(u,y)\in\Xi} I(u, y).$$

4. Existence of Optimal Solutions

In this section we focus on the solvability of optimal control problem (3.2)–(3.6). Hereinafter, we suppose that the space $L^1(\Omega) \times \mathbb{W}_p(\Omega)$ is endowed with the norm $\|(u, y)\|_{L^1(\Omega)\times\mathbb{W}_p(\Omega)} := \|u\|_{L^1(\Omega)} + \|y\|_{p,\Delta}$.

We begin with a couple of auxiliary results.

Lemma 4.1. *Let $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ be a sequence such that $(u_k, y_k) \xrightarrow{\tau} (u, y)$ in $L^1(\Omega) \times \mathbb{W}_p(\Omega)$. Then we have*

$$\lim_{k\to\infty} \int_\Omega u_k \Delta y_k \Delta\varphi \, dx = \int_\Omega u \Delta y \Delta\varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega; \Gamma_D). \tag{4.1}$$

Proof. Since $u_k \to u$ in $L^1(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, we get that $u_k \to u$ strongly in $L^r(\Omega)$ for every $1 \leqslant r < +\infty$. In particular, we have that $u_k \to u$ in $L^{p'}(\Omega)$ and $\Delta y_k \Delta\varphi \rightharpoonup \Delta y \Delta\varphi$ in $L^p(\Omega)$. Hence, it is immediate to pass to the limit and to deduce (4.1). $\square$

As a consequence, we have the following property.

Corollary 4.1. *Let* $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ *and* $\{\zeta_k \in \mathbb{W}_p(\Omega)\}_{k\in\mathbb{N}}$ *be sequences such that* $(u_k, y_k) \xrightarrow{\tau} (u, y)$ *in* $L^1(\Omega) \times \mathbb{W}_p(\Omega)$ *and* $\zeta_k \to \zeta$ *in* $\mathbb{W}_p(\Omega)$. *Then*

$$\lim_{k\to\infty} \int_\Omega u_k \Delta y_k \Delta\zeta_k \, dx = \int_\Omega u\Delta y \Delta\zeta \, dx.$$

Our next step concerns the study of topological properties of the set of feasible solutions Ξ to problem (3.6).

The following result is crucial for our further analysis.

Theorem 4.1. *Let* $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ *be a bounded sequence in* $BV(\Omega) \times \mathbb{W}_p(\Omega)$. *Then there is a pair* $(u, y) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ *such that, up to a subsequence,* $(u_k, y_k) \xrightarrow{\tau} (u, y)$ *and* $(u, y) \in \Xi$.

Proof. By Theorem 2.1 and reflexivity of the space $\mathbb{W}_p(\Omega)$, there exists a subsequence of $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$, still denoted by the same indices, and functions $u \in BV(\Omega)$ and $y \in \mathbb{W}_p(\Omega)$ such that

$$u_k \to u \text{ in } L^1(\Omega), \quad y_k \rightharpoonup y \text{ in } \mathbb{W}_p(\Omega), \text{ and, hence, } y_k \rightharpoonup y \text{ in } W_0^{1,p}(\Omega). \tag{4.2}$$

Then by Lemma 4.1, we have

$$\lim_{k\to\infty} \int_\Omega u_k \Delta\varphi \Delta y_k \, dx = \int_\Omega u\Delta\varphi\Delta y \, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega; \Gamma_D).$$

It remains to show that the limit pair (u, y) is related by inequality (3.7) and satisfies the state constraints (3.15). With that in mind we write down the Minty relation for (u_k, y_k):

$$\int_\Omega u_k \Delta\varphi \left(\Delta\varphi - \Delta y_k\right) dx \geqslant \langle f, \varphi - y_k \rangle, \quad \forall\, \varphi \in C_0^\infty(\Omega; \Gamma_D). \tag{4.3}$$

In view of (4.2) and Lemma 4.1, we have

$$\lim_{k\to\infty} \int_\Omega |\Delta\varphi|^2 u_k \, dx = \int_\Omega |\Delta\varphi|^2 u \, dx, \quad \lim_{k\to\infty} \int_\Omega u_k \Delta\varphi\Delta y_k \, dx = \int_\Omega u\Delta\varphi\Delta y \, dx.$$

Thus, passing in relation (4.3) to the limit as $k \to \infty$, we arrive at the inequality (3.7) which means that $y \in \mathbb{W}_2(\Omega)$ is a weak solution to the boundary value problem (3.2)–(3.3) in the sense of Minty. Since the injections (2.1) are compact and the cone $L^p_+(\Gamma_S)$ is closed with respect to the strong convergence in $L^p(\Gamma_S)$, it follows that $\frac{\partial y_k}{\partial \nu} \to \frac{\partial y}{\partial \nu}$ strongly in $L^p(\Gamma_S)$ and, hence,

$$\lim_{k\to\infty} \gamma_1(y_k) = \gamma_1(y) \in L^p_+(\Gamma_S) \text{ and } \gamma_1(y) \in \zeta^{max} - L^p_+(\Gamma_S).$$

This fact together with $u \in \mathfrak{A}_{ad}$ leads us to the conclusion: $(u, y) \in \Xi$, i.e. the limit pair (u, y) is feasible to optimal control problem (3.6). The proof is complete. □

In conclusion of this section, we give the existence result for optimal pairs to the problem (3.6).

Theorem 4.2. *Assume that, for given distributions* $f \in L^{p'}(\Omega)$, $y_d \in L^p(\Omega)$, *and* $\zeta^{max} \in L^p(\partial\Omega)$, *the Hypothesis* (H_1) *is valid. Then optimal control problem* (3.6) *admits at least one solution* $(u^{opt}, y^{opt}) \in BV(\Omega) \times \mathbb{W}_p(\Omega)$.

Proof. Since the set of feaasible pairs Ξ is nonempty and the cost functional is bounded from below on Ξ, it follows that there exists a minimizing sequence $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ to problem (3.6). Then the inequality

$$\inf_{(u,y)\in\Xi} I(u,y) = \lim_{k\to\infty} \left[\int_\Omega |y_k(x) - y_d(x)|^p \, dx + \int_\Omega |Du_k| \right] < +\infty,$$

implies the existence of a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}} \int_\Omega |Du_k| \leqslant C.$$

Hence, in view of the definition of the class of admissible controls $\mathfrak{A}_{ad}$ and a priori estimate (3.14), the sequence $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega) \times \mathbb{W}_p(\Omega)$. Therefore, by Theorem 4.1, there exist functions $u^* \in \mathfrak{A}_{ad}$ and $y^* \in \mathbb{W}_p(\Omega)$ such that $(u^*, y^*) \in \Xi$ and, up to a subsequence, $u_k \to u^*$ strongly in $L^1(\Omega)$ and $y_k \rightharpoonup y^*$ weakly in $\mathbb{W}_p(\Omega)$. To conclude the proof, it is enough to show that the cost functional I is lower semicontinuous with respect to the τ-convergence. Since $y_k \to y^*$ strongly in $L^p(\Omega)$ by Sobolev embedding theorem, it follows that

$$\lim_{k\to\infty} \int_\Omega |y_k(x) - y_d(x)|^p \, dx = \int_\Omega |y^*(x) - y_d(x)|^p \, dx,$$

$$\liminf_{k\to\infty} \int_\Omega |Du_k| \geqslant \int_\Omega |Du^*| \text{ by (2.5)}.$$

Thus,

$$I(u^*, y^*) \leqslant \liminf_{k\to\infty} I(u_k, y_k) = \inf_{(u,y)\in\Xi} I(u,y).$$

Hence, (u^*, y^*) is an optimal pair, and we arrive at the required conclusion. □

5. Regularization of OCP (3.6)

As was pointed out in [22], the p-Laplacian $\Delta_p(u, y)$ provides an example of a quasi-linear elliptic operator with a so-called degenerate nonlinearity for $p > 2$. In this context we have non-differentiability of the state y with respect to the control u. As follows from Theorem 4.2, this fact is not an obstacle to prove existence of optimal controls in the coefficients, but it causes certain difficulties when deriving the optimality conditions for the considered problem. On the other hand, the ordering cone of positive elements $L^p_+(\Gamma_S)$ is non-solid, i.e. it has an empty topological interior in L^p-space. Therefore, it is reasonably to apply a suitable relaxation of the pointwise state constraints in the form of some inequality conditions involving the so-called Henig approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ of $L^p_+(\Gamma_S)$, where B is a fixed closed base of $L^p_+(\Gamma_S)$. Since $L^p_+(\Gamma_S) \subset \left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ for all $\varepsilon > 0$, it allows us to replace the cone $L^p_+(\Gamma_S)$ by its approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$. In fact, it leads to some relaxation of the inequality constraints of the considered problem, and, hence, to the approximation of the feasible set to the original OCP. As a result, we introduce the following family of approximating control problems (see, for comparison, the approach of Casas and Fernandez [5] for quasi-linear elliptic equations with a distributed control in the right hand side and the approach of Kogut and Leugering [12], where the Henig regularization of pointwise state constraints have been proposed).

$$\text{Minimize } \left\{ I(u,y) = \int_\Omega |y - z_d|^p \, dx + \int_\Omega |Du| \right\} \tag{5.1}$$

subject to the constraints

$$\Delta^2_{\varepsilon,k,p}(u,y) = f \quad \text{in} \quad \Omega, \tag{5.2}$$

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on } \Gamma_D, \quad y = \Delta y = 0 \quad \text{on } \Gamma_S, \tag{5.3}$$

$$\frac{\partial y}{\partial \nu} \in \left(L^p_+(\Gamma_S)\right)_\varepsilon(B), \quad \zeta^{max} - \frac{\partial y}{\partial \nu} \in \left(L^p_+(\Gamma_S)\right)_\varepsilon(B), \tag{5.4}$$

$$u \in \mathfrak{A}_{ad} = \left\{ v \in BV(\Omega) \;\middle|\; \xi_1(x) \leqslant v(x) \leqslant \xi_2(x) \text{ a.e. in } \Omega \right\}. \tag{5.5}$$

Here, $k \in \mathbb{N}$, ε is a small parameter, which varies within a strictly decreasing sequence of positive numbers converging to 0,

$$\Delta^2_{\varepsilon,k,p}(u,y) = \Delta\left(u(x) \left(\varepsilon + \mathcal{F}_k\left(|\Delta y|^2\right)\right)^{\frac{p-2}{2}} \Delta y \right), \tag{5.6}$$

$\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing $C^1(\mathbb{R}_+)$-function such that

$$\begin{gathered} \mathcal{F}_k(t) = t, \text{ if } t \in \left[0, k^2\right], \quad \mathcal{F}_k(t) = k^2 + 1, \text{ if } t > k^2 + 1, \quad \text{and} \\ t \leqslant \mathcal{F}_k(t) \leqslant t + \delta, \text{ if } k^2 \leqslant t < k^2 + 1 \quad \text{for some } \delta \in (0,1), \end{gathered} \tag{5.7}$$

$$B := \left\{ \xi \in L^p_+(\Gamma_S) \middle| \int_{\Gamma_S} \xi \, d\mathcal{H}^{N-1} = 1 \right\} \tag{5.8}$$

is a closed base of ordering cone $\Lambda := L^p_+(\Gamma_S)$,

$$\begin{aligned} \left(L^p_+(\Gamma_S)\right)_\varepsilon(B) &:= \mathrm{cl}_{\|\cdot\|_{L^p(\Gamma_S)}}\left(\mathrm{cone}\left(B + B_\varepsilon(0)\right)\right) \\ &:= \mathrm{cl}_{\|\cdot\|_{L^p(\Gamma_S)}}\left(\left\{\mu z \mid \mu \geq 0, z \in B + B_\varepsilon(0)\right\}\right) \end{aligned}$$

is the *Henig dilating cone*, and $\frac{1}{\varepsilon}B_\varepsilon(0) := \left\{ v \in L^p(\Gamma_S) \mid \|v\|_{L^p(\Gamma_S)} \leqslant 1 \right\}$ is the closed unit ball in $L^p(\Gamma_S)$ centered at the origin.

As for the function $\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$, it can e.g. be defined by

$$\mathcal{F}_k(t) = \begin{cases} t, & \text{if } 0 \leqslant t \leqslant k^2, \\ (k^2 - t)^3 + (k^2 - t)^2 + t, & \text{if } k^2 \leqslant t \leqslant k^2 + 1, \\ k^2 + 1, & \text{if } t \geqslant k^2 + 1. \end{cases}$$

A direct calculation shows that in this case $\delta = 4/27$.

It is clear that the effect of such perturbations of $\Delta_p^2(u, y)$ is its regularization around critical points where $|\Delta y(x)|$ vanishes or becomes unbounded. In particular, if $y \in W_0^{2,p}(\Omega)$ and $\Omega_k(y) := \left\{ x \in \Omega \ : \ |\Delta y(x)| > \sqrt{k^2 + 1} \right\}$, then the following chain of inequalities

$$\begin{aligned} |\Omega_k(y)| &:= \int_{\Omega_k(y)} 1 \, dx \leqslant \frac{1}{\sqrt{k^2 + 1}} \int_{\Omega_k(y)} |\Delta y(x)| \, dx \\ &\leqslant \frac{1}{\sqrt{k^2 + 1}} |\Omega_k(y)|^{\frac{1}{p'}} \left(\int_\Omega |\Delta y|^p \, dx \right)^{\frac{1}{p}} = \frac{\|y\|_{W_0^{2,p}(\Omega)}}{\sqrt{k^2 + 1}} |\Omega_k(y)|^{\frac{p-1}{p}} \end{aligned}$$

shows that the Lebesgue measure of the set $\Omega_k(y)$ satisfies the estimate

$$|\Omega_k(y)| \leqslant \left(\frac{1}{\sqrt{k^2 + 1}} \right)^p \|y\|^p_{W_0^{2,p}(\Omega)} \leqslant \|y\|^p_{W_0^{2,p}(\Omega)} k^{-p}, \quad \forall \, y \in W_0^{2,p}(\Omega), \tag{5.9}$$

i.e. the approximation $\mathcal{F}_k(|\Delta y|^2)$ is essential on sets with small Lebesgue measure. The main goal of this section is to show that for each $\varepsilon > 0$ and $k \in \mathbb{N}$, the perturbed optimal control problem (5.1)–(5.5) is well posed and its solutions can be considered as a reasonable approximation of optimal pairs to the original problem (3.6). To begin with, we establish a few auxiliary results concerning monotonicity and growth conditions for the regularized p-harmonic operator $\Delta^2_{\varepsilon,k,p}$.

For our further analysis, we make use of the following the notation

$$\|\varphi\|_{\varepsilon,k,u} = \left(\int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta \varphi|^2) \right)^{\frac{p-2}{2}} |\Delta \varphi|^2 u \, dx \right)^{1/p} \quad \forall \varphi \in W_0^{2,2}(\Omega).$$

Remark 5.1. For an arbitrary element $y^* \in W_0^{2,2}(\Omega)$ let us consider the level set $\Omega_k(y^*) := \left\{ x \in \Omega \ : \ |\Delta y^*(x)| > \sqrt{k^2 + 1} \right\}$. Then

$$\begin{aligned} |\Omega_k(y^*)| &:= \int_{\Omega_k(y^*)} 1 \, dx \leqslant \frac{1}{\sqrt{k^2 + 1}} \int_{\Omega_k(y^*)} |\Delta y^*(x)| \, dx \\ &\leqslant \frac{1}{k} |\Omega_k(y^*)|^{\frac{1}{2}} \left(\int_{\Omega_k(y^*)} |\Delta y^*|^2 \, dx \right)^{\frac{1}{2}} \\ &= \frac{1}{k} \left(\frac{1}{\varepsilon + k^2 + 1} \right)^{\frac{p-2}{4}} \left(\int_{\Omega_k(y^*)} \left(\varepsilon + \mathcal{F}_k(|\Delta y^*|^2) \right)^{\frac{p-2}{2}} |\Delta y^*|^2 \, dx \right)^{\frac{1}{2}} |\Omega_k(y^*)|^{\frac{1}{2}} \\ &\leqslant \frac{1}{k^{\frac{p}{2}}} \, |\Omega_k(y^*)|^{\frac{1}{2}} \, \alpha^{-\frac{1}{2}} \|y^*\|^{\frac{p}{2}}_{\varepsilon,k,u}. \end{aligned}$$

Hence, the Lebesgue measure of the set $\Omega_k(y^*)$ satisfies the estimate

$$|\Omega_k(y^*)| \leqslant \frac{\alpha^{-1}}{k^p}\|y^*\|^p_{\varepsilon,k,u}, \quad \forall\, y^* \in W^{2,2}_0(\Omega). \tag{5.10}$$

Now, we establish the following results.

Proposition 5.1. For every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$, the operator

$$A_{\varepsilon,k,u} := -\Delta^2_{\varepsilon,k,p}(u,\cdot) : \mathbb{W}_2(\Omega) \to (\mathbb{W}_2(\Omega))^*$$

is bounded and $\|A_{\varepsilon,k,u}\| \leqslant \left(\varepsilon + k^2 + 1\right)^{\frac{p-2}{2}} \|\xi_2\|_{L^\infty(\Omega)}$, where

$$\mathbb{W}_2(\Omega) := W^{2,2}_0(\Omega;\Gamma_D).$$

Proof. From the assumptions on $\mathcal{F}_k$ and the boundedness of u we obtain

$$\begin{aligned}
\|A_{\varepsilon,k,u}\| &= \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1} \|A_{\varepsilon,k,u} y\|_{(\mathbb{W}_2(\Omega))^*}\\
&= \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1}\ \sup_{\|v\|_{W^{2,2}_0(\Omega)}\leqslant 1} \langle A_{\varepsilon,k,u} y, v\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)}\\
&= \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1}\ \sup_{\|v\|_{W^{2,2}_0(\Omega)}\leqslant 1} \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y|^2)\right)^{\frac{p-2}{2}} \Delta y \Delta v u\, dx\\
&\leqslant \left(\varepsilon + k^2 + 1\right)^{\frac{p-2}{2}} \|\xi_2\|_{L^\infty(\Omega)} \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1}\ \sup_{\|v\|_{W^{2,2}_0(\Omega)}\leqslant 1} \|y\|_{W^{2,2}_0(\Omega)} \|v\|_{W^{2,2}_0(\Omega)}\\
&= \left(\varepsilon + k^2 + 1\right)^{\frac{p-2}{2}} \|\xi_2\|_{L^\infty(\Omega)},
\end{aligned}$$

which concludes the proof. □

Proposition 5.2. For every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$, the operator $A_{\varepsilon,k,u}$ is strictly monotone.

Proof. To begin with, we make use of the following algebraic inequality:

$$\left[\left(\varepsilon + \mathcal{F}_k(|a|^2)\right)^{\frac{p-2}{2}} a - \left(\varepsilon + \mathcal{F}_k(|b|^2)\right)^{\frac{p-2}{2}} b\right](a-b) \geqslant \varepsilon^{\frac{p-2}{2}} |a-b|^2, \quad \forall\, a,b \in \mathbb{R}. \tag{5.11}$$

In order to prove it, we note that the left hand side of (5.11) can be rewritten as follows

$$\begin{aligned}
&\left(\left(\varepsilon + \mathcal{F}_k(|a|^2)\right)^{\frac{p-2}{2}} a - \left(\varepsilon + \mathcal{F}_k(|b|^2)\right)^{\frac{p-2}{2}} b\right)(a-b)\\
&\quad = \int_0^1 \frac{d}{ds}\left\{\left(\varepsilon + \mathcal{F}_k(|sa + (1-s)b|^2)\right)^{\frac{p-2}{2}} (sa + (1-s)b)\right\} ds(a-b)\\
&= \int_0^1 \left(\varepsilon+\mathcal{F}_k(|sa+(1-s)b|^2)\right)^{\frac{p-2}{2}} |a-b|^2\, dx + (p-2)\int_0^1 \left\{\left(\varepsilon+\mathcal{F}_k(|sa+(1-s)b|^2)\right)^{\frac{p-4}{2}}\right.\\
&\qquad \left.\times \mathcal{F}_k'(|sa + (1-s)b|^2)\, |(sa + (1-s)b)\,(a-b)|^2 \right\} ds = I_1 + I_2.
\end{aligned}$$

Since $p \geqslant 2$ and $\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing $C^1(\mathbb{R}_+)$-function, it follows that $I_2 \geqslant 0$ for all $a, b \in \mathbb{R}^N$. It remains to observe that

$$\big(\varepsilon + \mathcal{F}_k(|sa + (1-s)b|^2)\big) \geqslant \varepsilon, \quad \forall\, a, b \in \mathbb{R}.$$

Hence, $I_1 \geqslant \varepsilon^{\frac{p-2}{2}}|a-b|^2$ and we arrive at the inequality (5.11). With this we obtain

$$\begin{aligned}
&\big\langle -\Delta_{\varepsilon,k,p}(u,y) + \Delta_{\varepsilon,k,p}(u,v), y - v\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)}\\
&= \int_\Omega u(x)\left((\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}}\Delta y - (\varepsilon + \mathcal{F}_k(|\Delta v|^2))^{\frac{p-2}{2}}\Delta v\right)(\Delta y - \Delta v)\, dx\\
&\geqslant \alpha\varepsilon^{\frac{p-2}{2}}\int_\Omega |\Delta y - \Delta v|^2\, dx = \alpha\varepsilon^{\frac{p-2}{2}}\|y - v\|^2_{W_0^{2,2}(\Omega)} \geqslant 0.
\end{aligned}$$

Since the relation

$$\big\langle A_{\varepsilon,k,u}y - A_{\varepsilon,k,u}v, y - v\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)} = 0$$

implies that $y = v$ almost everywhere in Ω, it follows that the strict monotonicity property (3.9) holds in this case. □

Proposition 5.3. For every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$, the operator $A_{\varepsilon,k,u}$ is coercive (in the sense of relation (3.11)).

Proof. In order to check this property it is enough to observe that for any $y \in \mathbb{W}_2(\Omega)$, $k \in \mathbb{N}$, $\varepsilon > 0$, and $u \in \mathfrak{A}_{ad}$, we have

$$\begin{aligned}
\big\langle A_{\varepsilon,k,u}y, y\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)} &= \big\langle -\Delta_{\varepsilon,k,p}(u,y), y\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)}\\
&= \int_\Omega \big(\varepsilon + \mathcal{F}_k(|\Delta y|^2)\big)^{\frac{p-2}{2}}|\Delta y|^2 u\, dx \geqslant \alpha\varepsilon^{\frac{p-2}{2}}\|y\|^2_{W_0^{2,2}(\Omega)}.
\end{aligned}$$

□

We are now in a position to apply the abstract theorem on monotone operators (see Theorem 3.1) to the equation $A_{\varepsilon,k,u}y = f$ with $f \in L^{p'}(\Omega)$. Closely following the arguments of Section 3, we arrive at the following assertion.

Theorem 5.1. *For each $\varepsilon > 0$, $k \in \mathbb{N}$, $u \in \mathfrak{A}_{ad}$, and $f \in L^{p'}(\Omega)$, the boundary value problem* (5.2)–(5.3) *admits a unique weak solution $y_{\varepsilon,k} \in \mathbb{W}_2(\Omega)$, i.e.*

$$\int_\Omega u(\varepsilon + \mathcal{F}_k(|\Delta y_{\varepsilon,k}|^2))^{\frac{p-2}{2}}\Delta y_{\varepsilon,k}\Delta\varphi\, dx = \int_\Omega f\varphi\, dx, \quad \forall\, \varphi \in \mathbb{W}_2(\Omega), \tag{5.12}$$

or equivalently

$$\int_\Omega u(x)(\varepsilon + \mathcal{F}_k(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi\,(\Delta\varphi - \Delta y_{\varepsilon,k}) \tag{5.13}$$

$$\geqslant \int_\Omega f(\varphi - y_{\varepsilon,k})\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega;\Gamma_D). \tag{5.14}$$

For every $\varepsilon > 0$ and $k \in \mathbb{N}$, we denote the set of feasible pairs to the problem (5.1)–(5.5) as follows

$$\Xi_{\varepsilon,k} = \left\{ (u,y) \;\middle|\; \begin{array}{c} u \in \mathfrak{A}_{ad},\ y \in \mathbb{W}_2(\Omega), \\ (u,y) \text{ are related by equality (5.12)}, \\ \frac{\partial y}{\partial \nu} \text{ satisfies the inclusions (5.4).} \end{array} \right\}. \tag{5.15}$$

It is worth to notice that Hypothesis (H_1) about regularity of the original OCP (3.6) can be characterized by the non-emptiness properties of the sets of feasible solutions $\Xi_{\varepsilon,k}$ for approximating control problem (5.1)–(5.5). Indeed, we have the following result (see [12, Theorem 8]).

Theorem 5.2. *Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\delta)$ be a monotonically decreasing sequence converging to 0 as $k \to \infty$. Then, for given distributions $f \in L^{p'}(\Omega)$, $y_d \in L^p(\Omega)$, and $\zeta^{max} \in L^p(\Gamma_S)$, the Hypothesis (H_1) implies that the approximating control problem (5.1)–(5.5) has a nonempty set of feasible solutions $\Xi_{\varepsilon,k}$ for all $\varepsilon = \varepsilon_k$, $k \in \mathbb{N}$. And vice versa, if there exists a sequence $\left\{(u^k, y^k)\right\}_{k\in\mathbb{N}}$ satisfying conditions*

$$(u^k, y^k) \in \Xi_{\varepsilon_k,k} \quad \textit{for all} \quad k \in \mathbb{N}, \qquad \textit{and} \quad \sup_{k\in\mathbb{N}} I(u^k, y^k) < +\infty, \tag{5.16}$$

then the sequence $\left\{(u^k, y^k)\right\}_{k\in\mathbb{N}}$ is τ-compact and each of its τ-cluster pairs is a feasible solution to the original OCP (3.6).

Thus, in view of Theorem 5.2 and Hypothesis (H_1), we can suppose that the sets $\Xi_{\varepsilon,k}$ are always nonempty and, therefore, the approximating control problem

$$(\mathbb{P}_{\varepsilon,k}) \qquad \min_{(u,y)\in\Xi_{\varepsilon,k}} I(u,y) \tag{5.17}$$

is consistent.

Analogously to problem $(\mathbb{P})$, we can prove the following theorem

Theorem 5.3. *For every positive value $\varepsilon > 0$ and integer $k \in \mathbb{N}$, the optimal control problem $(\mathbb{P}_{\varepsilon,k})$ has at least one solution.*

The proof follows the steps of that of Theorem 4.2. Indeed, it is immediate to check that $\Xi_{\varepsilon,k}$ is not empty. Then, we can take a minimizing sequence $\{(u_i, y_i)\}_{i\in\mathbb{N}} \subset \Xi_{\varepsilon,k}$. The lower boundedness of I implies the boundedness of $\{(u_i, y_i)\}_{i\in\mathbb{N}}$ in $BV(\Omega) \times W_0^{2,2}(\Omega)$. Then, arguing as in the proof of Theorem 4.2, we deduce the existence of a subsequence, denoted in the same way, and a pair $(u^*, y^*) \in \Xi_{\varepsilon,k}$ such that $u_i \overset{*}{\rightharpoonup} u^*$ in $BV(\Omega)$ and $y_i \rightharpoonup y^*$ in $W_0^{2,2}(\Omega)$. Hence, $I(u^*, y^*) \leqslant \liminf_{i\to\infty} I(u_i, y_i)$. Since $\frac{\partial y}{\partial \nu} \in W^{1/2,2}(\Gamma_S)$ for any $y \in W_0^{2,2}(\Omega; \Gamma_D)$, the injection $W^{1/2,2}(\Gamma_S) \hookrightarrow L^2(\Gamma_S)$ is compact, and the Henig dilating cone $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ is closed with respect to the strong convergence in $L^2(\Gamma_S)$, it follows that $\frac{\partial y_k}{\partial \nu} \to \frac{\partial y^*}{\partial \nu}$ strongly in $L^2(\Gamma_S)$ and, hence,

$$\lim_{k\to\infty} \frac{\partial y_k}{\partial \nu} = \frac{\partial y^*}{\partial \nu} \in L^p_+(\Gamma_S) \quad \text{and} \quad \frac{\partial y^*}{\partial \nu} \in \zeta^{max} - \left(L^p_+(\Gamma_S)\right)_\varepsilon(B).$$

This fact together with $u \in \mathfrak{A}_{ad}$ leads us to the conclusion: $(u,y) \in \Xi_{\varepsilon,k}$, i.e. the limit pair (u^*, y^*) is optimal to the problem $(\mathbb{P}_{\varepsilon,k})$.

For our further analysis, we need to obtain some appropriate a priory estimates for the weak solutions to problem (5.2)–(5.3). With that in mind, we make use of the following auxiliary results.

Proposition 5.4. Let $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$ be given. Then, for arbitrary $g \in L^2(\Omega)$ and $y \in W_0^{2,2}(\Omega)$, we have

$$\left|\int_\Omega gy\,dx\right| \leqslant C_\Omega \|g\|_{L^2(\Omega)} \left[\alpha^{-\frac{1}{p}} |\Omega|^{\frac{p-2}{2p}} \|y\|_{\varepsilon,k,u} + \alpha^{-\frac{1}{2}} \|y\|^{\frac{p}{2}}_{\varepsilon,k,u}\right]. \tag{5.18}$$

Proof. Let us fix an arbitrary element y of $W_0^{2,2}(\Omega)$. We associate with this element the set $\Omega^k(y)$, where $\Omega^k(y) := \{x \in \Omega \;:\; |\Delta y(x)| > k\}$. Then, by Friedrich's inequality,

$$\begin{aligned}\int_\Omega gy\,dx &\leqslant \|g\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} \\ &\leqslant C_\Omega \|g\|_{L^2(\Omega)} \Big(\|\Delta y\|_{L^2(\Omega\setminus\Omega^k(y))} + \|\Delta y\|_{L^2(\Omega^k(y))}\Big).\end{aligned} \tag{5.19}$$

Using the fact that

$$\begin{aligned}\|\Delta y\|_{L^2(\Omega\setminus\Omega^k(y))} &\leqslant |\Omega|^{\frac{p-2}{2p}} \|\Delta y\|_{L^p(\Omega\setminus\Omega^k(y))} \\ &\leqslant |\Omega|^{\frac{p-2}{2p}} \left(\int_{\Omega\setminus\Omega^k(y)} (\varepsilon + |\Delta y|^2)^{\frac{p-2}{2}} |\Delta y|^2\,dx\right)^{\frac{1}{p}}\end{aligned}$$

and

$$\begin{gathered}\mathcal{F}_k(|\Delta y|^2) = |\Delta y|^2 \quad \text{a.e. in } \Omega \setminus \Omega^k(y), \text{ and} \\ k^2 \leqslant \mathcal{F}_k(|\Delta y|^2) \leqslant k^2 + 1 \quad \text{a.e. in } \Omega^k(y), \quad \forall\, k \in \mathbb{N},\end{gathered}$$

we obtain

$$\begin{aligned}\|\Delta y\|_{L^2(\Omega\setminus\Omega^k(y))} &\leqslant |\Omega|^{\frac{p-2}{2p}} \left(\int_{\Omega\setminus\Omega^k(y)} (\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}} |\Delta y|^2\,dx\right)^{\frac{1}{p}} \\ &\leqslant |\Omega|^{\frac{p-2}{2p}} \alpha^{-\frac{1}{p}} \|y\|_{\varepsilon,k,u},\end{aligned} \tag{5.20}$$

$$\|\Delta y\|_{L^2(\Omega^k(y))} \leqslant \left(\int_{\Omega^k(y)} (\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}} |\Delta y|^2\,dx\right)^{\frac{1}{2}} \leqslant \alpha^{-\frac{1}{2}} \|y\|^{\frac{p}{2}}_{\varepsilon,k,u}. \tag{5.21}$$

As a result, inequality (5.18) immediately follows from (5.19)–(5.21). The proof is complete. □

Definition 5.1. Let $\{u_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}} \subset \mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls. We say that a two-parametric sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}} \subset W_0^{2,2}(\Omega)$ is bounded with respect to the $\|\cdot\|_{\varepsilon,k,u_{\varepsilon,k}}$-quasi-seminorm if $\sup_{\substack{\varepsilon>0\\k\in\mathbb{N}}} \|y_{\varepsilon,k}\|_{\varepsilon,k,u_{\varepsilon,k}} < +\infty$.

To conclude this section, let us show that for every $u \in \mathfrak{A}_{ad}$ and $f \in L^{p'}(\Omega)$, the sequence $\{y_{\varepsilon,k} = y_{\varepsilon,k}(u,f)\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ of weak solutions to the boundary value problem (5.2)–(5.3) is bounded with respect to the $\|\cdot\|_{\varepsilon,k,u}$-quasi-seminorm in the sense of Definition 5.1.

Indeed, the integral identity (5.12) together with estimate (5.18) (for $g = f$) immediately lead us to the relation

$$\begin{aligned}
\|y_{\varepsilon,k}\|^p_{\varepsilon,k,u} &:= \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon,k}|^2 u\,dx \\
&\leqslant \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon,k}|^2 u\,dx = \int_\Omega f y_{\varepsilon,k}\,dx \\
&\leqslant C_\Omega \|f\|_{L^2(\Omega)} \left[\alpha^{-\frac{1}{p}} |\Omega|^{\frac{p-2}{2p}} \|y_{\varepsilon,k}\|_{\varepsilon,k,u} + \alpha^{-\frac{1}{2}} \|y_{\varepsilon,k}\|^{\frac{p}{2}}_{\varepsilon,k,u}\right].
\end{aligned} \tag{5.22}$$

As a result, it follows from (5.22) that

$$\|y_{\varepsilon,k}\|_{\varepsilon,k,u} \leqslant \max\left\{C_f^{\frac{2}{p}}, C_f^{\frac{1}{p-1}}\right\}, \quad \forall\,\varepsilon>0,\ \forall\,k\in\mathbb{N},\ \forall\,u\in\mathfrak{A}_{ad}, \tag{5.23}$$

where $C_f := C\|f\|_{L^2(\Omega)} = C_\Omega \left(\alpha^{-\frac{1}{p}}|\Omega|^{\frac{p-2}{2p}} + \alpha^{-\frac{1}{2}}\right) \|f\|_{L^2(\Omega)}$.

6. Asymptotic Analysis of the Approximating OCP $(\mathbb{P}_{\varepsilon,k})$

Our main intention in this section is to show that optimal solutions to the original OCP $(\mathbb{P})$ can be attained (in some sense) by optimal solutions to the approximated problems $(\mathbb{P}_{\varepsilon,k})$. With that in mind, we make use of the concept of variational convergence of constrained minimization problems (see [11]) and study the asymptotic behaviour of a family of OCPs $(\mathbb{P}_{\varepsilon,k})$ as $\varepsilon \to 0$ and $k \to \infty$. We also utilize the fact that the sequence of cones $\left\{\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B)\right\}_{k\in\mathbb{N}}$ converges to $L^p_+(\Gamma_S)$ in Kuratowski sense with respect to the norm topology of $L^p(\Gamma_S)$ as ε_k tends monotonically to zero (see Proposition 7 in [12]), that is

$$K\text{-}\liminf_{k\to\infty} \left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B) = L^p_+(\Gamma_S) = K\text{-}\limsup_{k\to\infty} \left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B), \tag{6.1}$$

where

$$\begin{aligned}K-\liminf_{k\to\infty}\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B) \\ &:= \big\{z\in L^p(\Gamma_S)\,\big|\,\text{for all neighborhoods } N \text{ of } z \text{ there is a} \\ &\qquad k_0\in\mathbb{N} \text{ such that } N\cap\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B)\neq\emptyset\ \forall\, k\geq k_0\big\},\\ K-\limsup_{k\to\infty}\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B) \\ &:= \big\{z\in L^p(\Gamma_S)\,\big|\,\text{for all neighborhoods } N \text{ of } z \text{ and every } k_0\in\mathbb{N} \\ &\qquad \text{there is a } k\geq k_0 \text{ such that } N\cap\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B)\neq\emptyset\big\}.\end{aligned}$$

We begin with some auxiliary results concerning the weak compactness in $W^{2,2}_0(\Omega)$ of $\|\cdot\|_{\varepsilon,k,u}$-bounded sequences.

Lemma 6.1. *Let $\{u_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset\mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls with associated states $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset W^{2,2}_0(\Omega;\Gamma_D)$, $y_{\varepsilon,k}=y_{\varepsilon,k}(u_{\varepsilon,k})$. Then the sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ is bounded in $W^{2,2}_0(\Omega)$. Moreover, each cluster point y of the sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ with respect to the weak convergence in $W^{2,2}_0(\Omega)$, satisfies: $y\in W^{2,p}_0(\Omega;\Gamma_D)$.*

Proof. The boundedness in $W^{2,2}_0(\Omega)$ immediately follows from (5.23) and the estimates

$$\begin{aligned}\|y_{\varepsilon,k}\|_{W^{2,2}_0(\Omega)} &\leqslant \|\Delta y_{\varepsilon,k}\|_{L^2(\Omega\setminus\Omega^k(y_{\varepsilon,k}))}+\|\Delta y_{\varepsilon,k}\|_{L^2(\Omega^k(y_{\varepsilon,k}))}\\ &\overset{\text{by (5.20)–(5.21)}}{\leqslant} C_\Omega\left[\alpha^{-\frac{1}{p}}|\Omega|^{\frac{p-2}{2p}}\|y_{\varepsilon,k}\|_{\varepsilon,k,u}+\alpha^{-\frac12}\|y_{\varepsilon,k}\|^{\frac{p}{2}}_{\varepsilon,k,u}\right],\end{aligned}$$

where $u\in\mathfrak{A}_{ad}$ is an admissible control and $\Omega^k(y_{\varepsilon,k}):=\{x\in\Omega\ :\ |\Delta y_{\varepsilon,k}(x)|>k\}$ for each $k\in\mathbb{N}$.

To establish the second part of the lemma, let us take a subsequence $\{y_{\varepsilon_i,k_i}\}_{i\in\mathbb{N}}$ of $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ (here, $\varepsilon_i\to0$ and $k_i\to\infty$ as $i\to\infty$) and a function $y\in W^{2,2}_0(\Omega;\Gamma_D)$ such that $y_{\varepsilon_i,k_i}\rightharpoonup y$ in $W^{2,2}_0(\Omega)$ as $i\to\infty$. Further, we fix an index $i\in\mathbb{N}$ and associate it with the following set

$$B_i:=\bigcup_{j=i}^{\infty}\Omega_{k_j}(y_{\varepsilon_j,k_j}),\ \text{where}\ \ \Omega_{k_j}(y_{\varepsilon_j,k_j}):=\left\{x\in\Omega\ :\ |\Delta y_{\varepsilon_j,k_j}(x)|>\sqrt{k_j^2+1}\right\}. \tag{6.2}$$

Due to estimates (5.10) and (5.23), we see that

$$\begin{aligned}|B_i| &\leqslant \alpha^{-1}\sum_{j=i}^{\infty}\frac{1}{k_j^p}\|y_{\varepsilon_j,k_j}\|^p_{\varepsilon_j,k_j,u_{\varepsilon_j,k_j}}\leqslant\alpha^{-1}\sup_{j\in\mathbb{N}}\|y_{\varepsilon_j,k_j}\|^p_{\varepsilon_j,k_j,u_{\varepsilon_j,k_j}}\sum_{j=i}^{\infty}\frac{1}{k_j^p}\\ &\leqslant\alpha^{-1}\max\left\{C_f^2,C_f^{\frac{p}{p-1}}\right\}\sum_{j=i}^{\infty}\frac{1}{k_j^p}<+\infty,\end{aligned}$$

and, therefore,

$$\lim_{i\to\infty} |B_i| = \mathcal{L}^N(\limsup_{i\to\infty} B_i) = 0. \tag{6.3}$$

Using again (5.23), we get

$$\begin{aligned}\int_{\Omega\setminus B_i} |\Delta y_{\varepsilon_j,k_j}|^p\,dx &\leqslant \int_{\Omega\setminus B_i} \left(\varepsilon_j + |\Delta y_{\varepsilon_j,k_j}|^2\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon_j,k_j}|^2\,dx \\ &\leqslant \alpha^{-1}\int_{\Omega\setminus B_i} \left(\varepsilon_j + \mathcal{F}_{k_j}(|\Delta y_{\varepsilon_j,k_j}|^2)\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon_j,k_j}|^2 u_{\varepsilon_j,k_j}\,dx \\ &\leqslant \alpha^{-1}\max\left\{C_f^2, C_f^{\frac{p}{p-1}}\right\}, \quad \forall j \geqslant i, \end{aligned} \tag{6.4}$$

hence $\{\Delta y_{\varepsilon_j,k_j}\}$ is bounded in $L^p(\Omega\setminus B_i)^N$. Since, $\Delta y_{\varepsilon_j,k_j} \rightharpoonup \Delta y$ in $L^2(\Omega)$, we infer $\chi_{\Omega\setminus B_j}\Delta y_{\varepsilon_j,k_j} \rightharpoonup \Delta y$ in $L^p(\Omega)$, where $\chi_{\Omega\setminus B_j}$ is the characteristic function of the set $\Omega\setminus B_j$. Hence, we obtain

$$\begin{aligned}\int_\Omega |\Delta y|^p\,dx &\overset{\text{by (6.3)}}{=} \lim_{i\to\infty}\int_{\Omega\setminus B_i} |\Delta y|^p\,dx \leqslant \lim_{i\to\infty}\liminf_{\substack{j\to\infty\\ j\geqslant i}} \int_{\Omega\setminus B_i} |\Delta y_{\varepsilon_j,k_j}|^p\,dx \\ &\overset{\text{by (6.4)}}{\leqslant} \alpha^{-1}\max\left\{C_f^2, C_f^{\frac{p}{p-1}}\right\}.\end{aligned}$$

Since $y\in W_0^{2,2}(\Omega;\Gamma_D)$, it follows from the last estimate that $y\in W_0^{2,p}(\Omega;\Gamma_D)$ and this concludes the proof. □

Lemma 6.2. *Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ and $\{u_i\}_{i\in\mathbb{N}}\subset\mathfrak{A}_{ad}$ be sequences such that*

$$\varepsilon_i\to 0,\quad k_i\to\infty,\quad u_i\to u \text{ strongly in } L^1(\Omega).$$

Let $y_i = y_{\varepsilon_i,k_i}(u_i)$ and $y = y(u)$ be the solutions of (5.3)-(5.5) and (3.2)-(3.3), respectively. Then

$$y_i \rightharpoonup y \quad in \quad W_0^{2,2}(\Omega) \quad as \quad i\to\infty, \tag{6.5}$$

$$\chi_{\Omega\setminus\Omega_k(y_i)}\Delta y_i \to \Delta y \text{ strongly in } L^p(\Omega), \tag{6.6}$$

$$\lim_{i\to\infty}\int_\Omega \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i\,dx = \int_\Omega |\Delta y|^p u\,dx, \tag{6.7}$$

where $\Omega_{k_i}(y_i)$ is defined by (6.2).

Proof. The proof is divided into five steps.

Step 1: $y_i \rightharpoonup y$ in $W_0^{2,2}(\Omega)$.- From Lemma 6.1 we deduce the existence of a subsequence, denoted in the same way $\{y_i\}_{i\in\mathbb{N}}\subset W_0^{2,2}(\Omega;\Gamma_D)$ and an element $y\in W_0^{2,p}(\Omega;\Gamma_D)$ such that $y_i\rightharpoonup y$ in $W_0^{2,2}(\Omega)$. Let us prove that y is the solution of (3.2)-(3.3). Let us fix an arbitrary test function $\varphi\in C_0^\infty(\Omega;\Gamma_D)$ and pass to the limit in the Minty inequality

$$\int_\Omega u_i(x)(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi\,(\Delta\varphi - \Delta y_i)\,dx \geqslant \int_\Omega f(\varphi - y_i)\,dx, \tag{6.8}$$

as $i \to \infty$. Taking into account that

$$\begin{gathered}(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi \to |\Delta\varphi|^{p-2}\Delta\varphi \ \text{ strongly in } L^r(\Omega), \text{ for all } 1 \leqslant r < \infty,\\ u_i \to u \ \text{ strongly in } L^r(\Omega), \text{ for all } 1 \leqslant r < \infty,\\ \Delta y_i \rightharpoonup \Delta y \ \text{ in } L^2(\Omega),\end{gathered}$$

we obtain

$$\begin{gathered}\lim_{i\to\infty}\int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}|\Delta\varphi|^2 u_i\,dx = \int_\Omega |\Delta\varphi|^p u\,dx,\\ \lim_{i\to\infty}\int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi\Delta y_i u_i\,dx = \int_\Omega |\Delta\varphi|^{p-2}\Delta\varphi\Delta y u\,dx.\end{gathered}$$

Thus, passing to the limit in relation (6.8) as $i \to \infty$, we arrive at the inequality (3.7) for every $\varphi \in C_0^\infty(\Omega;\Gamma_D)$. From density of $C_0^\infty(\Omega;\Gamma_D)$ in $W_0^{2,p}(\Omega;\Gamma_D)$, we infer that (3.7) holds for every $\varphi \in W_0^{2,p}(\Omega;\Gamma_D)$, and hence $y \in W_0^{2,p}(\Omega;\Gamma_D)$ is the solution to the boundary value problem (3.2)–(3.3) in the sense of distributions. Since the solution of (3.2)-(3.3) is unique, the whole sequence $\{y_i\}_{i\in\mathbb{N}}$ converges weakly to $y = y(u)$ in $W_0^{2,2}(\Omega)$.

Step 2: $\chi_{\Omega\setminus\Omega_k(y_i)}\Delta y_i \rightharpoonup \Delta y$ *in* $L^p(\Omega)$.- Following the definition of the sets $\Omega_{k_i}(y_i)$ and using (5.23), we obtain

$$\begin{aligned}\int_\Omega |\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i|^p\,dx &= \int_{\Omega\setminus\Omega_{k_i}(y_i)} |\Delta y_i|^p\,dx\\ &\leqslant \alpha^{-1}\int_{\Omega\setminus\Omega_{k_i}(y_i)} \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}}|\Delta y_i|^2 u_i\,dx,\\ &\leqslant \alpha^{-1}\|y_i\|^p_{\varepsilon_i,k_i,u_i} \leqslant C < +\infty, \quad \forall\, i \in \mathbb{N}.\end{aligned}$$

Hence, taking a new subsequence if necessary, we infer the existence of a function $g \in L^p(\Omega)$ such that $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i \rightharpoonup g$ in $L^p(\Omega)$ as $i \to \infty$. Since $u_i \to u$ in $L^{p'}(\Omega)$, we conclude that

$$\lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)} \Delta y_i\varphi u_i\,dx = \int_\Omega g\varphi u\,dx, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{6.9}$$

On the other hand, in view of the weak convergence $\Delta y_i \rightharpoonup \Delta y$ in $L^2(\Omega)$,

$$\begin{aligned}\int_\Omega \Delta y\varphi u\,dx &= \lim_{i\to\infty}\int_\Omega \Delta y_i\varphi u_i\,dx\\ &= \lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)} \Delta y_i\varphi u_i\,dx + \lim_{i\to\infty}\int_{\Omega_{k_i}(y_i)} \Delta y_i\varphi u_i\,dx.\end{aligned} \tag{6.10}$$

Since

$$\left|\int_{\Omega_{k_i}(y_i)} \Delta y_i \varphi u_i\, dx\right| \leqslant \|u_i\|_{L^\infty(\Omega)} \|\varphi\|_{C(\overline{\Omega})} \sqrt{|\Omega_{k_i}(y_i)|} \left(\int_{\Omega_{k_i}(y_i)} |\Delta y_i|^2\, dx\right)^{1/2}$$
$$\leqslant \frac{\|u_i\|_{L^\infty(\Omega)} \|\varphi\|_{C(\overline{\Omega})}}{\left(\varepsilon_i + k_i^2 + 1\right)^{\frac{p-2}{4}}} \sqrt{|\Omega_{k_i}(y_i)|} \|y_i\|^{\frac{p}{2}}_{\varepsilon_i, k_i, u_i}$$
$$\overset{\text{by (5.10),(5.23)}}{\leqslant} \|\xi_2\|_{L^\infty(\Omega)} \|\varphi\|_{C(\overline{\Omega})} \frac{C}{k_i^{p-1}} \to 0 \quad \text{as} \quad i \to \infty,$$

it follows from (6.9) and (6.10) that

$$\int_\Omega g \varphi u\, dx = \int_\Omega \Delta y \varphi u\, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Hence, $g = \Delta y$ almost everywhere in Ω and the convergence $\chi_{\Omega \setminus \Omega_k(y_i)} \Delta y_i \rightharpoonup \Delta y$ in $L^p(\Omega)$ holds.

Step 3: $\chi_{\Omega \setminus \Omega_k(y_i)} \Delta y_i \to \Delta y$ *in* $L^p(\Omega)$.- For each $i \in \mathbb{N}$, we have the energy equalities

$$\begin{aligned} \int_\Omega u_i (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2))^{\frac{p-2}{2}} |\Delta y_i|^2\, dx &= \int_\Omega f y_i\, dx, \\ \int_\Omega u(x) |\Delta y|^p\, dx &= \int_\Omega f y\, dx. \end{aligned} \tag{6.11}$$

From (6.11) and the fact that $y_i \rightharpoonup y$ in $W_0^{2,2}(\Omega)$, we deduce

$$\begin{aligned} \lim_{i\to\infty} \int_\Omega u_i (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2))^{\frac{p-2}{2}} |\Delta y_i|^2\, dx &= \lim_{i\to\infty} \left[\int_\Omega f y_i\, dx\right] \\ &= \int_\Omega f y\, dx \overset{\text{by } (6.11)_2}{=} \int_\Omega u |\Delta y|^p\, dx. \end{aligned} \tag{6.12}$$

Moreover, we have

$$\begin{aligned} &\int_\Omega u |\Delta y|^p\, dx = \lim_{i\to\infty} \int_\Omega \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i\, dx \\ &\geqslant \limsup_{i\to\infty} \int_{\Omega \setminus \Omega_{k_i}(y_i)} \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i\, dx \\ &\overset{\text{by (5.7)}}{\geqslant} \limsup_{i\to\infty} \int_{\Omega \setminus \Omega_{k_i}(y_i)} \left(\varepsilon_i + |\Delta y_i|^2\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i\, dx \\ &\geqslant \limsup_{i\to\infty} \int_\Omega \chi_{\Omega \setminus \Omega_{k_i}(y_i)} |\Delta y_i|^p u_i\, dx \geqslant \liminf_{i\to\infty} \int_\Omega \chi_{\Omega \setminus \Omega_{k_i}(y_i)} |\Delta y_i|^p u_i\, dx. \end{aligned} \tag{6.13}$$

Since $u_i \to u$ in $L^r(\Omega)$ for every $1 \leqslant r < +\infty$, $\{u_i\}_i$ is bounded in $L^\infty(\Omega)$ and $u_i(x) \geqslant \alpha$ for almost all $x \in \Omega$, it is easy to check that $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p} \rightharpoonup \Delta y u^{1/p}$ in $L^p(\Omega)$. Using this convergence and (6.13) we get

$$\int_\Omega u|\Delta y|^p\,dx \geqslant \limsup_{i\to\infty}\int_\Omega u_i\chi_{\Omega\setminus\Omega_{k_i}(y_i)}|\Delta y_i|^p\,dx$$
$$\geqslant \liminf_{i\to\infty}\int_\Omega u_i\chi_{\Omega\setminus\Omega_{k_i}(y_i)}|\Delta y_i|^p\,dx = \liminf_{i\to\infty}\|\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p}\|^p_{L^p(\Omega)}$$
$$\geqslant \|\Delta y u^{1/p}\|^p_{L^p(\Omega)} = \int_\Omega u|\Delta y|^p\,dx.$$

The weak convergence $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p} \rightharpoonup \Delta y u^{1/p}$ in $L^p(\Omega)$ and the convergence of their norms $\|\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p}\|_{L^p(\Omega)} \to \|\Delta y u^{1/p}\|_{L^p(\Omega)}$ imply the strong convergence $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p} \to \Delta y u^{1/p}$ in $L^p(\Omega)$. Now, it is a simple exercise to check the strong convergence $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i \to \Delta y$ in $L^p(\Omega)$.

Step 4: Proof of (6.7).- From (6.6) and (6.13) we obtain

$$\lim_{i\to\infty}\int_{\Omega_{k_i}(y_i)}\left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}}|\Delta y_i|^2 u_i\,dx = 0. \tag{6.14}$$

Let us prove that

$$\lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)}\left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}}|\Delta y_i|^2 u_i\,dx = \int_\Omega|\Delta y|^p u\,dx. \tag{6.15}$$

This is established as follows. From (5.7) we deduce

$$\left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}}|\Delta y_i|^2\chi_{\Omega\setminus\Omega_{k_i}(y_i)}$$
$$\leqslant (\varepsilon_i + \delta + |\Delta y_i|^2)^{\frac{p-2}{2}}|\Delta y_i|^2\chi_{\Omega\setminus\Omega_{k_i}(y_i)}$$
$$\leqslant 2^{\frac{p-2}{2}}((\varepsilon_i+\delta)^{\frac{p-2}{2}}|\Delta y_i|^2 + |\Delta y_i|^p)\chi_{\Omega\setminus\Omega_{k_i}(y_i)}.$$

From (6.6) we know that the last term converges in $L^1(\Omega)$. Taking a subsequence if necessary we can dominate it by a $L^1(\Omega)$ function. Then by a simple application of Lebesgue's dominated convergence theorem we deduce (6.15). Finally, (6.14) and (6.15) imply (6.7).

Step 5: $y_i \to y$ in $W_0^{2,2}(\Omega)$.- First, we apply (6.14) to deduce

$$\lim_{i\to\infty}\int_{\Omega_k(y_i)}|\Delta y_i|^2\,dx \leqslant \frac{1}{\alpha}\lim_{i\to\infty}\int_{\Omega_k(y_i)}(\varepsilon_i + \mathcal{F}_k(|\Delta y_i|^2))^{\frac{p-2}{2}}|\Delta y_i|^2 u_i\,dx = 0.$$

Now, combining this estimate and (6.6) we conclude that

$$\Delta y_i = \chi_{\Omega_k(y_i)}\Delta y_i + \chi_{\Omega\setminus\Omega_k(y_i)}\Delta y_i \to \Delta y \ \text{ strongly in } \ L^2(\Omega).$$

□

We are now in a position to show that optimal pairs to the approximated OCP $(\mathbb{P}_{\varepsilon,k})$ lead in the limit to optimal solutions of the original OCP $(\mathbb{P})$.

Theorem 6.1. *Let $\Big\{(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})\Big)\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ be an arbitrary sequence of optimal pairs to the approximating problems $(\mathbb{P}_{\varepsilon,k})$. Then, this sequence is bounded in $BV(\Omega)\times W^{2,2}_0(\Omega)$ and any cluster point (u^0, y^0) with respect to the (weak-*,weak) topology is a solution of the OCP $(\mathbb{P})$. Moreover, if for one subsequence we have $u^0_{\varepsilon,k} \overset{*}{\rightharpoonup} u^0$ in $BV(\Omega)$ and $y^0_{\varepsilon,k} \rightharpoonup y^0$ in $W^{2,2}_0(\Omega)$, then the following properties hold*

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} (u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) = (u^0, y^0) \ \ \textit{strongly in} \ \ L^1(\Omega)\times W^{2,2}_0(\Omega), \tag{6.16}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega |Du^0_{\varepsilon,k}| = \int_\Omega |Du^0|, \tag{6.17}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})}\Delta y^0_{\varepsilon,k} = \Delta y^0 \ \ \textit{strongly in} \ \ L^p(\Omega), \tag{6.18}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\Delta y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\, dx = \int_\Omega |\Delta y^0|^p u^0\, dx, \tag{6.19}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) = I(u^0, y^0). \tag{6.20}$$

Proof. The boundedness of $\Big\{(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})\Big)\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ in $BV(\Omega)\times W^{2,2}_0(\Omega)$ is an immediate consequence of the boundedness of $\mathfrak{A}_{ad}$ in $BV(\Omega)$ and Lemma 6.1. Let us take a subsequence, denoted in the same way, such that $u^0_{\varepsilon,k} \overset{*}{\rightharpoonup} u^0$ in $BV(\Omega)$ and $y^0_{\varepsilon,k} \rightharpoonup y^0$ in $W^{2,2}_0(\Omega)$. From compactness property of BV-bounded sequences, we get that

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} u^0_{\varepsilon,k} = u^0 \ \ \text{strongly in} \ \ L^1(\Omega) \ \ \text{and} \ \ \int_\Omega |Du^0| \leqslant \liminf_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega |Du^0_{\varepsilon,k}|. \tag{6.21}$$

From this convergence properties we infer that $u^0 \in \mathfrak{A}_{ad}$. Moreover, Lemma 6.2 implies that y^0 is the solution of (3.2)-(3.3) corresponding to $u = u^0$, therefore, in virew of (6.1), we deduce that $(u^0, y^0) \in \Xi$. Combining (6.5) and (6.21) we deduce (6.16). Convergences (6.18) and (6.19) follow from (6.6) and (6.7). Let us prove that (u^0, y^0) is a solution of $(\mathbb{P})$. Given an arbitrary element $(u, y) \in \Xi$, we define $u_{\varepsilon,k} = u$ and $y_{\varepsilon,k}$ as the solution of (5.2)-(5.3), hence $(u_{\varepsilon,k}, y_{\varepsilon,k}) \in \Xi_{\varepsilon,k}$. From (6.5) and (6.7) we get

$$I(u,y) = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u, y_{\varepsilon,k}) = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u_{\varepsilon,k}, y_{\varepsilon,k}).$$

Now, using (6.5), (6.16), (6.21), the above identity and the fact that $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})$ is a solution of $(\mathbb{P}_{\varepsilon,k})$, we get

$$\begin{aligned} I(u^0,y^0) &\leqslant \liminf_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^0_{\varepsilon,k},y^0_{\varepsilon,k}) \leqslant \limsup_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^0_{\varepsilon,k},y^0_{\varepsilon,k}) \\ &\leqslant \limsup_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u_{\varepsilon,k},y_{\varepsilon,k}) = I(u,y). \end{aligned}$$

Since (u,y) is arbitrary in Ξ, this implies that (u^0,y^0) is a solution of $(\mathbb{P})$. Moreover, taking $(u,y)=(u^0,y^0)$ in the above inequalities, (6.20) is proved. Finally, (6.17) is an immediate consequence of (6.20) and the convergence properties established before. □

Since Theorem 6.1 does not give an answer whether the entire set of solutions Ξ^{opt} to problem (3.2)–(3.6) can be attained in such a way, the following result shed some light on this matter.

Corollary 6.1. *Let $(u^0,y^0)\in\Xi^{opt}$ be an optimal solution to the OCP $(\mathbb{P})$ such that there is a closed neighborhood $\mathcal{U}(u^0)$ of u^0 in the norm topology of $L^1(\Omega)$ satisfying*

$$I(u^0,y^0) < I(u,y) \quad \forall\, u\in\mathfrak{A}_{ad}\cap\mathcal{U}(u^0) \text{ such that } (u,y)\in\Xi \text{ and } u\neq u^0. \tag{6.22}$$

Then there exists a sequence of local minima $(u^0_{\varepsilon,k},y^0_{\varepsilon,k})$ of problems $(\mathbb{P}_{\varepsilon,k})$ such that

$$(u^0_{\varepsilon,k},y^0_{\varepsilon,k}) \to (u^0,y^0) \quad \text{in the sense of Theorem 6.1.}$$

Proof. By the strict local optimality of (u^0,y^0), we have that it is the unique solution of

$$(\mathbb{Q}) \qquad \min_{(u,y)\in\Xi,\, u\in\mathcal{U}(u^0)} I(u,y).$$

For every ε and k let us consider the control problems

$$(\mathbb{Q}_{\varepsilon,k}) \qquad \min_{(u,y)\in\Xi_{\varepsilon,k},\, u\in\mathcal{U}(u^0)} I(u,y).$$

Since $(u^0,y_{\varepsilon,k}(u^0))\in\Xi_{\varepsilon,k}$, it follows that $(\mathbb{Q}_{\varepsilon,k})$ has feasible controls, hence there exists at least one solution $(u^0_{\varepsilon,k},y^0_{\varepsilon,k})$ of $(\mathbb{Q}_{\varepsilon,k})$ for every (ε,k). Now, arguing as in the proof of Theorem 6.1, we deduce that $(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\to(\tilde{u}^0,\tilde{y}^0)$ strongly in $L^1(\Omega)\times W^{2,2}_0(\Omega)$, and $(\tilde{u}^0,\tilde{u}^0)$ is the unique solution of $(\mathbb{Q})$. This implies the existence of ε^0 and k^0 such that $u^0_{\varepsilon,k}$ belongs to the interior of $\mathcal{U}(u^0)$ for every $\varepsilon\leqslant\varepsilon^0$ and $k\geqslant k^0$. Consequently, $(u^0_{\varepsilon,k},y^0_{\varepsilon,k})$ is a local minimum of $(\mathbb{P}_{\varepsilon,k})$ for every $\varepsilon\leqslant\varepsilon^0$ and $k\geqslant k^0$. This concludes the proof. □

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ON A REPRESENTATION OF THE SOLUTION TO THE DIRICHLET PROBLEM IN A DISK. THE POISSON INTEGRAL BASED SOLUTION IN POLYNOMIALS

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Communicated by Prof. P. I. Kogut

Abstract. The representation $u(\boldsymbol{x}) = F_2(\boldsymbol{x})\,Q_{m-2}(\boldsymbol{x}) + Q_m(\boldsymbol{x})$ for the solution to the Dirichlet problem for the Laplace equation in a disk: $F_2(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_0|^2 - c^2 \leqslant 0$, is proved using the Poisson integral; $Q_m(\boldsymbol{x})$ being the polynomial boundary function of degree m, $Q_{m-2}(\boldsymbol{x})$ being the uniquely determined polynomial of degree $m-2$.

Key words: the Dirichlet problem, the Poisson integral.

2010 Mathematics Subject Classification: 31A25, 31B20, 35A09, 35G15, 35J25.

1. Introduction

Consider the well known Dirichlet problem for the Laplace equation in a disk of radius c centered at point $\boldsymbol{x}_0$ in the plane $\mathbb{R}^2$ parameterized by the cartesian orthogonal coordinates $\boldsymbol{x} = (x_1, x_2)$

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0\,, & \boldsymbol{x} \in \mathcal{B}_c^2(\boldsymbol{x}_0) := \big\{\boldsymbol{x}\colon |\boldsymbol{x} - \boldsymbol{x}_0|^2 < c^2\big\}\,, \\ u(\boldsymbol{x}) = u_0(\boldsymbol{x})\,, & \boldsymbol{x} \in \mathcal{S}_c^2(\boldsymbol{x}_0) := \big\{\boldsymbol{x}\colon |\boldsymbol{x} - \boldsymbol{x}_0|^2 = c^2\big\}\,, \end{cases} \tag{1.1}$$

where the boundary function $u_0(\boldsymbol{x}) \in \mathscr{PC}\big(\mathcal{S}_c^2(\boldsymbol{x}_0)\big)$.

The unique solution $u(\boldsymbol{x}) \in \mathscr{C}^a\big(\mathcal{B}_c^2(\boldsymbol{x}_0)\big) \bigcap \mathscr{PC}\big(\overline{\mathcal{B}_c^2(\boldsymbol{x}_0)}\big)$ to the problem is known to have some representations [3], for example, $a)$ as the trigonometric series

$$\mathring{u}(r,\varphi) = \frac{a_0}{2} + \sum_{\mu=1}^{\infty} \left(\frac{r}{c}\right)^{\mu} \Big(a_\mu \cos(\mu\varphi) + b_\mu \cos(\mu\varphi)\Big), \tag{1.2}$$

where the circle over the function name indicates changing the cartesian coordinates to the polar ones: $x_1 = x_{1,0} + r\cos\varphi$, $x_2 = x_{2,0} + r\sin\varphi$, $(r,\varphi) \in \overline{\mathcal{B}_c^2(\boldsymbol{x}_0)}$; a_0, a_μ, and b_μ are the Fourier coefficients for $\mathring{u}_0(\varphi)$; $b)$ as the Poisson integral

$$\mathring{u}(r,\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathring{u}_0(\theta)\,(c^2 - r^2)\,\mathrm{d}\theta}{c^2 - 2cr\cos(\theta - \varphi) + r^2}\,, \tag{1.3}$$

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being the convolution of the boundary function and the Poisson kernel, or c) as the real part of the Cauchy integral.

But what happens to the solution to the Dirichlet problem (1.1) when the boundary function is a polynomial

$$u_0(\boldsymbol{x}) = Q_m(\boldsymbol{x}) = \sum_{p+q=0}^{m} a_{p,q}\, x_1^p\, x_2^q = \{(x_1, x_2) \to (c, \varphi)\} = \mathring{u}_0(\varphi)\,, \tag{1.4}$$

where $p, q \in \mathbb{Z}\backslash\mathbb{Z}_-$, $a_{p,q} \in \mathbb{R}$? It is a very simple question to be answered quickly. The solution is of course a polynomial of the same degree m as the boundary function $Q_m(\boldsymbol{x})$ (1.4). But has the polynomial solution a morphology suitable for checking the solution to be valid? The question had puzzled us in academic year 2012 – 2013 we started as the lecturer and the instructor in the partial differential equations course at the Faculty of Mech & Math of DNU. Setting up the tutorial Dirichlet problems in polynomials we tried to compose the solution manual in such a way to check the solutions to the problems not pointwise but functionally. The linear boundary functions are exactly the solutions to the problems provided the domains of definition of the boundary functions are extended form $\mathcal{S}_c^2(\boldsymbol{x}_0)$ to $\overline{\mathcal{B}_c^2(\boldsymbol{x}_0)}$. The quadratic boundary functions $Q_2(\boldsymbol{x})$ lead to the quadratic solutions and are easily represented as follows

$$u(\boldsymbol{x}) = F_2(\boldsymbol{x})\, b_0 + Q_2'(\boldsymbol{x})\,, \tag{1.5}$$

where b_0 are the uniquely determined constants, and the prime over the boundary functions is explained below in the formulation of the proposition 1.1.

But what about the solution to the Dirichlet problem when the boundary function is a polynomial of the degree higher than second? We had thoroughly studied all the known to us textbooks and solution manuals on the subject in Russian and English but in vain. We had been amazed that no one of the above textbooks or solution manuals answers the question. Therefore, we had to conjecture that the morphology of the solution remains the same as that given by the formula (1.5) where the constant b_0 is replaced with a polynomial $P_{m-2}(\boldsymbol{x})$ of the order $m-2$. The conjecture had been formulated in [2] as the following

Proposition 1.1. Solution to the Dirichlet problem (1.1), where the boundary function is a polynomial $Q_m(\boldsymbol{x})$ (1.4), admits the following representation

$$u(\boldsymbol{x}) = F_2(\boldsymbol{x})\, Q_{m-2}(\boldsymbol{x}) + Q_m'(\boldsymbol{x})\,, \tag{1.6}$$

where the polynomial $F_2(\boldsymbol{x})$ of second degree specifies the boundary of the disk: $F_2(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_0|^2 - c^2 = 0$, $Q_{m-2}(\boldsymbol{x})$ is the uniquely determined polynomial of degree $m-2$, and $Q_m'(\boldsymbol{x})$ is the evident extension of the boundary function $Q_m(\boldsymbol{x})$ from $\mathcal{S}_c^2(\boldsymbol{x}_0)$ to $\overline{\mathcal{B}_c^2(\boldsymbol{x}_0)}$ (this means that c is replaced with $r \in [0, c]$ in (1.4)).

Applying some environments allowing symbolic algebraic manipulations we had successfully tested the conjecture using a lot of the boundary functions, including those of very high degree. Finally, we had proved the proposition using the direct approach based on the trigonometric series representation (1.2) for the solution to the Dirichlet problem and published the proof in [2].

Then we had succeeded in proving the proposition in quite different ways, say, applying the symmetry methods [1, 5, 6]. But when proving the above proposition we had turned out to be involved in the problem of finding the morphology of the Neumann problem in a disk. We had tried to find the representation formula to the Neumann problem posed in polynomials and had found that the integral formula for the solution known in $\mathbb{R}^2$ as the Dini integral suits well for this. So, completing our exercises with the Dirichlet problem in a disk posed in polynomials we'd like to present a proof of the above statement fully based on the Poisson integral for the Dirichlet problem.

2. Proving the representation

Proof. Firstly, we expand the Poisson kernel into the series [3]

$$\begin{aligned} P(\theta; r, \varphi) &= \frac{c^2 - r^2}{c^2 - 2cr\cos\theta + r^2} \\ &= \frac{1-\varrho^2}{1 - 2\varrho\,\cos\theta + \varrho^2} = \left\{ z = \varrho\, e^{i\theta} \right\} \\ &= -1 + \frac{1}{1-z} + \frac{1}{1-\bar{z}} = -1 + \sum_{\gamma=0}^{\infty} z^\gamma + \sum_{\gamma=0}^{\infty} \bar{z}^\gamma \\ &= 1 + 2\sum_{\gamma=1}^{\infty} \varrho^\gamma \cos(\gamma\theta) = 1 + 2\sum_{\gamma=1}^{\infty} \left(\frac{r}{c}\right)^\gamma \cos(\gamma\theta), \end{aligned}$$

absolutely and uniformly convergent in the disk $\mathcal{B}_c^2(\boldsymbol{x}_0)$ and rewrite the Poisson integral (1.3) as follows

$$\mathring{u}(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \mathring{u}_0(\theta)\, P(\theta-\varphi; r, \varphi)\,\mathrm{d}\theta = I_0(\varphi) + 2\sum_{\gamma=1}^{\infty}\left(\frac{r}{c}\right)^\gamma I_\gamma(\varphi), \qquad (2.1)$$

where the integral terms are

$$I_\gamma(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \mathring{u}_0(\theta)\, \cos[\gamma\,(\theta-\varphi)]\,\mathrm{d}\theta, \qquad \gamma \in \mathbb{Z}\backslash\mathbb{Z}_- . \qquad (2.2)$$

Secondly, we consider the monomials $x_1^p\, x_2^q \subseteq Q_m(\boldsymbol{x}) = u_0(\boldsymbol{x})$ (1.4), where $2 \leqslant p+q \leqslant m$, separately, accounting for the following cases to be possible: 1) $p+q$ is an odd: a) p is an odd, $q=0$; b) $p=0$, q is an odd; c) p is an odd, q is an even; d) p is an even, q is an odd; 2) $p+q$ is an even: a) p is an even, $q=0$; b) $p=0$, q is an even; c) p is an odd, q is an odd; d) p is an even, q is an even, and for the sake of brevity assume that $\boldsymbol{x}_0=\mathbf{0}$.

Let p is an odd: $p \geqslant 3$, and $q=0$, then [4]

$$u_0(\boldsymbol{x}) = x_1^p = c^p \cos^p \varphi = \frac{c^p}{2^{p-1}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \cos\left[(p-2\mu)\,\varphi\right] = \mathring{u}_0(\varphi)\,, \tag{2.3}$$

and we calculate the integral terms (2.2) to be

$$I_0(\varphi) = \frac{1}{4\pi}\,\frac{c^p}{2^{p-1}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \int_0^{2\pi} \cos\left[(p-2\mu)\,\theta\right] \mathrm{d}\theta = 0\,,$$

$$\begin{aligned} I_\gamma(\varphi) &= \frac{1}{4\pi}\,\frac{c^p}{2^{p-1}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \int_0^{2\pi} \cos\left[(p-2\mu+\gamma)\,\theta - \gamma\varphi\right] \mathrm{d}\theta \\ &+ \frac{1}{4\pi}\,\frac{c^p}{2^{p-1}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \int_0^{2\pi} \cos\left[(p-2\mu-\gamma)\,\theta + \gamma\varphi\right] \mathrm{d}\theta \\ &= \frac{1}{2}\,\frac{c^p}{2^{p-1}}\, C_p^{\frac{p-\gamma}{2}} \cos(\gamma\varphi), \quad \gamma = p-2\mu, \quad 2\mu = 0,\,\ldots,\,p-1\,. \end{aligned}$$

Since $(p-2\mu) \in \mathbb{N}$, all the integrals of the formula for the term $I_0(\varphi)$ vanish. The same is true for all the integrals in the first sum of the formula for the terms $I_\gamma(\varphi)$, and for $p-2\mu-\gamma \in \mathbb{N}$ and the corresponding integrals in the second sum of the formula. But when $\gamma = p-2\mu$, then $\cos\left[(p-2\mu-\gamma)\,\theta+\gamma\varphi\right] = \cos(\gamma\varphi)$, and the corresponding integrals in the second sum are equaled to $2\pi \cos(\gamma\varphi)$.

Thirdly, we substitute the above non-zero integral terms $I_\gamma(\varphi)$ into the integral formula (2.1)

$$\begin{aligned} \mathring{u}(r,\varphi) &= \frac{c^p}{2^{p-1}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \left(\frac{r}{c}\right)^\gamma \cos(\gamma\varphi) \\ &= \frac{r^p}{2^{p-1}} \cos(p\varphi) + \frac{1}{2^{p-1}} \sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^{p-2\mu}\, c^{2\mu} \cos\left[(p-2\mu)\,\varphi\right] \end{aligned}$$

and rearrange the last sum as follows

$$\sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^{p-2\mu}\, c^{2\mu} \cos\left[(p-2\mu)\,\varphi\right] = \sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^{p-2\mu} \left(c^{2\mu} \mp r^{2\mu}\right) \cos\left[(p-2\mu)\,\varphi\right]$$

$$= \sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^p \cos\left[(p-2\mu)\,\varphi\right] + \sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^{p-2\mu} \left(c^{2\mu} - r^{2\mu}\right) \cos\left[(p-2\mu)\,\varphi\right]$$

$$= \sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^p \cos\left[(p-2\mu)\,\varphi\right] - \left(r^2 - c^2\right) \sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^{p-2\mu}\, A_\mu(r) \cos\left[(p-2\mu)\,\varphi\right],$$

where the factorization of the binoms $c^{2\mu} - r^{2\mu}$ is used, and

$$A_\mu(r) = \begin{cases} 1\,, & \mu = 1\,, \\ r^{2\mu-2} + c^2 r^{2\mu-4} + \ldots + c^{2\mu-4} r^2 + c^{2\mu-2}\,, & \mu > 1\,. \end{cases}$$

Gathering all the terms, we obtain the solution to the Dirichlet problem (1.1)

$$\begin{aligned} \mathring{u}(r,\varphi) &= \frac{r^p}{2^{p-1}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \cos\left[(p-2\mu)\,\varphi\right] \\ &- \frac{r^2 - c^2}{2^{p-1}} \sum_{\mu=1}^{\frac{p-1}{2}} C_p^\mu\, r^{p-2\mu}\, A_\mu(r) \cos\left[(p-2\mu)\,\varphi\right] \\ &\overset{(2.3)}{=} \mathring{u}_0'(r,\varphi) + \mathring{F}_2(r,\varphi)\, \mathring{Q}_{p-2}(r,\varphi)\,. \end{aligned}$$

The other monomials $x_1^p\, x_2^q$ are treated the same way. □

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ON SOME TYPES OF INVERSE PROBLEMS FOR DIFFERENTIAL EQUATIONS

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Abstract. The inverse problems for differential equations are investigated, the solutions of which do not use information about the exact characteristics of the physical process. Such inverse problems have not yet become widespread, but they are of great practical importance. Some approaches to solving inverse problems of this type are suggested.

Key words: differential equations,inverse problems, classification, solution algorithms.

2010 Mathematics Subject Classification: 34A55, 45D05.

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1. Introduction

Consider some types of inverse problems for differential equations [9], [10].

Let the physical process be characterized in the general case by a certain number of variables $x_1, x_2, ..., x_n$(state variables). The choice of the physical process characteristics is determined by the ultimate research goals. Let us assume that the variables $x_1, x_2, ..., x_n$ satisfy a linear system of ordinary differential equations with constant coefficients:

$$\dot{x} = Ex + Fz, \tag{1.1}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ is a vector function of state variables, $z(t) = (z_1(t), z_2(t), \ldots, z_m(t))^T$ is a vector function of external loads (some of components are unknown), $(.)^T$ is transpose sign; $E = \{e_{ik}\}, 1 \leq i, k \leq n, F = \{f_{jl}\}, 1 \leq j \leq n, 1 \leq l \leq m$ are matrices with constant coefficients of the corresponding dimension, $t \in [0, T], [0, T]$ is interval of time where the solution of inverse problem is investigated. By mathematical description of the physical process we consider the set of the system of equations (1.1) with symbols of external loads, the vector function of the external loads as $z(t) = (z_1(t), z_2(t), \ldots, z_m(t))^T$ in the special form for individual problem and the initial condition $x(0) = x^0$. Thus, the mathematical description is a collection of mathematical model, vector function of external loads and initial conditions.

Briefly concerning the justification of such the definition. In inverse problems for the system (1.1) there are two main classes of inverse problems: - the determination of the coefficients of the matrices E, F using the some experimentally determined components of the vector function x(t), the given initial conditions and the

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vector function $z(t)$; - the unknown components of the vector function $z(t)$ are determined under the some experimentally determined of the components of the vector function $x(t)$, the given initial conditions and the coefficients of the matrices E, F [7]. For the purpose of separating these two basic problems it is necessary to introduce the mathematical description of the physical process in the form of a collection of mathematical model (differential equations (1.1)) with symbols of external loads (general form of external loads) and separately the vector function of external loads $z(t)$ in the special form for individual problem [9].

The combination of these two inverse problems into one inverse problem is undesirable since it leads to a high degree of uncertainty. This is confirmed by the results of a review of the literature. Among the mathematical descriptions, one can single out descriptions that give the results of mathematical simulation which coincide with the given experimental measurements of the components of the vector function $x(t)$ with the experiment accuracy. Such descriptions are called adequate mathematical descriptions (AMD) [11].

2. Statement of problem

For system (1.1) it is possible to obtain several types of inverse problems in frame of second class of inverse problems for ordinary differential equations [9], [10], [11]. Let us investigate only two inverse problems from them.

Inverse problems of type I: it is necessary to define all unknown components $z_1(t), z_2(t), \dots, z_m(t)$ of vector function of external loads $z(t)$ on segment $[0, T]$ using given from experiment functions of state variables $x_1(t), x_2(t), \dots, x_n(t)$ on segment $[0, T]$, matrixes E, F and initial conditions aimed at constructing of AMD. Such inverse problems have not yet become widespread, but they have important mean for problems of mathematical modeling [11], as well as for problems of physical processes prediction [16], [15] and control problems [8], [4], [2].

Inverse problems of type II:it is necessary to define all unknown components of vector function of external loads $z_1(t), \dots, z_m(t)$ on segment $[0, T]$ using some components of vector function of state variables $x_1(t), \dots, x_n(t)$ on segment $[0, T]$ (which are given from experiment) and initial condition $x(0) = x^0$ aimed at obtaining the useful information about exact characteristics of real unknown external loads on system (1.1).

These inverse problems have applications for problems of diagnostics [3], for problems where it is necessary to obtain a new knowledge about the world around us [12].

Suppose that system (1.1) has only one unknown component $z_j(t), 1 < j \leq m$ of vector function $z(t)$ and only one given component $x_1(t)$ of vector function $x(t)$.

Using the linearity of the system (1.1), Volterra's integral equation of the first kind with respect to the unknown function $z_1(t)$ can be obtained with use the additional condition (see later)

$$\int_0^t K(t, \tau) z_j(\tau)\, d\tau = u_{\delta,j}(t), or A_{p,j} z_j = u_{\delta,j}, t \in [0, T], \tag{2.1}$$

where $z_j(t) \in Z, u_{\delta,j} \in U; U, Z$ are some normed functional spaces.

The function $u_{\delta,j}(t)$ is defined in terms of the initial conditions $x(0) = x^0$, the given function $x_1(t)$ and the known components $z_2(t), \dots, z_m(t)$ of the vector function of the external loads $z(t)$ with a predetermined error

$$\| u_{\delta,j} - u_{\delta,j}^{ex} \|_U \leq \delta_j, \tag{2.2}$$

where $u_{\delta,j}^{ex}$ are exact right parts in (2.1).

Notice that inverse problems of type I, II have the same equations for unknown functions $z_j(t)$ [9], [11].

The properties of operators $A_{p,j}$ are saved for inverse problems of both types: operators $A_{p,j}$ are compact operators for typical cases of functional spaces choice. Let us consider the set of possible solutions $Q_{\delta,p,j}$ of equation (2.1):

$$Q_{\delta,p,j} = \{z \in Z : \| A_{p,j} z_j - u_{\delta,j} \|_U \leq \delta_j\}. \tag{2.3}$$

The sets $Q_{\delta,p,j}$ are not bounded at any δ_j while operators $A_{p,j}$ are compact operators. So inverse problems of type I belongs to class of incorrect problems [9], [17] and special methods have to be used for their solution [9], [17].

In inverse problem of type I it is enough to obtain any function from the set $Q_{\delta,p,j}$. The inverse problems with such ultimate goals will be called as the inverse problems of synthesis [9], [11]. Let us consider exact equation z_j^{ex} of exact equation (2.1)

$$A_{p,j}^{ex} z_j^{ex} = u_j^{ex}, 1 < j \leq m, \tag{2.4}$$

where $A_{p,j}^{ex}$ is the exact operator in (2.1), u_j^{ex} is exact right part (initial data).

Besides the exact solution z_j^{ex} of the equation (2.4) may not belong to set of possible solutions $Q_{\delta,p,j}$ since the operator $A_{p,j}$ is being described inexactly of the real physical process.

In such inverse problems there is no need to require the convergence of the approximate solution of the problem to an exact solution z_j^{ex} of equation (2.4) by $\delta_j \to 0$ [10]. Therefore, the approximate solution cannot have properties of regularization [10], [11].

In addition these inverse problems have the following features:

- the approximate solution can considerably differ from the exact solution z_j^{ex} as $z_j^{ex} \notin Q_{\delta,p,j}$;
- the size of an error of the approximate solution in relation to the exact solution z_j^{ex} has no importance for further use of the approximate solution;
- the exact solution z_j^{ex} of an inverse problem in aggregate with initial mathematical model can give worse results of mathematical modeling than the approximate solution as $z_j^{ex} \notin Q_{\delta,p,j}$;

- the error of the operator $A_{p,j}$ to the exact operator $A^{ex}_{p,j}$ is possible not to take into account, as the initial inexact mathematical model of physical process will be used at mathematical modeling further.

The solution of such problems can interpret only as a good model for purposes of mathematical modeling.

In [9], [17] it is assumed that exact solution of equation (2.1) z^{ex}_j belongs to the set of possible solutions $Q_{\delta,p,j}$. This property was used by construction of regularized algorithms. So in case of inverse problems of type I ($z^{ex}_j \notin Q_{\delta,p,j}$) it is necessary to choose another approaches.

Now the conditions will be obtained when the set of possible solution is not empty.

Let us assume that in system (1.1) there is only one state variable $\tilde{x}_1$, which is obtained by experiment and only one unknown component $z_j(t), 1 < j \leq m$ of vector function $z_(t)$ of external loads. The matrix $F = \hat{F} = f_{k,i}, 1 \leq k \leq n, 1 \leq i \leq m$, has the special form: by fixed $l, j\,(1 \leq j \leq m, 2 \leq l \leq m)$ element $\hat{f}_{lj} \neq 0, \hat{f}_{ki} = 0, k \neq l, i \neq j$.

Then first equation of system (1.1) with matrices $E, \hat{F}$ is differentiates $(n-1)$-times on t:

$$\left.\begin{aligned} &\frac{d^2\tilde{x}_1}{dt^2} = B_1E^2\tilde{x}(t) + \hat{f}_{lj}z_j(t),\\ &\frac{d^3\tilde{x}_1}{dt^3} = B_1E^3\tilde{x}(t) + \hat{f}_{lj}\dot{z}_j(t),\\ &\dots\\ &\frac{d^{(n-1)}\tilde{x}_1}{dt^{(n-1)}} = B_1E^{(n-1)}\tilde{x}(t) + \hat{f}_{lj}z_j^{(n-3)}(t), \end{aligned}\right\} \tag{2.5}$$

where matrix-string B_1 is defined as $B_1 = \{b^1_i\}, 1 \leq i \leq n, b^1_1 = 1, b^1_k = 0, k \neq 1, 1 \leq k \leq n$.

System (2.5) and first equation of (1.1) are solved relatively state variables $\tilde{x}_2(t), ..., \tilde{x}_n(t)$ through variables $\tilde{x}_1(t), \dot{\tilde{x}}_1(t), ..., \tilde{x}_1^{(n-1)}(t)$ and is substituted in following equation

$$\frac{d^n\tilde{x}_1}{dt^n} = B_1E^n\tilde{x}(t) + \hat{f}_{lj}z_j^{(n-2)}(t)$$

.

Consequently the next equation was obtained

$$\frac{d^n\tilde{x}_1}{dt^n} = \Phi(t, \tilde{x}_1, \dot{\tilde{x}}_1, \dots, \tilde{x}_1^{(n-1)}(t), \hat{f}_{lj}z_j, \dots, \hat{f}_{lj}z_j^{(n-2)}). \tag{2.6}$$

For solving system from equations (2.5) and first equation of (1.1) relatively state variables $\tilde{x}_2(t), \dots, \tilde{x}_n(t)$ it is sufficient that the Jacobian of such a transformation is nonzero.

Hence this condition has the form

$$J = \frac{D(B_1 E\tilde{x}, B_1 E^2\tilde{x}, \ldots, B_1 E^{n-1}\tilde{x})}{D(\tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n)} =$$

$$= \det \begin{pmatrix} \frac{\partial B_1 E\tilde{x}}{\partial \tilde{x}_2} & \frac{\partial B_1 E\tilde{x}}{\partial \tilde{x}_3} & \ldots & \frac{\partial B_1 E\tilde{x}}{\partial \tilde{x}_n} \\ \frac{\partial B_1 E^2\tilde{x}}{\partial \tilde{x}_2} & \frac{\partial B_1 E^2\tilde{x}}{\partial \tilde{x}_3} & \ldots & \frac{\partial B_1 E^2\tilde{x}}{\partial \tilde{x}_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial B_1 E^{n-1}\tilde{x}}{\partial \tilde{x}_2} & \ldots & \ldots & \frac{\partial B_1 E^{n-1}\tilde{x}}{\partial \tilde{x}_n} \end{pmatrix} \neq 0. \quad (2.7)$$

The differential equation of n -th order (2.6) is obtained under condition (2.7).

$$L_n[\tilde{x}_1(t)] = L_{n-2}[z_j(t)], \quad (2.8)$$

where $L_n[\tilde{x}_1(t)], L_{n-2}[z_j(t)]$ are linear differential operators of n -order and $(n-2)$ -order respectively.

Thus, the sufficient condition for the reduction of the inverse problem for system (1.1) to the inverse problem for one high-order equation (2.8) is condition (2.7).

In inverse problem for equation (2.8) it is necessary to define the function of external load $z_j(t)$, using function of state variable $\tilde{x}_1(t)$ that is obtained from experiment data with error.

Let the inaccuracy of function $\tilde{x}_1(t)$ relatively to exact function $x_1^{ex}(t)$ in metric $C[0,T]$ is given as follow

$$\| \tilde{x}_1(t) - x_1^{ex} \|_{C[0,T]} \leq \delta_1. \quad (2.9)$$

Theorem 2.1. *The set of possible solutions $Q_{\delta,p,j}$ of inverse problem for system* (1.1) *is non-empty if condition* (2.7) *is valid.*

Proof. The inverse problem for system (1.1)(1.1) can be reduced to inverse problem for equation (2.8) if the condition (2.7) is satisfied.

It is well-known that any continuous function $\tilde{x}_1(t)$ can be approximate in metric $C[0,T]$ with any accuracy δ by polynomial of q -order $x_{1,q}(t)$ [1]:

$$\| \tilde{x}_1(t) - x_{1,q}(t) \|_{C[0,T]} \leq \delta, q = q(\delta). \quad (2.10)$$

Initial conditions for function $x_{1,q}(t)$ are values

$$x_{1,q}(0) = x_{1,q}^{0,0}, \dot{x}_{1,q}(0) = x_{1,q}^{1,0}, \ddot{x}_{1,q}(0) = x_{1,q}^{2,0}, ..., x_{1,q}^{(n-1)}(0) = x_{1,q}^{(n-1),0}. \quad (2.11)$$

Let us consider the function $\gamma_1(t)$

$$\gamma_1(t) = L_n[x_{1,q}(t)]. \quad (2.12)$$

It is evident that $\gamma_1(t) \in C[0, T]$.

The solutions $x_{1,q}(t)$ of nonhomogeneous differential equation

$$\gamma_1(t) = L_{n-2}[z_{j,q}(t)]. \tag{2.13}$$

with different initial conditions form the set $Z_{j,q}$, $z_{j,q}(t) \in Z_{j,q}$.

The solution $\hat{x}_{1,q}(t)$ of differential equation (2.8) with initial conditions (2.11) and any functions from set $Z_{j,q}$ will satisfy the inequality (2.10).

The right part $u_{\delta,j}$ of equation (2.1) is determined with the help of function $\tilde{x}_1(t)$ only through continuous operations. So the set of possible solutions $Q_{\delta,p,j}$ for equation (2.1) is non-empty for any δ and this set contains the functions continuously differentiable any number number of times. □

3. Possible approaches of solving synthesis inverse problems

Let us consider the possibility of constructing stable algorithms for solving inverse synthesis problems without assuming that the exact solution z_j^{ex} of the inverse problem belongs to the set of possible solutions $Q_{\delta,p,j}$.

If the functional spaces Z, U are Banach spaces and the operator $A_{p,j}$ is linear, then the set of functions $Q_{\delta,p,j}$ is convex, closed, and unbounded. Let's show it.

Let $z_1, z_2 \in Q_{\delta,p,j}$. Then for $z_\alpha = \alpha z_1 + (1-\alpha) z_2 \in Q_{\delta,p,j}$ inequality is realized

$$\|A_{p,j} z_\alpha - u_{\delta,j}\|_U \leq \| \alpha A_{p,j} z_1 + (1+\alpha) A_{p,j} z_2 - \alpha u_{\delta,j} - (1-\alpha) u_{\delta,j} \|_U \leq \delta_j.$$

Consequently, the sets $Q_{\delta,p,j}$ are convex.

Let the sequence $z_k \in Q_{\delta,p,j}$ strongly converges to the element z_0.

Then

$$\begin{aligned} \| A_{p,j} z_0 - u_{\delta,j} \|_U &\leq \| A_{p,j} z_k - A_{p,j} z_0 \|_U + \| A_{p,j} z_k - u_{\delta,j} \|_U \\ &\leq \| A_{p,j} \| \| z_k - z_0 \| + \delta j. \end{aligned}$$

Let us pass to the limit in the last inequality when $k \to \infty$. Then we have

$$\| A_{p,j} z_0 - u_{\delta,j} \|_U \leq \delta j.$$

Thus, the sets $Q_{\delta,p,j}$ are closed.

In inverse synthesis problems the exact solution z_j^{ex} of the inverse problem does not belong to the set of possible solutions $Q_{\delta,p,j}$. Therefore, it is impossible to construct the regularizing algorithms which give solutions converging to an exact solution. However, in inverse problems of synthesis there are no need in such algorithms property, but it is sufficient to find any function from the set of possible solutions $Q_{\delta,p,j}$. This set is unbounded for arbitrary δ because of the compactness of the operator $A_{p,j}$. Therefore, it makes sense to choose from the set $Q_{\delta,p,j}$ a non-arbitrary element but an element with additional properties which are suitable for further research. For example, we can choose as the solution of inverse problem the most "convenient"element from the set $Q_{\delta,p,j}$ for the mathematical modeling purposes. It is possible to choose from $Q_{\delta,p,j}$ the simplest element [9], [11] that is the most stable to small changes of the initial data, the best element for prediction purposes [9], [11] and so on. Some algorithms of a choice from set $Q_{\delta,p,j}$ of an element with additional properties based on a variation principle can are proposed.

Consider now the following extreme problem

$$\Omega[z_{\delta_j,p}] = \inf_{z\in Q_{\delta,p,j}} \Omega[z], \tag{3.1}$$

where functional $\Omega[z]$ has been defined on set Z [17].

Theorem 3.1. *Assume that system* (1.1) *has only one state variable $\tilde{x}_1(t)$, which is obtained by experiment, and only one unknown component $z_j(t), 1 \le j \le m$ of vector function of external loads. The matrix $F = \hat{F} = f_{k,i}, 1 \le k \le n, 1 \le i \le m$, has the special form: by fixed $l, j\,(1 \le j \le m, 2 \le l \le m)$ element $\hat{f}_{lj} \ne 0, \hat{f}_{ki} = 0, k \ne l, i \ne j$. Suppose that Z is a reflexive Banach function space, that the functional $\Omega[z]$ is convex and lower semi continuous on $Q_{\delta,p,j}$, that the Lebesgue set $M(v)$ bounded for a certain function from $v \in Q_{\delta,p,j}$: $M(v) = \{z \in Q_{\delta,p,j} : \Omega[z] \le \Omega[v]\}$. Then the solution of the extreme problem (3.1) exists and belongs to $Q_{\delta,p,j}$.*

Proof. It is obvious that the exact lower bound of the functional $\Omega[z]$ on $Q_{\delta,p,j}$ can be achieved only from the points of the set $M(v)$ [19].

We show that the set $M(v)$ closed. Let the sequence $z_k \subset M(v)$ strongly converges to w . Since the set $Q_{\delta,p,j}$ is closed we have $w \in Q_{\delta,p,j}$. The following inequality holds since the functional $\Omega[z]$ is lower semi continuous on $Q_{\delta,p,j}$:

$$\Omega[w] \le \underline{\lim}_{k\to\infty} \Omega[z_k] \le C = \Omega[v].$$

Hence we have $w \in M(v)$. The closedness of the set $M(v)$ is proved.

Let us show that the set $M(v)$ is convex. Let $z_1, z_2 \in M(v)$. Since the functional $\Omega[z]$ is convex, we have

$$\Omega[z_\alpha] = \Omega[\alpha z_1 + (1-\alpha)z_2] \le \alpha\Omega[z_1] + (1-\alpha)\Omega[z_2] \le$$

$$\le \alpha\Omega[v] + (1-\alpha)\Omega[v] = \Omega[v].$$

Consequently the element $z_\alpha = \alpha z_1 + (1-\alpha)z_2 \in M(v)$. The convexity of set $M(v)$ is proved.

It is known that any bounded closed convex set from a reflexive Banach space Z is weakly compact. So the set $M(v)$ is weakly compact set.

We choose an arbitrary minimizing sequence $z_k \in M(v)$

$$\lim_{k\to\infty} \Omega[z_k] = \Omega^*.$$

Since the set $M(v)$ is weakly compact, it follows that there is at least one subsequence $z_{k_m} \in M(v)$ that weakly converges to some point z^*

$$z_{k_m} \longrightarrow z^* \text{ as } m \to \infty, \; z^* \in M(v).$$

The following inequality is valid

$$\Omega^* \leq \Omega[z^*] \leq \underline{\lim}_{k\to\infty} \Omega[z_{k_m}] = \lim_{k\to\infty} \Omega[z_k] = \Omega^*, \Omega^* = \Omega[z^*].$$

□

Theorem 3.2. *Let us assumed that system* (1.1) *has only one state variable* $\tilde{x}_1(t)$*, which is obtained by experiment, and there is only one unknown component* $z_j(t), 1 \leq j \leq m$ *of vector function of external loads. The matrix* $F = \hat{F} = f_{k,i}, 1 \leq k \leq n, 1 \leq i \leq m$, *has the special form: by fixed* $l, j\,(1 \leq j \leq m, 2 \leq l \leq m)$ *element* $\hat{f}_{lj} \neq 0, \hat{f}_{ki} = 0, k \neq l, i \neq j$. *Suppose that* Z *is a Gilbert functional space, that the functional* $\Omega[z]$ *is strongly convex and lower semi continuous on* $Q_{\delta,p,j}$ *, that the Lebesgue set* $M(v)$ *is bounded for a certain function from* $v \in Q_{\delta,p,j}$*:* $M(v) = \{z \in Q_{\delta,p,j} : \Omega[z] \leq \Omega[v]\}$ *. Then the solution of the extreme problem* (3.1) *exists, unique and belongs to* $Q_{\delta,p,j}$.

Proof. According to Theorem 3.1, the set $Z^* = z \in Q_{\delta,p,j} : \Omega[z] = \Omega^*$ is not empty. Since the functional $\Omega[z]$ is strictly convex the set Z^* consists of a single point.

□

The choice of the function v is determined from physical considerations.

Thus, there is a principal possibility of constructing solutions of inverse synthesis problems on the basis of the variation principle. Satisfaction of additional conditions is determined by a special choice of the functional $\Omega[z]$. These solutions should be interpreted only as functions necessary for the subsequent mathematical modeling of physical processes in order to predict the behavior of physical processes, optimize the characteristics of these processes, etc.

In inverse problems of type II it is necessary to find such function from the set $Q_{\delta,p,j}$ which gives the useful information about exact solution of equation (2.1). However, such information cannot be obtained if the exact solution z_j^{ex} does not belong to the set $Q_{\delta,p,j}$.

Thus, it is necessary to take into account the error of operator $A_{p,j}$ in relation to the exact operator A_j^{ex}.

Let the characteristic of the deviation of the exact operator A_j^{ex} from the approximate operator $A_{p,j}$ be given [17] for linear operators $A_{p,j}$, A_j^{ex} in case U is normed functional space:

$$\| A_{p,j} - A_j^{ex} \|_{Z\to U} = \sup_{\|z\|\leq 1} \| A_{p,j}z - A_j^{ex}z \|_U \leq h_j. \tag{3.2}$$

A similar error characteristic can be determined in another way

$$\| A_{p,j} - A_j^{ex} \|_\Omega = \sup_{z \in Z_1} \frac{\rho_U(A_{p,j}z, A_j^{ex}z)}{\{\Omega[z]\}^{0.5}} \leq h_j \tag{3.3}$$

for fixed positive functional $\Omega[z]$ defined on $Z_1 \subset Z, Z_1$ everywhere dense in Z.

In this case the set of possible solutions of (2.1) $Q_{\delta,h,p,j}$ should be expanded so that the exact solution z_j^{ex} belongs to it with guarantee $z_j^{ex} \in Q_{\delta,h,p,j}$ [17]:

$$Q_{\delta,h,p,j} = \{z : \| A_{p,j}z - u_{\delta,j} \|_U \leq \delta_j + h_j \|z\|_Z\}, \tag{3.4}$$

where h_j is the error characteristic of the operator $A_{p,j}$.

The sets $Q_{\delta,h,p,j}$ are not bounded at any δ and any h_j while operators $A_{p,j}$ are compact operators. So inverse problems of type II belong to class of incorrect problems [9], [17] and special methods are used for their solutions [9], [10], [11], [17], [6].

To this type of inverse problems should be attributed the problems of diagnosis in various areas of activity [5], definition of real properties of physical objects [12] and etc. The inverse problems with such ultimate goals will be called as the inverse problems of measurement (or interpretation) [9], [10].

Consider now the following extreme problem:

$$\Omega[z_{h_j,\delta_j,p}] = \inf_{z \in Q_{\delta,h,p,j} \cap Z_1} \Omega[z], \tag{3.5}$$

where functional $\Omega[z]$ has been defined on set $Z_1 \subset Z, Z_1$ everywhere dense in Z [17].

During solving the practical inverse problems there are a big difficulties of definition of the value h_j since the structure and parameters of the exact operator A_j^{ex} cannot be determined in principle. Consequently, the exact solution z_j^{ex} of the inverse problem does not belong in an expanded set of possible solutions $Q_{\delta,h,p,j}$ with a guarantee as a rule.

In addition, the approximate solution obtained in this way after substituted into equation (2.1), gives a big value of deviation from the experimental data which excludes an objective evaluation of the results of solving the inverse problem.

A certain different approach for solving of inverse measurement problems is proposed in the works [13], [18]. To obtain the useful information on the exact solution z_j^{ex}, a hypothesis is proposed [14]. Such an approach makes it possible to obtain objective estimates from below of the exact solution z_j^{ex} in the sense of a priori given functional $\Omega[z]$.

When solving the well known inverse problem of astrodynamics (the measurement problem), Adams and Leverier did not take into account the error of the operator $A_{p,j}$ [12]. Nevertheless, the solution was obtained which turned out to be quite accurate. This contradiction gives a strong impetus for the search of new methods for solving measurement inverse problems.

4. Conclusion

It is shown that the algorithms for finding approximate solutions of inverse synthesis problems and inverse measurement problems for differential equations can be created without assuming that an exact solution of inverse problem belongs to the set of possible solutions.

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